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Lectures on $\mathcal{N} = 2$ gauge theory

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Abstract: In a series of four lectures I will review classic results and recent developments in four dimensional $\mathcal{N} = 2$ supersymmetric gauge theory. Lecture 1 and 2 review the standard lore: some basic definitions and the Seiberg-Witten description of the low energy dynamics of various gauge theories. In Lecture 3 we review the six-dimensional engineering approach, which allows one to define a very large class of $\mathcal{N} = 2$ field theories and determine their low energy effective Lagrangian, their S-duality properties and much more. Lecture 4 will focus on properties of extended objects, such as line operators and surface operators.

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Field theories with $\mathcal{N} = 2$ supersymmetry in four dimensions are a beautiful theoretical playground: this amount of supersymmetry allows many exact computations, but does not fix the Lagrangian uniquely. Rather, one can write down a large variety of asymptotically free $\mathcal{N} = 2$ supersymmetric Lagrangians. In this lectures we will not include a review of supersymmetric field theory. We will not give a systematic derivation of the Lagrangian for $\mathcal{N} = 2$ field theories either. We will simply describe whatever facts we need to know at the beginning of each lecture.

Part I

Lecture 1

Supersymmetric gauge theories have a rather simple structure. A canonical example is four dimensions $\mathcal{N} = 1$ super Yang-Mills theory. It consists of a gauge field coupled to a massless Majorana fermion valued in the adjoint representation of the gauge group, the gaugino. A necessary condition for supersymmetry is that the number of propagating bosonic and fermionic degrees of freedom should be the same. This happens for a four dimensional gauge field together with a fermion: both have two massless degrees of freedom. This theory has the minimal possible amount of supersymmetry available in four dimensions: four real conserved supercharges organized in a Weyl spinor Q_α and its conjugate $\bar{Q}_{\hat{\alpha}}$. They anticommute to the translations, $\{Q, \bar{Q}\} = \gamma^\mu P_\mu$. Schematically, they act on the fields as $QA_\mu = \gamma_\mu \psi$ and $Q\psi = F_{\mu\nu}\gamma^{\mu\nu}$.

In order to describe a simple gauge theory which admits a larger amount of supersymmetry, we can use a simple trick: consider the combination of a gauge field and an adjoint fermion in more than four dimensions. The number of degrees of freedoms of bosons and fermions will match in 6 and 10 dimensions. In 6 dimensions a Weyl fermion has four complex components, and a total of 4 degrees of freedom, matching the 4 polarization of the gauge field. The system has 4 complex supercharges. In 10 dimensions a Majorana-Weyl fermion has 16 components, and 8 degrees of freedom. The resulting theory has 16 supercharges.

From the Lagrangian of a theory in $4+d$ dimensions, one can derive a Lagrangian for a four-dimensional theory with the same amount of supersymmetry, by *dimensional reduction*: take all fields to be functions of four coordinates $x^0 \cdots x^3$ only (and restrict the gauge group to gauge transformations whose parameter depends on those coordinates $x^0 \cdots x^3$ only). The components of the connection along the remaining d coordinates behave as $4d$ scalar fields, and the higher dimensional fermions give rise to several $4d$ gauginoes. The result is a Lagrangian for four dimensional gauge theory with 8 supercharges ($\mathcal{N}=2$ in 4d) or 16 ($\mathcal{N}=4$ in 4d). More concretely, if we start from the 6d gauge theory, the A_4 and A_5 components of the gauge connection behave as two real $4d$ scalar fields valued in the adjoint representation of the gauge group. We will combine them into a complex scalar field $\Phi = A_4 + iA_5$. The $6d$ fermions give rise to a pair of four-dimensional gauginoes. The 4d Lagrangian has some interesting scalar couplings, which arise from the 6d terms containing covariant derivatives in the $4,5$ directions. There are Yukawa couplings to the fermions, from the $\psi D_4 \Gamma^4 \psi + \psi D_5 \Gamma^5 \psi$ pieces of the kinetic energy, and a quartic self-interaction term $[A_4, A_5]^2 \sim [\Phi, \bar{\Phi}]^2$ from the 6d gauge kinetic term.

The resulting Lagrangian is rather rigid. Unless more fields are added (we'll do that in the second lecture) It can be shown that all couplings are fixed by supersymmetry given the gauge kinetic term. For an asymptotically free theory the only possible couplingss are the gauge coupling g and the theta angle θ. We will often encounter the holomorphic combination $\tau = \frac{\theta}{2\pi} + \frac{4\pi^2}{g^2}$. The scalar potential has exact flat directions where Φ acquires an arbitrary diagonal expectation value: the Coulomb branch of vacua. At a generic location on the Coulomb branch, the gauge symmetry is Higgsed to the abelian subgroup $U(1)^r$, where r is the rank of the gauge group. All W-bosons acquire a mass through the Higgs mechanism, and so do the off-diagonal components of the fermions, through the Yukawa couplings to the scalar field. In the infrared, one is left with an abelian $\mathcal{N}=2$ gauge theory. We will denote the surviving scalar fields in the Cartan of the gauge group as a. The mass of a W-boson associated to a root vector α of the gauge group can be written as $|(\alpha, a)|$.

Something interesting is happening here to the supersymmetry algebra, which deserves some explanation. There is a basic textbook exercise in supersymmetry: count the number of particles in a multiplet of the supersymmetry algebra. If one considers a massive particle, brings it at rest, and assumes an anticommutator $Q, Q = E\gamma^0$, the whole SUSY multiplet can be built by treating half of the supercharges as creation operators, half as destruction operators. The result is a supermultiplet of dimension $2^{n/2}$, where n is the number of real supercharges. For a $\mathcal{N}=2$ theory that's 16 degrees of freedom, but the massive W-boson we saw in the Coulomb branch sits in a shorter multiplet which contains four bosonic and four fermionic degrees of freedom. How does such short representation appear? A natural way to obtain shorter multiplets is to consider massless particles, which cannot be brought to rest, and write the anticommutator as $Q, \bar{Q} = E(\gamma^0 + \gamma_1)$: half of the supercharge simply squares to zero, and the multiplet's dimension drops down.

From the point of view of the original 6d Lagrangian, this is what happens to the W-boson. Indeed, the translation operators which enter the 6d SUSY algebra act on the fields as covariant derivatives. On the Coulomb branch we gave an expectation value to A_4, A_5, hence the covariant derivatives P_4, P_5 can be non-zero even if the fields do not depend on x^4, x^5. Upon dimensional reduction, they appear in SUSY algebra of the $\mathcal{N} = 2$ 4d gauge theory. Schematically, $\{Q, \bar{Q}\} = \gamma^\mu P_\mu$ and $\{Q, Q\} = P_4 + iP_5$. The combination $Z = P_4 + iP_5$ is called "central charge" in the 4d theory, as it commutes with all the other generators of the algebra. Another useful point of view is that the anticommutator of supercharges gives a gauge transformation with parameter Φ, which will act non-trivialy on a charged state in the Coulomb branch of the theory. Either way it is easy to compute the central charge Z as the integral on the sphere at infinity of $\mathrm{Tr}\Phi F^+$. Here F^+ is the self-dual part of the field strength. For a W-boson of charge α it is (α, a). Hence the $6d$ momentum is null: $P^2 = m_{4d}^2 - |Z|^2 = 0$. This allows the W-boson to sit in a reduced multiplet.

In the presence of a central charge Z, the energy of all states is bounded from below by the BPS bound $E \geq |Z|$. Particles like the W-boson, which saturate the BPS bound are called BPS particles. BPS particles play a crucial role in our understanding of theories with extended SUSY. The mass of a BPS particle can move away from $|Z|$ only if it combines with several other BPS articles in groups which assemble a non-BPS SUSY multiplet. It is easy to write down an index (helicity supertrace) $Tr(-1)^F J_3^2$ which is zero for long (non-BPS) multiplets, and non-zero for BPS multiplets only. This index will count the number of "unpaired" BPS particles. Naively, such an index will not vary as the parameters of the theory are varied. This statement has a crucial loophole, which will play an important role later.

Which other BPS objects can exist in the gauge theory, besides W-bosons? It is useful to look first at the classical theory. The BPS bound arises naturally in the study of monopoles in the Higgs branch of the gauge theory, i.e. static classical configurations of the Higgs field and gauge fields, which end up carrying abelian magnetic charges. Indeed for a static configuration of a magnetic field B_i and a scalar field Φ the energy density $\mathrm{Tr}\left(B_i^2 + D_i\Phi^2\right)$ can be reorganized in a suggestive form $\mathrm{Tr}\left((B_i - D_i\Phi)^2 + 2B_iD_i\Phi\right)$. The first term is minimized by a BPS configuration, which satisfy the first order equation of motion $B_i = D_i\Phi$, while the second is a total derivative ($D_iB_i = 0$ because of Bianchi identity) which can be integrated to a surface term, the integral on the sphere at infinity of $\mathrm{Tr}\Phi F$. In the Coulomb branch, that is proportional to the magnetic charge of the configuration. Including the overall factor of g^{-2} and a contribution from the theta angle, this classical energy bound can be expressed in terms of a contribution $\tau(\beta, a)$ to the central charge, where β is the magnetic charge. The field equations for a BPS configuration with monopole charge have a moduli space of classical solutions, which depends on the specific choice of magnetic charge. Dynamical configurations are possible as well, which can carry electric charges α as well, and have a central charge $Z = (\alpha + \tau\beta, a)$. The classical configurations take the form of localized solitons carrying electric and magnetic charges. In the full quantum theory, these will behave as BPS particles. At weak coupling, one can compute the index which counts the number of such particles for a given charge, by looking for appropriate wavefunctions on the moduli space of classical solutions.

We aim to understand the low energy theory on the Coulomb branch of vacua. We have seen that the original non-abelian gauge theory reduces to a $U(1)^r$ abelian gauge theory. Classically, the Lagrangian for the abelian theory is rather trivial: if we denote the scalar fields in the Cartan subalgebra by a^I, the corresponding gauge fields as A^I, the Killing form as k_{IJ}, and denote τk_{IJ} as τ_{IJ}, then the Lagrangian takes the schematic form

$$\mathrm{Im}\tau_{IJ}\left(-\partial_\mu a^I \partial^\mu \bar{a}^J - F^I_{\mu\nu} F^{J,\mu\nu}\right) + \mathrm{Re}\tau_{IJ} F \wedge F \tag{1}$$

It is also useful to consider a set of gauge invariant order parameters on the Coulomb branch, the expectation values of the Casimirs of Φ $u_i = \langle C^{(i)}[\Phi]\rangle$. For example, for a $SU(n)$ gauge theory we can take $u^i = \langle \mathrm{Tr}\Phi^i\rangle$ for $i = 2, \cdots n$.

Quantum mechanically, we need to integrate away the massive W-bosons in order to get an effective low energy Lagrangian. This gives a one-loop correction to the gauge couplings. Non-renormalization theorems prevent perturbative higher loop corrections, but leave the possibility of instanton corrections. As a result, the gauge couplings of the IR theory become functions of the scalar fields a_I which parameterize the Coulomb branch, which are the the scalar superpartners of the $U(1)^r$ IR abelian gauge fields A_I. $\mathcal{N} = 2$ supersymmetry strongly restricts the possible form of the corrections. The whole low energy Largangian is encoded in a single holomorphic function of the a^I, the prepotential $\mathcal{F}(a)$. Then the holomorphic gauge coupling is $\tau_{IJ} = \partial_I \partial_J \mathcal{F}$. The form of the Lagrangian is still 1, with this field-dependent coupling. The functional relation between the scalar order parameters u_i and the scalar fields a_I may also be deformed. The central charge of the theory can be computed from the effective Lagrangian: the electric contribution is still linear in a^I, but but the magnetic contribution involves some $a^D_I = \partial_I \mathcal{F}$: $Z = q_{e,I} a^I + q^I_m a^D_I$.

Notice that the kinetic terms must be positive definite, i.e. $\mathrm{Im}\tau_{IJ}(a)$ should be positive definite. There is a certain degree of tension between this requirement and the holomorphicity of $\tau_{IJ}(a)$. This lies at the core of the beautiful work of Seiberg and Witten. Before we review the SW solution for the pure $SU(2)$ gauge theory, I'd like to present one last useful fact about the IR gauge theory, more precisely about its lattice of electric and magnetic charges. There is a symplectic pairing on the charge lattice: the gauge fields sourced by a configuration of two static objects of charges $\gamma_1 = (q_m, q_e)$ and $\gamma_2 = (q'_m, q'_e)$ carry an angular momentum $2j = q_e \cdot q'_m - q'_e \cdot q_m$. We will denote that antisymmetric pairing as $\langle\gamma_1, \gamma_2\rangle$. Dirac quantization requires it to be an integer for all pairs of charges. In the pure gauge theory, the Dirac quantization may not be saturated: electric and magnetic charges of the BPS particles live both in the root lattice of the gauge group, which may not be self-dual.

1. Seiberg-Witten solution of pure $SU(2)$ gauge theory

For a pure $SU(2)$ gauge theory, we have a single parameter a on the Coulomb branch. We normalize a as in arXiv:hep-th/9408099, so that the eigenvalues of the adjoint valued scalar field are $a, -a$ and the single order parameter $u = \langle \mathrm{Tr}\Phi^2 \rangle = 2a^2$. The coordinate u is a good parameterization of the Coulomb branch. At one loop, we have

$$a = \sqrt{u/2} \tag{1.1}$$

$$a_D = \frac{i}{\pi} a \log \frac{a^4}{\Lambda^4} \tag{1.2}$$

Here $\Lambda^4 = \exp 2\pi i \tau_{UV}$ is the strong coupling scale. In this normalization, the electric charge of a W boson is $q_e = 2$, and the charge of a magnetic monopole is $q_m = 1$. The spectrum of dyons at weak coupling includes once all the charges $\gamma = \pm(1, 2n)$ for all integers n. [1]

The gauge coupling of the IR abelian theory is $tau = \frac{\partial a_D}{\partial a} \sim \frac{i}{\pi} \log \frac{a^4}{\Lambda^4}$. It is weak at large $u \gg \Lambda^2$. As we go around the origin once in the u plane the "periods" (a, a_D) undergo an important monodromy due to the logarithm branch cut: $a \to -a$ and $a_D \to 4a - a_D$. The central charge $Z = q_e a + q_m a_D$ is unaffected by the monodromy: the IR theta angle is shifted as $\theta \to \theta + 8\pi$ and then Witten's effect causes a shift of the electric charges of dyons. The final result is a monodromy $q_e \to -q_e + 4q_m$ and $q_m \to -q_m$.

The gauge coupling becomes stronger towards the origin of the u plane. In the absence of instanton corrections, the imaginary part of τ would become negative as $|u| \sim \Lambda^2$. Instanton corrections must be rather drastic to solve this problem: the complex u plane cannot be mapped holomorphically inside the unit disk, but the function $\exp 2\pi i \tau$ is trying to just do that. Hence we need to map the u plane to the τ upper-half plane through a function with a more interesting pattern of cuts. We are severely limited by the constraint that the physics should be continuous across a cut in the u plane. The key observation to solve the problem is to notice that the naive one-loop expression for a_D goes to zero at $u = \Lambda^2$, indicating that the monopole particle may become light in the troublesome region of the u plane. If this is true in the full theory, the IR Lagrangian around $u = \Lambda^2$ should show the telltale effect of integrating away a light particle carrying magnetic charge!

In presence of a light magnetic particle it is natural to use the electric-magnetic duality of the IR theory and work with the dual gauge fields. Their superpartner turns to to be a_D, and the prepotential is Legendre-transformed, so that $-a = \frac{\partial \mathcal{F}_D}{\partial a_D}$. The dual gauge coupling is $\tau_D = -1/\tau$. Integrating away the electrically charged W boson of central charge $2a$ led to a logarithmic term in $\tau \sim \frac{4i}{\pi} \log a$. Dually, near a massless monopole singularity, $\tau_D \sim -1/\tau \sim -\frac{i}{2\pi} \log a_D$. Notice the different prefactor: the W-boson contribution had a $q_e^2 = 4$ prefactor, and another factor of -2 from its BPS index. The magnetic monopole sits in a different SUSY multiplet which has BPS index 1. Notice that the magnetic gauge coupling becomes weak in the region where the magnetic monopole is light. Moreover, a has a logarithmic behavior $a \sim -\frac{i}{2\pi} a_D \log a_D$ and we see a monodromy $a \to a + a_D$, i.e.

[1] In arXiv:hep-th/9407087 a different normalization was used, where the electric charge of the W-boson was set to 1.

$q_m \to q_m + q_e$ around the massless monopole singularity in the u-plane. You may wonder how such a monodromy could be consistent with the known spectrum of BPS particles at weak coupling: the answer is that the BPS spectrum is allowed to jump at certain special codimension 1 walls in the u plane. We will come back to this Lecture 4.

With a natural choice for the location of the cut in the logarithm at large u, of all the other dyons only the one with electric charge $(1,2)$ become light in the one-loop approximation, around $u = -\Lambda^2$. A similar discussion applies to the region where this dyon is light. In general, a light BPS particle of charge γ_0 will induce a singularity in the u plane around which we have a monodromy $\gamma \to \gamma + \langle \gamma, \gamma_0 \rangle \gamma_0$ of the charge lattice. Now, a beautiful observation: the monodromies of the periods and of τ induced by these two tentative singularities in the u plane combine to give the correct monodromy at large u! Indeed we have $(q_m, q_e) \to (q_m + q_e, q_e)$ around the massless monopole singularity, and then $(q_m + q_e, q_e) \to (q_m + q_e, q_e) + (q_e - 2q_m - 2q_e)(1,2) = (-q_m, -q_e - 4q_m)$ around the massless dyon singularity. It is consistent, and almost necessary, to assume that no other such singularities are present in the u-plane.

Let's review the physical content of this analysis. Semiclassically, we would expect a singularity at $u = 0$ where the W-boson becomes light, and non-abelian gauge theory is restored. The W-boson contribution to the β function makes the gauge coupling large at small u, and allows non-perturbative effects to take over before the gauge coupling is driven to unphysical values. The semiclassical singularity "splits" into two better behaved singularities, where magnetically charged solitonic degrees of freedom of spin < 1 become light, which drive the dual magnetic gauge coupling to be small, but keep it in the physical region.

In terms of the holomorphic map $\tau(u)$, this means that the image of the map will be a region with cusps at $\tau = \infty$, $\tau = 0$, $\tau = 2$, and sides glued together by $\tau \to \tau + 4$, $1/\tau \to 1 + 1/\tau$ and $1/(\tau - 2) \to 1 + 1/(\tau - 2)$. These transformations actually close to a subgroup $\Gamma_0(4)$ of $SL(2,Z)$ (upper right entry divisible by 4). The quotient of the upper half plane by $\Gamma_0(4)$ is a sphere, and $\tau(u)$ defines a holomorphic map from the u plane to the sphere. The map is unique once we specify the location of the singularities $u \pm \Lambda^2$. We could easily allow for a different location of the singularities, but a rescaling of u would simply amount to a different renormalization convention, and there is a certain Z_2 $u \to -u$ symmetry which is expected to be exact. We can find more than $\tau(u)$: even the periods a, a_D can be derived in a straightforward way: if we consider the matrix $g(u)$ whose columns are (a, a_D) and their u derivatives, then it has specific $SL(2,Z)$ monodromies M_0, M_1 and M_∞ around the singularities in the u plane: it transform as $g \to M_i g$. Then $A = g^{-1}\partial_u g$ is a meromorphic function on the u plane, with well defined poles at the singularities. It can be readily reconstructed from its poles, and then (a, a_D) are computed as solutions of $\partial_u g = gA$. They are certain hypergeometric functions.

Seiberg and Witten found useful to express this solution in terms of a family of elliptic curves of modular parameter τ. Although this may appear to be simply a shortcut towards writing the solution, the parameterization by the SW curve is key to solve a very wide class of $\mathcal{N} = 2$ theories. An elliptic curve is typically defined by a cubic equation $y^2 = p_3(x)$. In order to define the correct family of elliptic curves, we need to find the correct parameterization of the degree 3 polynomial $p_3(x)$ in terms of u. The family which gives rise to the correct $\Gamma_0(4)$ monodromies and modular region in the u plane is available in standard math literature, or can be guessed with a little bit of patience: $y^2 = \Lambda^2 z^3 + 2uz^2 + \Lambda^2 z$. The curve is a branched cover of the z plane, with singularities at $0, \infty, z_\pm = -\frac{u}{\Lambda^2} \pm \sqrt{\frac{u^2}{\Lambda^4} - 1}$. The periods of the holomorphic differential on the curve, $\omega = \frac{dz}{y}$, along appropriate cycles of the elliptic curve have all the correct properties to be the derivatives of (a, a_D) with respect to u: they transform correctly under monodromies and their ratio is τ. Be γ_e the cycle which winds around $0, z_+$: at large $u \gg \Lambda^2$ $z_+ \to 0$, and if we compute the contour integral $\frac{1}{4\pi i} \oint_{\gamma_e} \omega$ around the circle $|z| = 1$, the term proportional to u in y^2 dominates, $y \sim \sqrt{2u}z + \cdots$ and the period of ω is $\frac{1}{2\sqrt{2u}} + \cdots$, which has the correct behavior for $\partial_u a$. We can fill the dots by expanding the square root

$$y^{-1} \sim \frac{\sqrt{2u}z}{(} 1 + \sum_n \binom{-1/2}{n} \left(\frac{\Lambda^2}{2u}\right)^n (z + 1/z)^n \tag{1.3}$$

and extract the residue from the $1/z$ terms to recover the series expansion for a in powers of Λ^4/u^2.

A second natural cycle γ_m winds around z_+ and z_-. At large u, $z_- = 1/z_+ \sim -\frac{2u}{\Lambda^2}$ and the period of ω receives a logarithmic contribution from the integral ω, which is still correctly reproduced by $\omega \sim \frac{1}{2\sqrt{2u}} \frac{dz}{z}$: $\frac{1}{i\pi\sqrt{2u}} \log \frac{2u}{\Lambda^2}$ which agrees with the desired behavior of $\partial_u a_D$. The other singularities, at $u = \pm\Lambda^2$, correspond to $z_+ = z_-$. We can compute $\partial_u a_D$ near $u = \Lambda^2$ with a small loop around the pair of zeroes , it has a finite limit $\frac{1}{2\Lambda}$. On the other hand, $\partial_u a \sim \frac{1}{4\pi i\Lambda} \log(u - \Lambda^2)$. We see the expected monodromy.

It is useful to introduce the notion of Seiberg-Witten differential, a meromorphic λ such that $\partial_u \lambda = \omega$ up to exact forms, so that that the periods on γ_e, γ_m reproduce a, a_D directly. A simple choice is $\lambda = y\frac{dz}{z^2}$. For future reference, it is useful to change variables slightly to $x = y/z$, so that $\lambda = xdz/z$ and the curve is $x^2 - 2u = \Lambda^2(z + 1/z)$.

2. Solution of $SU(N)$ gauge theory

The latter form of the curve and differential is quite suggestive. If we make Λ^2 very small, for intermediate values of $|z|$ x is almost constant, with value $\pm a$, and the period of λ around the origin on either sheet is close to $\pm a$ as well. A simple generalization of the $SU(2)$ curve would be a curve of the form $P_N(x) = \Lambda^N(z + 1/z)$ for some degree N polynomial $P_N(x)$. Then for small Λ (the power of Λ in the curve is selected for dimensional reason) at intermediate values of $|z| \sim 1$ the value of x on the various sheets is quite close to the roots of $P_N(x)$. The period of λ on a cycle γ_i^e winding around the origin on one of the N sheets is then also very close to the corresponding root of P_N. We could tentatively identify the coefficients of P_N as the order parameters of an $SU(N)$ gauge theory, by setting $P_N(x) = \langle \det x - \Phi \rangle$. The Coulomb branch parameters a_i are expected to be close to the roots of $P_N(x)$ at weak coupling, and could be identified with the periods of λ on γ_i^e. From now on we take P_N to be monic, and set the coefficient of z^{N-1} to zero, so that the a_i add to zero. The cycles γ_i^e modded by $\sum \gamma_i^e = 0$ generate a lattice which can be identified with the weight lattice of $SU(N)$. The W-boson charges will be of the form $\gamma_i^e - \gamma_j^e$.

At small and at large $|z|$, the N sheets meet at ramification points defined by $P'_N(x_i) = 0$, $z_i + 1/z_i = P_N(x_i)/\Lambda^N$. There are $N-1$ ramification points near the origin, and $N-1$ at large $|z|$, from the $N-1$ roots of $P'_N(x)$. To visualize the remaining homology cycles, it is useful to consider z as a function of x. Away from $x \sim a_i$, there are two sheets which are well approximated by $z = \frac{P_N(x)}{\Lambda^N}$ and $z = \frac{\Lambda^N}{P_N(x)}$. They meet in the regions $x \sim a_i$. The electric cycles γ_e simply wind around any of the $x \sim a_i$ tubes, and the dual magnetic cycles γ^m_{ij} go from $x = a_i$ to $x = a_j$ on one sheet, and back on the other sheet. They behave like another root lattice of $SU(N)$, as we expected for the magnetic monopole charges of pure $SU(N)$ gauge theory, and the intersection of electric and magnetic cycles reproduces the expected symplectic pairing: $\gamma^e_i \cdot \gamma^m_{jk}$ equals 1 if $i = j$, -1 if $i = k$ and zero otherwise.

The periods of λ on the magnetic cycles receives a logarithmic contribution at small Λ, which generalizes the $SU(2)$ result: away from a small regions near $x \sim a_i$, $z \sim P_N(x_i)/\Lambda^N$ and $\lambda = \sum_k \frac{a_k dx}{x-a_k} + N dx$. For a contour going from a_i to a_j, the terms with $k \neq i, j$ are readily integrated to $a_k \log(a_j - a_k) - a_k \log(a_i - a_k)$. The divergence of the term $k = i$ at $x = a_i$ is cutoff in the full curve and the logarithmic piece is just $a_i \log(a_j - a_i)/\Lambda$. A similar term comes from $k = j$. All in all, these expressions take the form of $a^j_D - a^i_D$ where a^i_D are computed from a prepotential $\mathcal{F} = \frac{1}{4\pi i} \sum_{i,j} (a_i - a_j)^2 \log(a_i - a_j)/\Lambda$: the expected result for the one loop effect of the W-bosons in the pure $SU(N)$ theory.

Notice that if u_a is the a-th coefficient of $P_N(x)$, the derivatives $\partial_{u_a}\lambda = -\frac{\partial_{u_a} P}{\partial_x P}\frac{dz}{z} = \frac{x^a dx}{z - 1/z}$ are holomorphic differentials on the curve: they are regular at finite x and behave as $\frac{dx}{x^{N-a}}$ as $x \to \infty$, $z \to 0$ or $x \to \infty$, $z \to \infty$ (remember $a \geq 2$). Hence the gauge couplings of the theory coincide with the period matrix of the curve, and $\mathrm{Im}\tau_{IJ}$ is positive definite. A priory, there was no reason for τ_{IJ} to be the period matrix of a curve: the period matrices of Riemann surfaces are a very low dimensional submanifold of all possible period matrices. The claim that this SW curve and differential describe the IR Lagrangian of the $SU(N)$ gauge theory is much more surprising than it was for $SU(2)$.

What are the physical consequences of this proposal? Semiclassically, the locus in the Coulomb branch where two eigenvalues a_i of Φ coincide supports a massless W-boson, and a non-abelian $SU(2)$ gauge symmetry is restored. This singularity is problematic, and should be resolved non-perturbatively. The SW curve proposes a resolution we are familiar with: two nicer singularities where a monopole or dyon becomes massless, at the loci where either two roots of $P_N - 2\Lambda^N$ or two roots of $P_N + 2\Lambda^N$ coincide.

Part II

Lecture 2

In this lecture I will add matter fields to the Lagrangian. In $\mathcal{N} = 2$ gauge theories matter fields are organized in *hypermultiplets*. Each hypermultiplet contains set of fermions, and two complex scalar fields Q and $\tilde{Q}$, sitting in conjugate representations of the gauge group, R and $\bar{R}$. I will not write the full Lagrangian here, I will need a single coupling $|\Phi Q|^2 + |\Phi \tilde{Q}|^2$ to the vector multiplet scalar field, where Φ is acting on the gauge indices. This implies that on the Coulomb branch, where Φ has a generic diagonal expectation value, most of the hypermultiplets are massive, and should be integrated away in the IR. At special points in the Coulomb branch, some hypermultiplet masses may go to zero, and the hypermultiplet scalar fields may be allowed expectation values, giving rise to other branches of the theory called Higgs branches. The geometry of Higgs branches s strongly protected by supersymmetry. It is possible to put stringent conditions on the structure of the IR theory by comparing the Higgs branch computed in the IR and the Higgs branch computed in the UV. Due to our time constraints, we will not be able to discuss such considerations in detail.

It is also possible to introduce explicit mass terms for hypermultiplets. This is possible only if the theory has a flavor symmetry group. The mass parameters take the form of a complex element of the flavor symmetry Lie algebra, and appear in the Lagrangian as $|MQ|^2 + |M\tilde{Q}|^2$, where M is acting on the flavor indices. The commutator $[M, \bar{M}]$ has to vanish, so M can be taken in the Cartan subalgebra of the flavor Lie algebra. A generic M breaks the flavor symmetry to the abelian Cartan subgroup. We denote the components of M as m_i. If R is a generic complex representation, hypermultiplets in n copies of R always carry a $U(n)$ flavor symmetry. If R is a real representation, one may naively expect a larger $U(2n)$ flavor symmetry rotating q into $\tilde{q}$, but the actual couplings break that to $USp(2n)$. On the other hand, if R is pseudoreal, the flavor symmetry is $SO(2n)$. Sometimes, if R is a pseudoreal representation on can impose a reality condition which leads to a single "half" hypermultiplet, or an odd number of them, with flavor symmetry $SO(2n+1)$. Quantum mechanically, there may by anomalies preventing this possibility.

Our canonical example is $SU(2)$ gauge theory. We can add a set of N_f hypermultiplets in a doublet representation. As the doublet is a pseudoreal representation, the resulting theory has $SO(2N_f)$ flavor symmetry. This is a canonical example where N_f cannot be a half-integer because of an anomaly (REFERENCE). The fundamental flavors contribute to the one-loop beta function, and if we add an excessive amount of matter we will loose asymptotic freedom. A borderline case is $SU(2)$ $N_f = 4$: the one-loop beta functins is exactly zero, and SUSY prevents higher loop corrections, hence the theory is exactly (super)conformal. The gauge coupling does not flow, and is an exactly marginal parameter of the theory. We can define the theory at small values of the coupling, and wonder what happens as the coupling increases. This leads to the subject of S-duality, which is the focus of my third lecture.

In the presence of a mass parameter m, the IR central charge is modified by an extra term $q_f m$: the mass of a BPS particle which carry flavor charge receives a contribution from the UV mass m. We will include q_f in the charge vectors $\gamma = (q_e, q_m, q_f)$. The flavor charges do not contribute to the pairing $\langle\gamma_1, \gamma_2\rangle$. We will denote the central charge $q_e a + q_m a_D + q_f m = a_\gamma$. The monodromies of the charge lattice around a singular locus caused by a massless particle carrying gauge and flavor charges γ are still given by $\tilde{\gamma} \to \tilde{\gamma} + \langle\gamma, \tilde{\gamma}\rangle\gamma$.

3. $N_f = 1$ $SU(2)$

We can start with the simple example of a $N_f = 1$ theory with no mass parameter turned on for the $SO(2)$ flavor symmetry. The perturbative expression for the periods is

$$a = \sqrt{u/2} \tag{3.1}$$

$$a_D = \frac{i}{\pi} a \log \frac{a^3}{\Lambda^3} \tag{3.2}$$

The expression for a_D combines the one-loop contribution from the W-boson, which has charge 2 and as a vector multiplet contributes with a factor of -2 to supersymmetric computations, and the two particles in the hypermultiplet doublet, which have electric charge 1, flavor charge ± 1 and contribute each with a factor of 1. The monodromy at infinity is $\tau \to \tau + 3$. Again, naively a_D vanishes around $u \sim \Lambda^2$, and then $a_D + a$ and $a_D + 2a$ vanish respectively around $u = \Lambda^2 \exp \frac{2\pi i}{3}$ and $u = \Lambda^2 \exp \frac{4\pi i}{3}$. Can the correct singularity structure in the u plane consist really of these three points? This picture passes a very strong test: if we concatenate the monodromies for the three singular points we recover the monodromy at infinity! In detail, ignoring for now the flavor charges,

$$(q_e, q_m) \to (q_e, q_m + q_e) \to (q_e - q_m, q_e) \to (-q_e - 3q_m, -q_m). \tag{3.3}$$

If we accept this picture, we could in principle hope to determine $\partial_u a$ and $\partial_u a_D$ as solutions of some differential equation on the u plane, with singularities at three points, and infinity. This problem is much harder than the one we met for $SU(2)$ $N_f = 0$. This is because only the conjugacy classes of the monodromy matrices are simple to encode in the differential equation, and the differential equation with four singularities has some undetermined coefficients which can be fixed in terms of the actual monodromies only by solving the differential equation. A much better strategy is to look directly for a Seiberg-Witten curve.

With only a bit of guesswork, one can get to a curve $y^2 = \Lambda^2 z^3 - 2uz^2 + \Lambda^2$, a mild variation of the $N_f = 0$ curve. The discriminant of the curve is $32u^3 - 27\Lambda^6$, hence we have three singularities, and at each singularity two branch points of y meet and the curve degenerates in a simple way. At large u, intermediate z, $y \sim (-2u)^{1/2} z$ and the periods of dz/y along the cycle $|z| = 1$ reproduce $\partial_u a$. At large z a branch point sits at $z \sim 2u/\Lambda^2$. At small z two branch points sit at $z \sim \sqrt{\Lambda^2/2u}$. Notice that the cycle $|z| = 1$ wraps around these two branch points. The period of dz/y along a cycle which wraps around the branch point at large z and either one of the two branch points at small z has a logarithmic contribution which matches to $\partial_u a_D$. We can integrate the holomorphic differential rather easily to $\lambda = y dz/z^2$, and verify patiently that it gives the appropriate expressions for a, a_D at weak coupling.

How does the picture change if we turn on the mass parameter? If the mass is very large, $m \geq \Lambda$, the naive central charge, $m \pm a$ suggests that an electrically charged particle should be massless at $u \sim 2m^2$, where the one loop approximation can be trusted. At scales well below m, we should recover the u plane of the $N_f = 0$ theory, with the singularities associated to gauge charges $(1, 0)$ and $(1, 2)$. Hence we see three singularities as at $m = 0$, and two of them have the same gauge charges, but the singularity due to gauge charges $(1, 1)$ is now associated to the elementary electrically charged hypermultiplets. This has a simple, if surprising, explanation: if we bring the $(1, 1)$ singularity of the $m = 0$ theory all the way to large u, passing in between the other two singularities, we cross the monopole monodromy line which transforms it into the desired particle of electric charge 1.

Can we reconstruct the flavor charge content of the particles which are massless at the singularities? We should keep in mind that the question has a certain degree of ambiguity: when we assign flavor charges in the IR to the magnetically charged particles, we can always shift our choice by multiples of the magnetic charges. In other words, we have a freedom to shift a_D by multiples of m without affecting the IR Lagrangian. At small m, if we assign flavor charges q_0, q_1 and q_2 respectively, the monodromies will act as

$$\begin{aligned}(q_e, q_m, q_f) &\to (q_e, q_m + q_e, q_f + q_e q_0) \to (q_e - q_m, q_e, q_f + q_e q_0 - q_m q_1) \\ &\to (-q_e - 3q_m, -q_m, q_f + q_e q_0 - q_m q_1 - (q_e + q_m) q_2\end{aligned} \tag{3.4}$$

It is easy to see from the one-loop a_D that the monodromy at infinity is not expected to shift q_f, hence $q_0 = q_2 = -q_1$. When we move to large m, the charge of the fundamental hypermultiplet is obtained by the monodromy $(1, 1, q_1) - (0, 1, q_0) = (1, 0, -2q_0)$. The answer $q_0 = -q_1 = q_2 = 1/2$ is actually quite sensible: the magnetic monopoles carry flavor charges due to the quantization of some fermion zeromodes from the matter hypermultiplets. If the matter fields are in a fundamental irrep of $SO(2N_f)$, the monopoles will be in a spinor representation of $SO(2N_f)$, with chirality determined by the parity of the electric charge.

It is easy to write down a curve which interpolates between small and large mass: $y^2 = \Lambda^2 z^3 - 2uz^2 - 2m\Lambda z - \Lambda^2$. We keep the same $\lambda = ydz/z^2$. At large mass, the last term in the equation can be disregarded, and a simple rescaling brings forth the $N_f = 0$ curve. We see that the branch points at large u, small z are at the roots of $2uz^2+2m\Lambda z+\Lambda^2$, and collide as desired at $u \sim 2m^2$. Notice that at small z, $y \sim \pm\Lambda \pm mz + \cdots$, hence λ has a pole at $z = 0$ with charge m. This is important: if λ has a pole there, the periods of λ along a cycle γ will depend on how γ winds around 0. Different choices would differ by integer multiples of the m. Hence the periods of λ take the desired form $Z_\gamma = aq_e + a_D q_m + mq_f$.

Finally, there is an useful way to rewrite the SW curve: shift $y \to xz + \Lambda$ so that $x^2 + 2u = \Lambda^2 z - 2(x + m)\frac{\Lambda}{z}$. The SW differential would go to $xdz/z + \Lambda/z^2$, but we can drop the latter term: it is single valued and has zero periods on all integration cycles. Notice also for future reference that the transformation $z \to 2z(x + m)/\Lambda$ moves some factors around in an interesting way, to $x^2 + 2u = 2\Lambda(x+m)z - \frac{\Lambda^2}{z}$, which is simply related to the original curve. In the process, $xdz/z \to xdz/z + x/(x + m)dx$. The last term is interesting: it will contribute to the period integrals only through the residue at $x = -m$, which is m. This shift of periods by a multiple of m is equivalent to a shift of the flavor charge by a multiple of the gauge charges.

4. Solution of $SU(N)$ gauge theory with fundamental flavors

The latter form of the curve suggests an interesting generalization: $P_N(x) = Q_M(x)\Lambda^{N-M}z+$

$\tilde{Q}_{M'}(x)\Lambda^{N-M'}/z$, $\lambda = xdz/z$. P,Q, $\tilde{Q}$ are monic polynomals. Let us observe the main features of this SW curve. The SW differential λ has poles at $z \to 0$ or $z \to \infty$ if x goes to the zeroes of Q_M and $\tilde{Q}_{M'}$. The residues are the roots of Q_M and $\tilde{Q}_{M'}$, which we should identify with mass parameters of the field theory. If we send the masses all to infinity, the curve reduces to the curve of pure $SU(N)$ gauge theory. We would like to argue that this curve is associated to the $SU(N)$ gauge theory with $M + M'$ fundamental flavors. Notice that we can freely move factors from the polynomial Q_M to $\tilde{Q}_{M'}$ and back, by the redefinitions $z \to zp(x)$. This shifts the flavor charges of the theory by multipletof the gauge charges, and is harmless. In order to have an asymptotically free theory, $2N > M + M'$. The $2N = M + M'$ case is also very interesting, as the theory is exactly conformal. In the following we will restrict to $2N > M + M'$. It is also useful to move factors around until $M \leq N$ and $M' \leq N$.

If Λ is very small, there is a large range of $|z|$ for which the N sheets of the equation lie close to $x \sim a_i$, with $P(x) \sim 0$. The periods along curves of constant $|z|$ recover the electric periods a_i, which are close to the roots of $P(x)$. We can repeat the exercise we did for pure $SU(N)$ gauge theory, and determine the logarithmic behavior of magnetic periods, and of the gauge couplings. If we look at z as a function of x, we have two sheets. One sheet keeps at large z, and away from a small regions near $x \sim a_i$, $z \sim P_N(x)/Q_M(x)\Lambda^{M-N}$ and $\lambda = \sum_k \frac{a_k dx}{x-a_k} - \sum_s \frac{m_s dx}{x-m_s} + (N-M)dx$. The other sheets keeps at small z, and away from a small regions near $x \sim a_i$, $1/z \sim P_N(x)/\tilde{Q}_{M'}(x)\Lambda^{M'-N}$ and $\lambda = \sum_k \frac{a_k dx}{x-a_k} - \sum_s \frac{\tilde{m}_s dx}{x-\tilde{m}_s} + (N-M')dx$.

We consider magnetic contours which go from $x = a_i$ to $x = a_j$ on one sheet, and back on the other sheet. All in all, these expressions take the form of $a_D^j - a_D^i$ where a_D^i are computed from a prepotential $\mathcal{F} = \frac{1}{4\pi i}\sum_{i,j}(a_i - a_j)^2 \log(a_i - a_j)/\Lambda - \frac{1}{8\pi i}\sum_{i,s}(a_i - m_s)^2 \log(a_i - m_s)/\Lambda + (m \to \tilde{m})$: the expected result for the one loop effect of the W-bosons and massive hypermultiplets in the pure $SU(N)$ theory.

5. Linear quivers of unitary gauge groups

The form of the curves we saw until now suggests further generalizations. Consider the curve $\sum_{i=0}^{n} c_i P_{N_i}(x)\Lambda^{N-N_i}z^i = 0$. If the polynomials P_{N_i} are monic, and $2N_i > N_{i-1} + N_{i+1}$, then at sufficiently small Λ there will be ranges of $|z|$ where a single term in the sum dominates the others, and there will be N_i roots for x which are close to the roots of the corresponding polynomial P_{N_i}. The periods of λ along cycles which wrap constant $|z|$ in the appropriate region and sheet will behave like the Coulomb branch parameters of a weakly coupled $SU(N_i)$ gauge group (flavor group for $i = 0$ or $i = n$).

If we look at z as a function of x there will be n sheets, and (with some help from a theorem by Newton), each sheet is well described by balancing two terms of the whole sum, $z_i \sim -P_{N_{i-1}}/P_{N_i}\Lambda^{N_i - N_{i-1}}$. Sheets meet at x near the roots of the P_{N_i} and we can describe convenient magnetic cycles which run along the $i-th$ sheet between zeroes of P_{N_i} and back along the $(i+1)$-th sheets. At small Λ we can compute the logarithmic contributions to the periods of λ along these magnetic periods, and recognize a simple matter content: bifundamental hypermultiplets between the $SU(N_i)$ and $SU(N_{i+1})$ gauge groups! The SW curve and differential appear to describe a linear quiver of special unitary gauge groups. The condition $2N_i > N_{i-1} + N_{i+1}$ ensures asymptotic freedom at all nodes. A little bit of extra work shows that the gauge coupling scale of the i-th gauge group is given as

$$\exp 2\pi i \tau_i = \Lambda_i^{2N_i - N_{i-1} - N_{i+1}} = \Lambda^{2N_i - N_{i-1} - N_{i+1}} \frac{c_{i-1} c_{i+1}}{c_i^2} \tag{5.1}$$

Again, the asymptotic freedom condition could be relaxed to $2N_i \geq N_{i-1} + N_{i+1}$ to include gauge groups with exactly marginal couplings with small changes to the analysis.