USING THE FDU METHOD AT NLO

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Content

- Basic introduction
- Loop-tree duality
- Massless Feynman integrals
  - Location of IR singularities and local UV counterterms
- Loop-tree duality with massive particles
- Physical example: $A^* \to q\bar{q}(g)$ @ NLO
- Conclusions and perspectives

Basic references (i.e. starting point)

1. *Catani et al, JHEP 09 (2008) 065*
2. *Rodrigo et al, Nucl.Phys.Proc.Suppl. 183:262-267 (2008)*
3. *Buchta et al, JHEP 11 (2014) 014*

Specific references (technical details)

Rodrigo et al, JHEP 02 (2016) 044; JHEP 08 (2016) 160; JHEP 10 (2016) 162

Basic introduction

Theoretical motivation

- When computing **IR-safe observables,** divergences cancel combining the real and virtual corrections (**KLN** theorem)
- For IR singularities, **phase-space integrals of real radiation** should originate the same structures that appear in **Feynman integrals for loop diagrams** ⇨ *Loop-tree theorems!*

Physical observable ⇨

Virtual corrections (loop integrals)

$$\int \frac{d^D q}{(2\pi)^D}$$

\+ *Real corrections (PS integrals)*

$$\int \frac{d^{D-1}\vec{q}}{(2\pi)^{D-1}\, 2q_0} = \int \frac{d^D q}{(2\pi)^D}(2\pi)\delta\left(q^2\right)\theta(q_0)$$

\+ *Renormalization counter-terms (ε poles times leading order)*

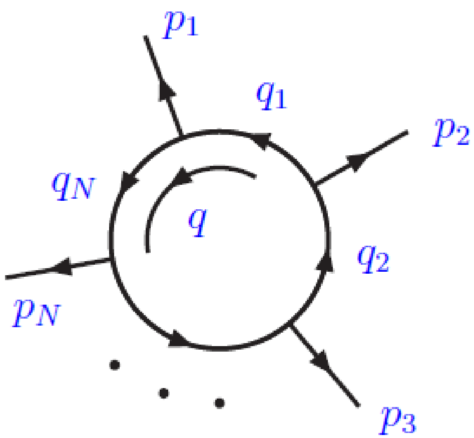
$$\frac{C_r}{\epsilon} \times d\sigma^{(0)}$$

Pole cancellation AFTER performing real-virtual integrals!!

WE WANT INTEGRAND LEVEL CANCELLATION!!!

Basic introduction

Feynman integrals and propagators



$$L_R^{(N)}(p_1, p_2, \cdots, p_N) = -\imath \int \frac{d^d\ell}{(2\pi)^d} \prod_{i=1}^{N} G_R(q_i)$$

Generic one-loop Feynman integral

$$q_i = \ell + \sum_{k=1}^{i} p_k$$

Momenta definition

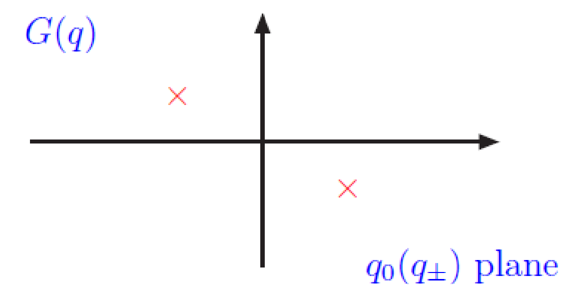
Prescriptions are useful to avoid poles. Different prescriptions are possible; connection between FTT and LTD theorems!

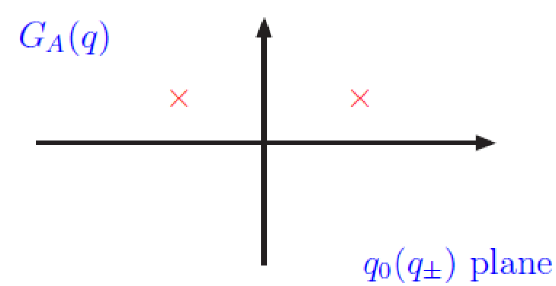
Feynman propagator

$$G(q) \equiv \frac{1}{q^2 + \imath 0}$$

Advanced propagator

$$G_A(q) \equiv \frac{1}{q^2 - \imath 0\, q_0}$$





Loop-tree duality

Derivation (one-loop)

- **Idea:** *«Sum over all possible 1-cuts»* (but with a **modified prescription…**)
  - Apply Cauchy's residue theorem to the Feynman integral:

$$L^{(N)}(p_1, p_2, \ldots, p_N) = \int_{\mathbf{q}} \int dq_0 \prod_{i=1}^{N} G(q_i) = \int_{\mathbf{q}} \int_{C_L} dq_0 \prod_{i=1}^{N} G(q_i) = -2\pi i \int_{\mathbf{q}} \sum \mathrm{Res}_{\{\mathrm{Im}\, q_0 < 0\}} \left[\prod_{i=1}^{N} G(q_i) \right]$$

  - Compute the residue in the poles with negative imaginary part:

$$\mathrm{Res}_{\{i\mathrm{-th\ pole}\}} \left[\prod_{j=1}^{N} G(q_j) \right] = \left[\mathrm{Res}_{\{i\mathrm{-th\ pole}\}}\, G(q_i) \right] \left[\prod_{\substack{j=1 \\ j\neq i}}^{N} G(q_j) \right]_{\{i\mathrm{-th\ pole}\}}$$

$$\left[\mathrm{Res}_{\{i\mathrm{-th\ pole}\}} \frac{1}{q_i^2 + i0} \right] = \int dq_0\, \delta_+(q_i^2)$$

$$\left[\prod_{j\neq i} G(q_j) \right]_{\{i\mathrm{-th\ pole}\}} = \prod_{j\neq i} \frac{1}{q_j^2 - i0\, \eta(q_j - q_i)}$$

Put on-shell the particle crossed by the cut

Introduction of «dual propagators» (η prescription, a future- or light-like vector)

Catani et al, JHEP09(2008)065; Rodrigo et al, JHEP02(2016)044

Loop-tree duality

Derivation (general facts)

- *It is crucial to keep track of the prescription!* Duality relation involves the presence of dual propagators:

$$L^{(N)}(p_1, p_2, \dots, p_N) = -\int_q \sum_{i=1}^{N} \tilde{\delta}(q_i) \prod_{\substack{j=1 \\ j\neq i}}^{N} \frac{1}{q_j^2 - i0\,\eta(q_j - q_i)}$$

  - The prescription involves a future- or light-like vector (arbitrary) and could depend on the loop momenta (at 1-loop is always independent of *q*). It is related with the finite value of **i0** in intermediate steps
  - *Connection with Feynman Tree Theorem*: **dual prescription** encodes the information contained in **multiple cuts**
  - Implement a shift in each term of the sum to have the same measure: the loop integral becomes a phase-space integral!
  - The unification of coordinates allows a cancellation of singularities among dual components (*UV and soft/collinear divergences remaining*)

Catani et al, JHEP09(2008)065; Rodrigo et al, JHEP02(2016)044

Loop-tree duality

Dual representation of one-loop integrals

Loop Feynman integral

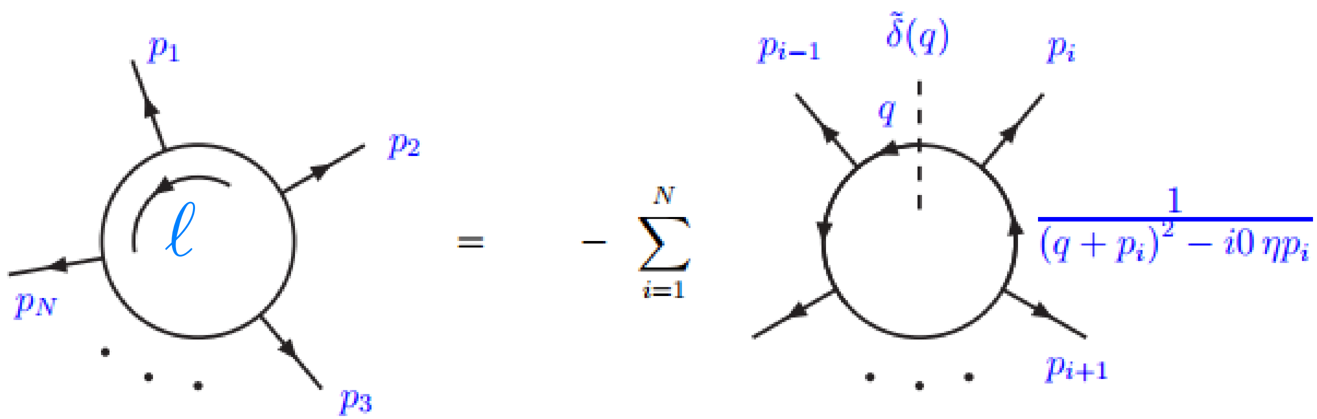
$$L^{(1)}(p_1,\ldots,p_N) = \int_\ell \prod_{i=1}^{N} G_F(q_i) = \int_\ell \prod_{i=1}^{N} \frac{1}{q_i^2 - m_i^2 + i0}$$

Dual integral

$$L^{(1)}(p_1,\ldots,p_N) = -\sum_{i=1}^{N} \int_\ell \tilde{\delta}(q_i) \prod_{j=1, j\neq i}^{N} G_D(q_i;q_j)$$

Sum of phase-space integrals!

$$G_D(q_i,q_j) = \frac{1}{q_j^2 - m_j^2 - i0\eta(q_j - q_i)} \qquad \tilde{\delta}(q_i) = i2\pi\,\theta(q_{i,0})\,\delta(q_i^2 - m_i^2)$$

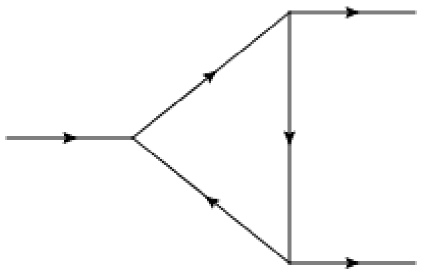


Catani et al, JHEP09(2008)065; Rodrigo et al, JHEP02(2016)044

Massless Feynman integrals

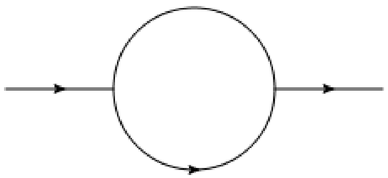
- **Idea I:** *apply loop-tree duality directly to Feynman integrals, in order to write them as PS integrals.*
- **Idea II:** *write counter-terms and perform integrand-level subtraction. This will lead to purely 4-dimensional integrable expressions!!*
- Two different kinds of physical singularities: UV and IR
  - IR divergent integral: massless triangle

$$L^{(1)}(p_1, p_2, -p_3) = \int_\ell \prod_{i=1}^{3} G_F(q_i) = -\frac{c_\Gamma}{\epsilon^2 \, s_{12}} \left(\frac{-s_{12} - \imath 0}{\mu^2} \right)^{-\epsilon}$$



  - UV divergent integral: bubble with massless propagators

$$L^{(1)}(p, -p) = \int_l \prod_{i=1}^{2} G_F(q_i) = c_\Gamma \frac{\mu^{2\epsilon}}{\epsilon\,(1-2\epsilon)} (-p^2 - \imath 0)^{-\epsilon}$$



Rodrigo et al, JHEP02(2016)044; JHEP08(2016)160; JHEP10(2016)162

Massless Feynman integrals

- To find the dual representation of Feynman integrals, we follow some steps:
  - ✓ If there are only single poles, we replace standard propagators with dual ones. Otherwise, we compute the residue and remove the energy integral:

$$\mathrm{Res}\,(f, z_0) = \frac{1}{(n-1)!}\left[\frac{\partial^{n-1}}{\partial z^{n-1}}\left((z-z_0)^n f(z)\right)\right]_{z=z_0} \Longrightarrow \int d\vec{q}_i\ \mathrm{Res}\left(\prod_j G_F(q_j), q_{i,0}^{(+)}\right)$$

  - ✓ Parametrize momenta; for instance, for 1->2 processes we used

$$\begin{aligned} p_1^\mu &= \frac{\sqrt{s_{12}}}{2}(1,0,0,1) \\ p_2^\mu &= \frac{\sqrt{s_{12}}}{2}(1,0,0,-1) \\ q_i^\mu &= \xi_{i,0}\frac{\sqrt{s_{12}}}{2}\left(1, \sqrt{1-y^2}\hat{e}_T^i, y\right) \end{aligned} \Longrightarrow \begin{aligned} y &\in [-1,1] \\ \xi_{i,0} &\in [0,\infty) \\ y &= 1-2v \end{aligned}$$

Scalar variables

in the massless case (analogous expressions when massive particles are present)

  - ✓ Factorize the measure in D-dimensions

$$d[\xi_{i,0}] = \frac{\mu^{2\epsilon}(4\pi)^{\epsilon-2}}{\Gamma(1-\epsilon)}\, s_{12}^{-2\epsilon}\, \xi_{i,0}^{-2\epsilon}\, d\xi_{i,0}$$

$$d[v_i] = (v_i(1-v_i))^{-\epsilon}\, dv_i$$

IMPORTANT: *We implement the method within DREG to establish a comparison with traditional results!*

Rodrigo et al, JHEP02(2016)044; JHEP08(2016)160; JHEP10(2016)162

Massless Feynman integrals

- *Reference example: Massless scalar three-point function in the time-like region*

$$L^{(1)}(p_1, p_2, -p_3) = \int_\ell \prod_{i=1}^{3} G_F(q_i) = -\frac{c_\Gamma}{\epsilon^2}\left(-\frac{s_{12}}{\mu^2} - i0\right)^{-1-\epsilon} = \sum_{i=1}^{3} I_i$$

LTD

$$I_1 = \frac{1}{s_{12}} \int d[\xi_{1,0}]\, d[v_1]\, \xi_{1,0}^{-1}\, (v_1(1-v_1))^{-1}$$

$$I_2 = \frac{1}{s_{12}} \int d[\xi_{2,0}]\, d[v_2]\, \frac{(1-v_2)^{-1}}{1-\xi_{2,0}+i0}$$

$$I_3 = \frac{1}{s_{12}} \int d[\xi_{3,0}]\, d[v_3]\, \frac{v_3^{-1}}{1+\xi_{3,0}-i0}$$

To regularize threshold singularity

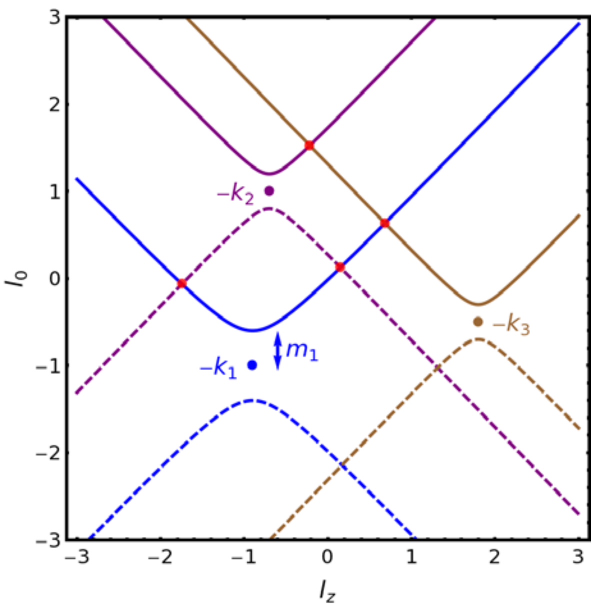
- This integral is UV-finite (power counting); there are only IR-singularities, associated to soft and collinear regions
- **OBJECTIVE:** Define a *IR-regularized* loop integral by adding real corrections at integrand level (i.e. no epsilon should appear, 4D representation)

Rodrigo et al, JHEP02(2016)044; JHEP08(2016)160; JHEP10(2016)162

Massless Feynman integrals

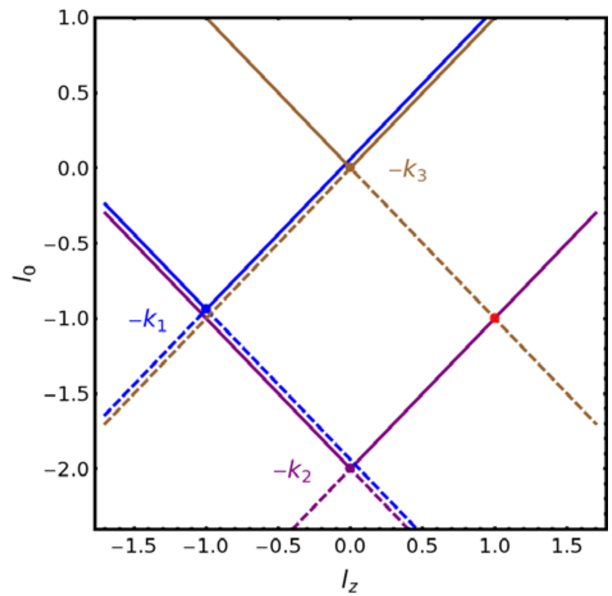
- Analize the dual integration region. It is obtained as the positive energy solution of the on-shell condition:

$$G_F^{-1}(q_i) = q_i^2 - m_i^2 + i0 = 0 \quad \Rightarrow \quad q_{i,0}^{(\pm)} = \pm\sqrt{\mathbf{q}_i^2 + m_i^2 - i0}$$



Massive case: hyperboloids (more details later)

- **Forward** (backward) on-shell hyperboloids associated with **positive** (negative) energy solutions.
- Degenerate to light-cones for massless propagators.
- *Dual integrands become singular at intersections* (two or more on-shell propagators)



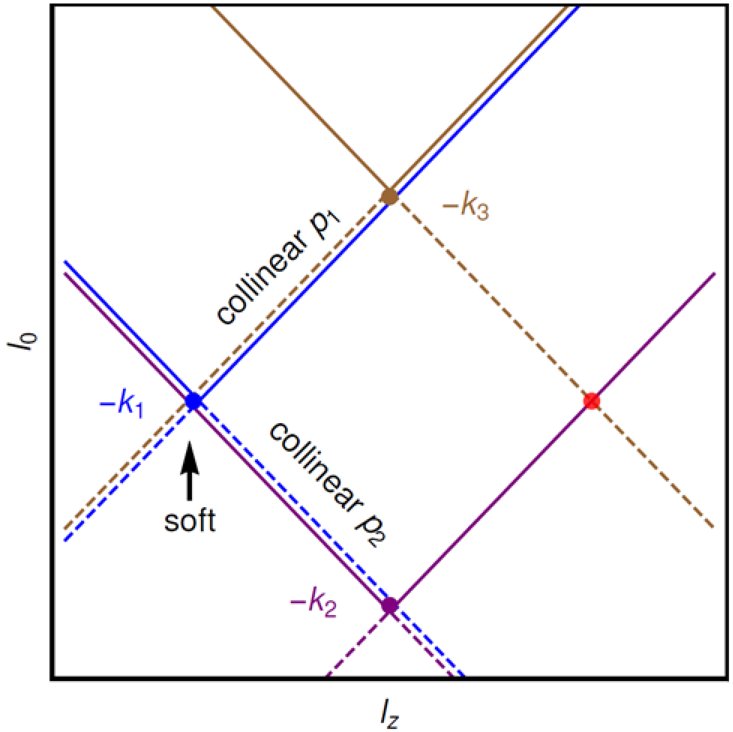
Massless case: light-cones

Buchta et al, JHEP11(2014)014; Rodrigo et al, JHEP02(2016)044

Massless Feynman integrals

IR singularities

- Analize the integration region. Application of LTD converts loop-integrals into PS ones: **integrate in forward light-cones**.



- Only **forward-backward** interference originate **threshold or IR poles.**
- **Forward-forward** cancel among dual contributions
- Threshold and IR singularities associated with finite regions (i.e. contained in a **compact region**)
- No threshold or IR singularity at large loop momentum

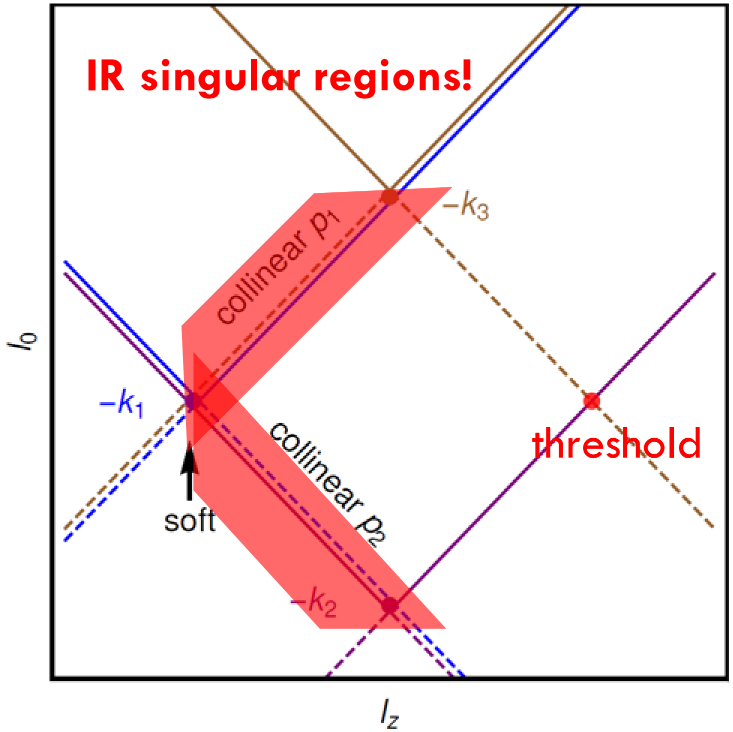
- This structure suggests how to perform real-virtual combination! Also, how to overcome threshold singularities (integrable but numerically unstable)

Rodrigo et al, JHEP02(2016)044; JHEP08(2016)160; JHEP10(2016)162

Massless Feynman integrals

IR singularities

- Analize the integration region. Application of LTD converts loop-integrals into PS ones: **integrate in forward light-cones**.



- From this plot, we conclude that singularities contained in $\mathbf{I_3}$ cancel among the other dual contributions.
- However, forward-backward singularities in $\mathbf{I_1}$ and $\mathbf{I_2}$ remains uncancelled: they must be combined with suitable **real** counterterms!
- When $\mathbf{q_2}$ and $\mathbf{q_3}$ become simultaneously on-shell, threshold singularities appear (*equivalent to Cutkosky rule!*)

- This structure suggests how to perform real-virtual combination! Also, how to overcome threshold singularities (integrable but numerically unstable)

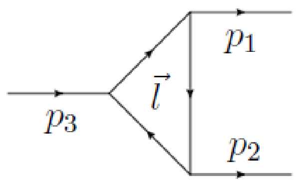
Rodrigo et al, JHEP02(2016)044; JHEP08(2016)160; JHEP10(2016)162

Massless Feynman integrals

Finite real+virtual integration

- Now, we must add **real** contributions. Suppose **one-loop** scalar scattering amplitude given by the triangle

Virtual

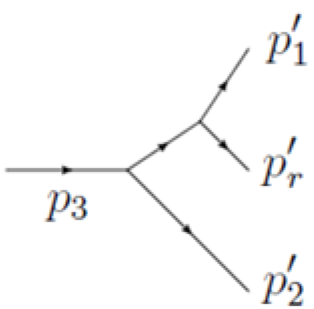


$$|\mathcal{M}^{(0)}(p_1,p_2;p_3)\rangle = ig$$
$$|\mathcal{M}^{(1)}(p_1,p_2;p_3)\rangle = -i\,g^3\,L^{(1)}(p_1,p_2,-p_3) \Rightarrow \mathrm{Re}\,\langle\mathcal{M}^{(0)}|\mathcal{M}^{(1)}\rangle$$

- **1->2 one-loop process** → **1->3 with unresolved extra-parton**
- Add scalar tree-level contributions with one extra-particle; consider interference terms:

Opposite sign!

Real



$$|\mathcal{M}^{(0)}_{ir}(p'_1,p'_2,p'_r;p_3)\rangle = -ig^2/s'_{ir} \Rightarrow \mathrm{Re}\,\langle\mathcal{M}^{(0)}_{ir}|\mathcal{M}^{(0)}_{jr}\rangle = \frac{g^4}{s'_{ir}\,s'_{jr}}$$

- Generate 1->3 kinematics starting from 1->2 configuration plus the loop three-momentum $\vec{l}$!!!

Rodrigo et al, JHEP02(2016)044; JHEP08(2016)160; JHEP10(2016)162

Massless Feynman integrals

Finite real+virtual integration

- **Mapping of momenta:** generate **1->3 real** emission kinematics (**3 external on-shell momenta**) starting from the variables available in the dual description of **1->2 virtual** contributions (**2 external on-shell momenta and 1 free three-momentum**)
  - ✓ Split the real phase space in two regions, i.e. $y'_{1r}<y'_{2r}$ and $y'_{2r}<y'_{1r}$, to separate the possible collinear singularities
  - ✓ Implement an optimized mapping in each region, to allow a fully local cancellation of IR singularities with those present in the dual terms

REGION 1:

$$p'^{\mu}_r = q^{\mu}_1 \,, \qquad p'^{\mu}_1 = p^{\mu}_1 - q^{\mu}_1 + \alpha_1 \, p^{\mu}_2 \,,$$
$$p'^{\mu}_2 = (1-\alpha_1)\, p^{\mu}_2 \,, \qquad \alpha_1 = \frac{q_3^2}{2 q_3 \cdot p_2} \,,$$

$\Rightarrow$

$$y'_{1r} = \frac{v_1 \, \xi_{1,0}}{1-(1-v_1)\,\xi_{1,0}} \qquad y'_{12} = 1 - \xi_{1,0}$$
$$y'_{2r} = \frac{(1-v_1)(1-\xi_{1,0})\,\xi_{1,0}}{1-(1-v_1)\,\xi_{1,0}}$$

REGION 2:

$$p'^{\mu}_2 = q^{\mu}_2 \,, \qquad p'^{\mu}_r = p^{\mu}_2 - q^{\mu}_2 + \alpha_2 \, p^{\mu}_1 \,,$$
$$p'^{\mu}_1 = (1-\alpha_2)\, p^{\mu}_1 \,, \qquad \alpha_2 = \frac{q_1^2}{2 q_1 \cdot p_1} \,,$$

$\Rightarrow$

$$y'_{1r} = 1 - \xi_{2,0} \qquad y'_{2r} = \frac{(1-v_2)\,\xi_{2,0}}{1-v_2\,\xi_{2,0}}$$
$$y'_{12} = \frac{v_2\,(1-\xi_{2,0})\,\xi_{2,0}}{1-v_2\,\xi_{2,0}}$$

Rodrigo et al, JHEP02(2016)044; JHEP08(2016)160; JHEP10(2016)162

Massless Feynman integrals

Finite real+virtual integration

- **Mapping of momenta:** generate **1->3 real** emission kinematics (**3 external on-shell momenta**) starting from the variables available in the dual description of **1->2 virtual** contributions (**2 external on-shell momenta and 1 free three-momentum**)
  - Express interference terms using the proper mapping → **Real and virtual contributions are described using the same integration variables!**
  - **Only I_1 and I_2 need to be combined with the real part;** define the following dual-cross sections:

$$\tilde{\sigma}_{i,R} = \sigma_0^{-1}\, 2\mathrm{Re} \int d\Phi_{1\to 3}\, \langle \mathcal{M}_{2r}^{(0)} | \mathcal{M}_{1r}^{(0)} \rangle\, \theta(y'_{jr} - y'_{ir})$$

$$\tilde{\sigma}_{i,V} = \sigma_0^{-1}\, 2\mathrm{Re} \int d\Phi_{1\to 2}\, \langle \mathcal{M}^{(0)} | \mathcal{M}_{i}^{(1)} \rangle\, \theta(y'_{jr} - y'_{ir})$$

$$\tilde{\sigma}_1 = \tilde{\sigma}_{1,V} + \tilde{\sigma}_{1,R} = \mathcal{O}(\epsilon)$$

$$\tilde{\sigma}_2 = \tilde{\sigma}_{2,V} + \tilde{\sigma}_{2,R} = -c_\Gamma \frac{g^2}{s_{12}} \frac{\pi^2}{6} + \mathcal{O}(\epsilon)$$

  - **Four-dimensional implementation at integrand level reproduces the four-dimensional limit of the standard DREG result!!!**

Rodrigo et al, JHEP02(2016)044; JHEP08(2016)160; JHEP10(2016)162

Massless Feynman integrals

UV singularities

- *Reference example: two-point function with massless propagators*

$$L^{(1)}(p,-p) = \int_\ell \prod_{i=1}^{2} G_F(q_i) = \frac{c_\Gamma}{\epsilon(1-2\epsilon)} \left(-\frac{p^2}{\mu^2} - i0 \right)^{-\epsilon} = \sum_{i=1}^{2} I_i$$

LTD

$$I_1 = - \int_\ell \frac{\tilde{\delta}(q_1)}{-2q_1 \cdot p + p^2 + i0}$$

$$I_2 = - \int_\ell \frac{\tilde{\delta}(q_2)}{2q_2 \cdot p + p^2 - i0}$$

To regularize threshold singularity

- In this case, the integration regions of dual integrals are two energy-displaced forward light-cones. This integral contains UV poles only!
- **OBJETIVE:** Define a *UV-regularized* loop integral by adding unintegrated UV counter-terms, and find a purely 4-dimensional representation of the loop integral

Rodrigo et al, JHEP02(2016)044; JHEP08(2016)160; JHEP10(2016)162

Massless Feynman integrals

UV counter-term

- Divergences arise from the high-energy region (UV poles) and can be cancelled with a suitable renormalization counter-term. For the scalar case, we use

$$I_{\mathrm{UV}}^{\mathrm{cnt}} = \int_\ell \frac{1}{(q_{\mathrm{UV}}^2 - \mu_{\mathrm{UV}}^2 + i0)^2}$$

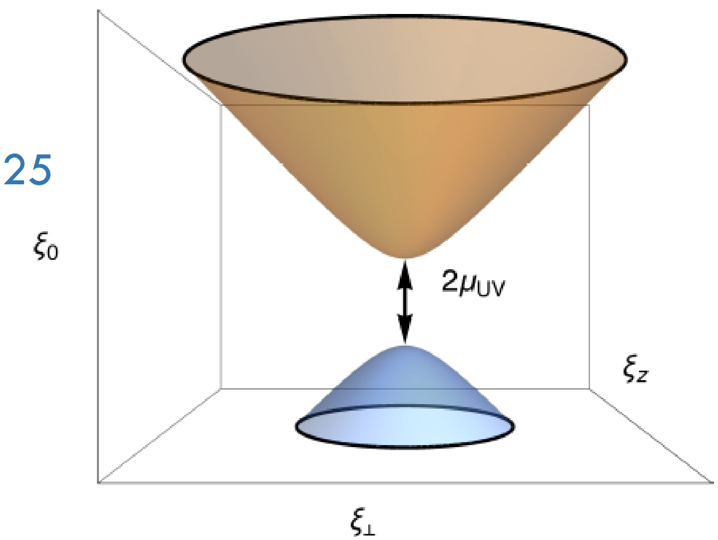
Becker, Reuschle, Weinzierl, JHEP 12 (2010) 013

- Dual representation (**new: double poles in the loop energy**)

$$I_{\mathrm{UV}}^{\mathrm{cnt}} = \int_\ell \frac{\tilde{\delta}(q_{\mathrm{UV}})}{2\left(q_{\mathrm{UV},0}^{(+)}\right)^2}$$

Bierenbaum *et al.* JHEP 03 (2013) 025

$$q_{\mathrm{UV},0}^{(+)} = \sqrt{\mathbf{q}_{\mathrm{UV}}^2 + \mu_{\mathrm{UV}}^2 - i0}$$



- Loop integration for loop energies larger than μ_{UV}

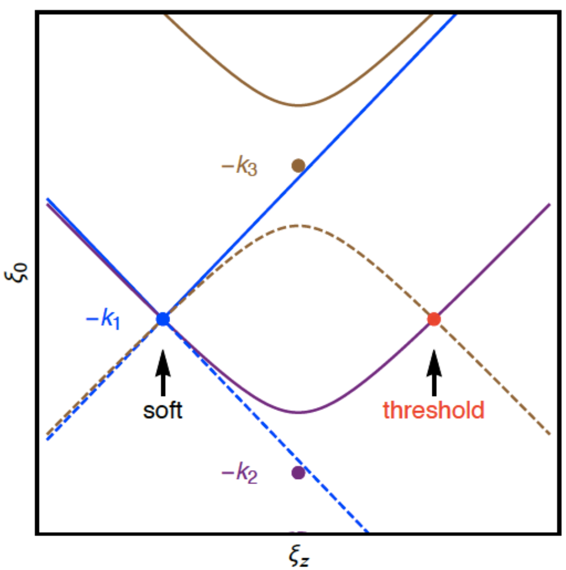
Rodrigo et al, JHEP02(2016)044; JHEP08(2016)160; JHEP10(2016)162

LTD with massive particles

Location of IR singularities

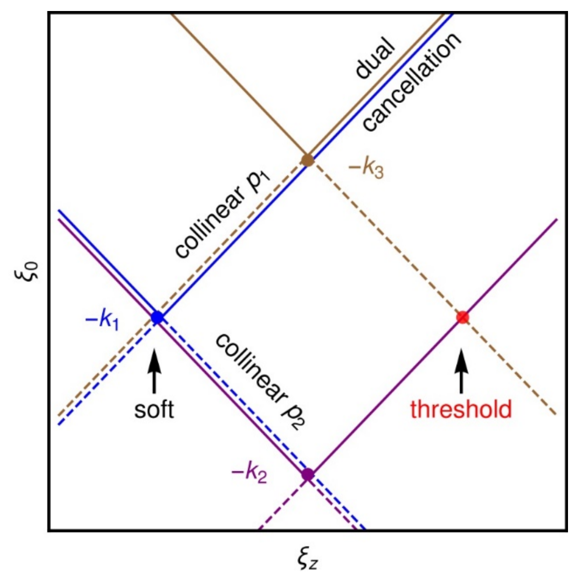
- Analize the dual integration region. It is obtained as the positive energy solution of the on-shell condition;

$$G_F^{-1}(q_i) = q_i^2 - m_i^2 + i0 = 0 \quad \Longrightarrow \quad q_{i,0}^{(\pm)} = \pm\sqrt{\mathbf{q}_i^2 + m_i^2 - i0}$$



Massive case: on-shell hyperboloids

- **Forward** (backward) on-shell hyperboloids associated with **positive** (negative) energy mode.
- Degenerate to light-cones for massless propagators.
- *Dual integrands become singular at intersections* (two or more on-shell propagators)



Massless case: light-cones

Rodrigo et al, JHEP02(2016)044; JHEP08(2016)160; JHEP10(2016)162

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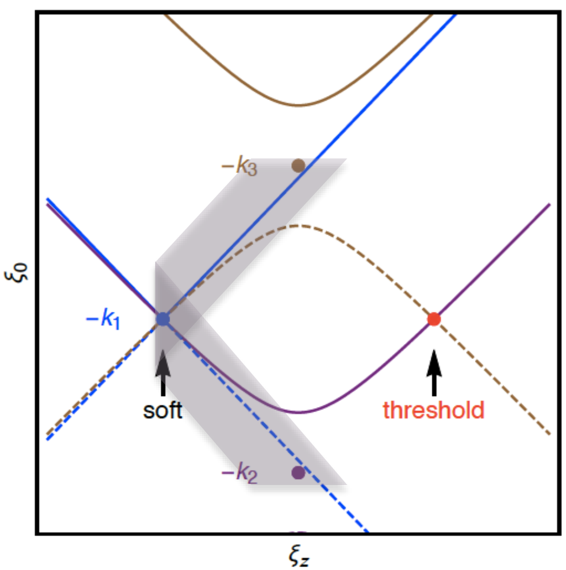
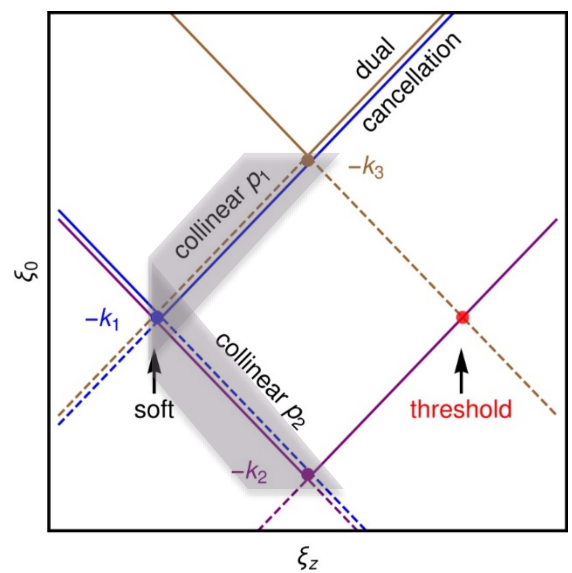


Massive case: on-shell hyperboloids

- **Forward** (backward) on-shell hyperboloids associated with **positive** (negative) energy mode.
- Degenerate to light-cones for massless propagators.
- *Dual integrands become singular at intersections* (two or more on-shell propagators)
- **Quasi-collinear** configurations lead to **Log(m^2)**, which is singular in the massless limit



Massless case: light-cones

Rodrigo et al, JHEP02(2016)044; JHEP08(2016)160; JHEP10(2016)162

LTD with massive particles

Real-virtual momentum mapping

- **Real-virtual momentum mapping with massive particles**
  - Consider **1** the **emitter**, **r** the **radiated particle** and **2** the **spectator**
  - Apply the PS partition and restrict to the only region where **1//r** is allowed (i.e. $\mathcal{R}_1 = \{y'_{1r} < \min y'_{kj}\}$)
  - Propose the following mapping:

$$
\boxed{
\begin{aligned}
p'^{\mu}_r &= q^{\mu}_1 \\
p'^{\mu}_1 &= (1-\alpha_1)\,\hat{p}^{\mu}_1 + (1-\gamma_1)\,\hat{p}^{\mu}_2 - q^{\mu}_1 \\
p'^{\mu}_2 &= \alpha_1\,\hat{p}^{\mu}_1 + \gamma_1\,\hat{p}^{\mu}_2
\end{aligned}
}
$$

Impose on-shell conditions to determine mapping parameters

  - *Express the loop three-momentum with the same parameterization used for describing the dual contributions!*

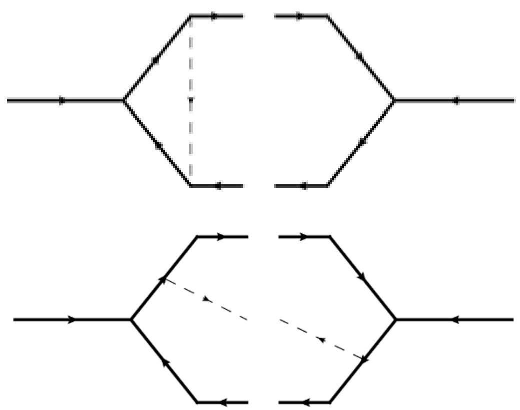
Repeat in each region of the partition…

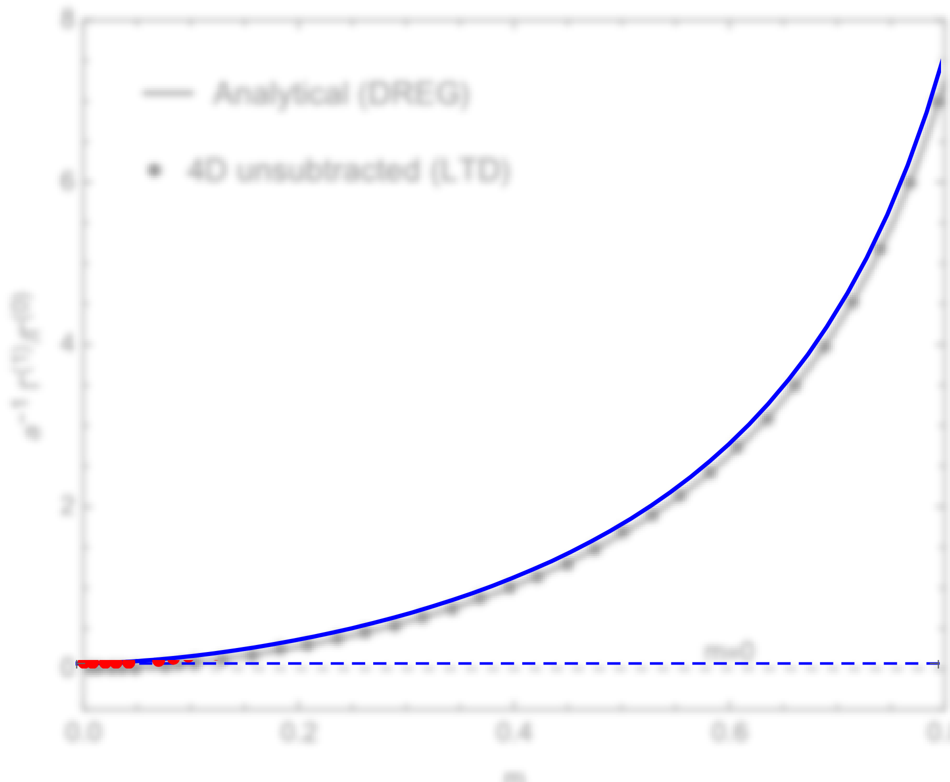
Rodrigo et al, JHEP02(2016)044; JHEP08(2016)160; JHEP10(2016)162

LTD with massive particles

Example: massive scalar three-point function (DREG vs LTD)

- We combine the dual contributions with the real terms (after applying the proper mapping) to get the total decay rate in the scalar toy-model.
  - The result agrees ***perfectly*** with standard DREG.
  - **Massless limit** is **smoothly** approached due to proper treatment of **quasi-collinear** configurations in the **RV mapping**





Rodrigo et al, JHEP10(2016)162

LTD with massive particles

- LTD must be applied to deal with **UV singularities** by building **local** versions of the usual UV counterterms.
- **1: Expand** internal propagators around the *"UV propagator"*

$$\frac{1}{q_i^2 - m_i^2 + \imath 0} = \frac{1}{q_{\rm UV}^2 - \mu_{\rm UV}^2 + \imath 0} \times \left[1 - \frac{2q_{\rm UV}\cdot k_{i,{\rm UV}} + k_{i,{\rm UV}}^2 - m_i^2 + \mu_{\rm UV}^2}{q_{\rm UV}^2 - \mu_{\rm UV}^2 + \imath 0} + \frac{(2q_{\rm UV}\cdot k_{i,{\rm UV}})^2}{(q_{\rm UV}^2 - \mu_{\rm UV}^2 + \imath 0)^2}\right] + \mathcal{O}\left((q_{\rm UV}^2)^{-5/2}\right)$$

Becker, Reuschle, Weinzierl, JHEP12(2010)013

- **2:** Apply LTD to get the **dual representation** for the expanded UV expression, and **subtract** it from the **dual+real** combined integrand.

LTD extended to deal with multiple poles

(use residue formula to obtain the dual representation)

- **3:** Take into account **wave-function and vertex renormalization** constants (not trivial in the massive case!)

Rodrigo et al, JHEP10(2016)162

LTD with massive particles

UV counterterms and renormalization

- Since we work at integrand level, we need **un-integrated** wave-function, mass and vertex renormalization constants!
- Self-energy corrections with **on-shell renormalization** conditions

$$\Sigma_R(\not p_1 = M) = 0 \qquad \left.\frac{d\Sigma_R(\not p_1)}{d\not p_1}\right|_{\not p_1=M} = 0$$

- ***Wave-function renormalization constant*** (**both IR and UV poles**):

$$\Delta Z_2(p_1) = -g_S^2 C_F \int_\ell G_F(q_1)G_F(q_3) \left((d-2)\frac{q_1 \cdot p_2}{p_1 \cdot p_2} + 4M^2 \left(1 - \frac{q_1 \cdot p_2}{p_1 \cdot p_2}\right) G_F(q_3) \right)$$

- ***Vertex renormalization*** (**only UV**):

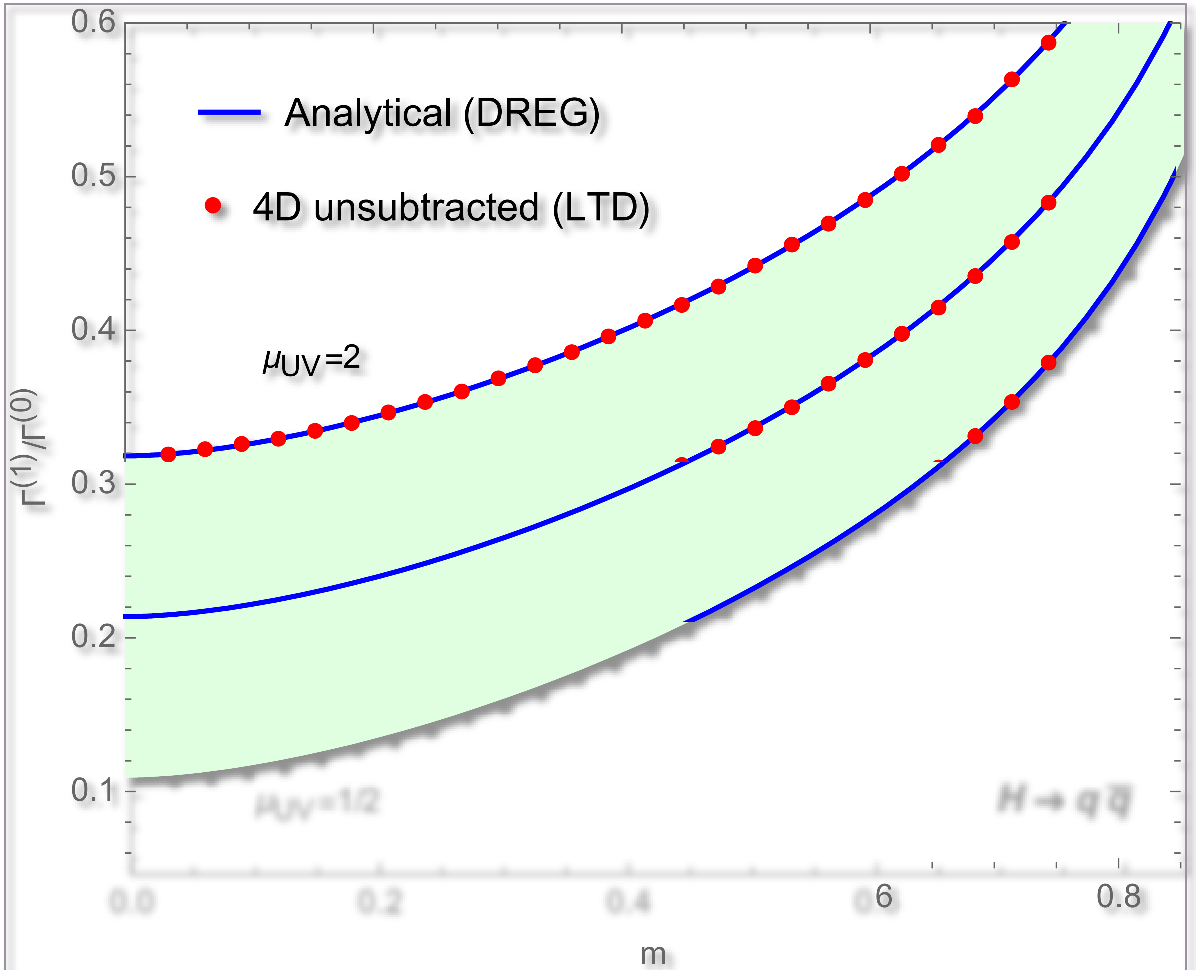
$$\mathbf{\Gamma}^{(1)}_{A,\mathrm{UV}} = g_S^2\, C_F \int_\ell (G_F(q_{\mathrm{UV}}))^3 \left[\gamma^\nu \not q_{\mathrm{UV}}\, \mathbf{\Gamma}^{(0)}_A\, \not q_{\mathrm{UV}}\, \gamma_\nu - d_{A,\mathrm{UV}}\, \mu^2_{\mathrm{UV}}\, \mathbf{\Gamma}^{(0)}_A\right]$$

- Integrated form of local counterterms agrees with standard UV counter-terms!

Rodrigo et al, JHEP10(2016)162

Physical example: $A^* \to q\bar{q}(g)$@NLO

Results and comparison with DREG

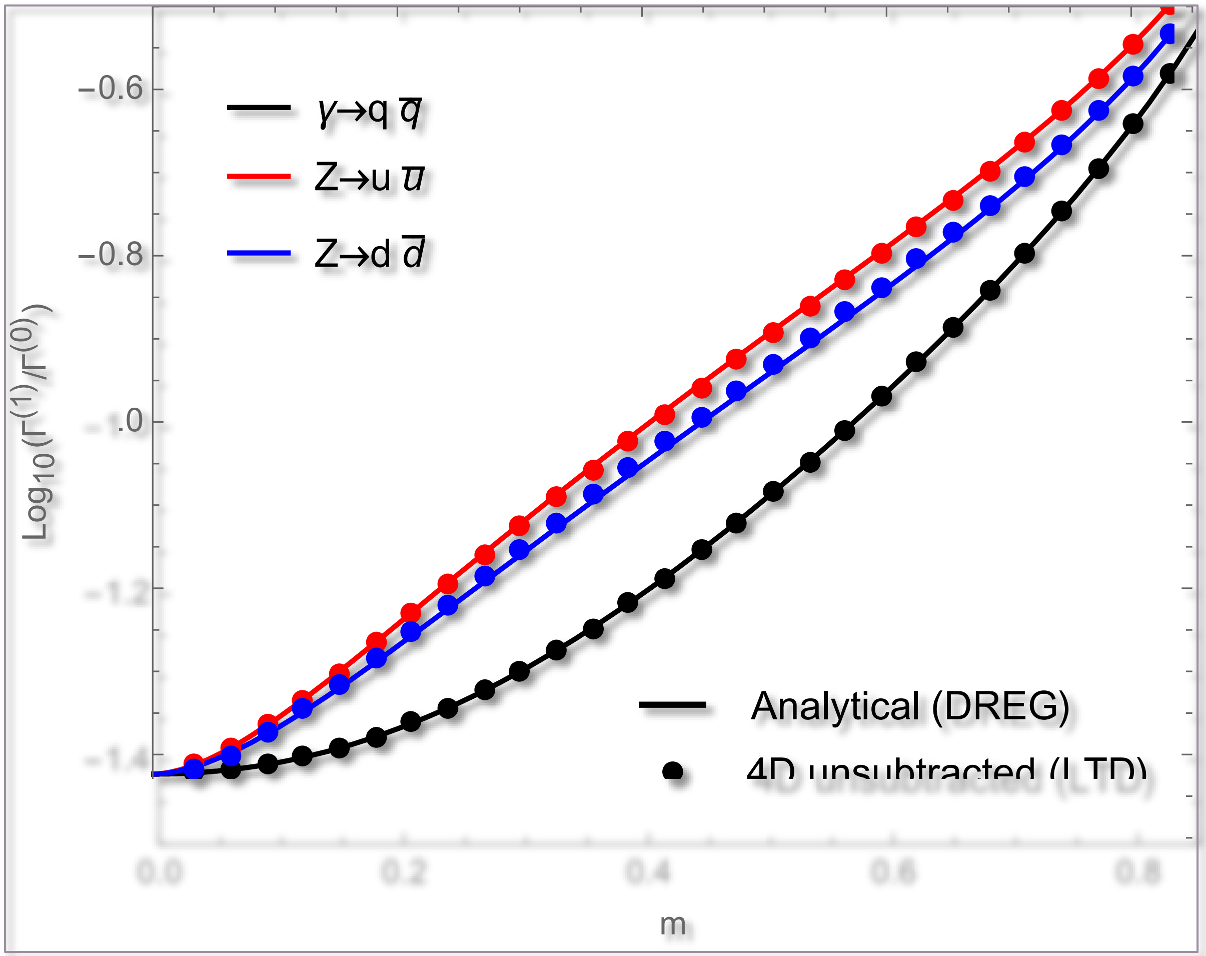


- *Total decay rate for Higgs into a pair of massive quarks:*
  - Agreement with the standard DREG result
  - Smoothly achieves the massless limit
  - Local version of UV counterterms succesfully reproduces the expected behaviour
  - Efficient numerical implementation

Rodrigo et al, JHEP10(2016)162

Physical example: $A^* \to q\bar{q}(g)$@NLO

Results and comparison with DREG



- *Total decay rate for a vector particle into a pair of massive quarks:*
  - Agreement with the standard DREG result
  - Smoothly achieves the massless limit
  - Efficient numerical implementation

Rodrigo et al, JHEP10(2016)162

Physical example: $A^* \to q\bar{q}(g)$@NLO

Final remarks

- The total decay-rate can be expressed using purely **four-dimensional integrands**
- We recover the total NLO correction, **avoiding to deal with DREG**

- **Main advantages:**
  - ✓ Direct **numerical** implementation (integrable functions for **ε=0**)

    Finite integral for ε=0 ~~⇒~~ Integrability with ε=0

    With FDU is true!
  - ✓ No need of tensor reduction (**avoids the presence of Gram determinants,** which could introduce numerical instabilities)
  - ✓ **Smooth transition** to the massless limit (due to the efficient treatment of **quasi-collinear** configurations)
  - ✓ **Mapped real-contribution used as a fully local IR counter-term for the dual contribution!**

Rodrigo et al, JHEP10(2016)162

Conclusions and perspectives

- ✓ Loop-tree duality allows to treat virtual and real contributions in the same way (simplification of implementation)
- ✓ Physical interpretation of **IR/UV singularities** in loop integrals (light-cone diagrams)
- ✓ Expression of *loop integrals* as *phase-space integrals* (take into account the proper *prescription*! Explicit example with I_2…)
- ✓ **Combined virtual-real terms are integrable in 4D!!**
- ✓ **First (realistic) physical implementation!!!**

- **Perspectives:**
  - Apply the technique to compute other physical observables (including heavy particles and multi-leg processes)
  - Extend the procedure to higher orders!!!

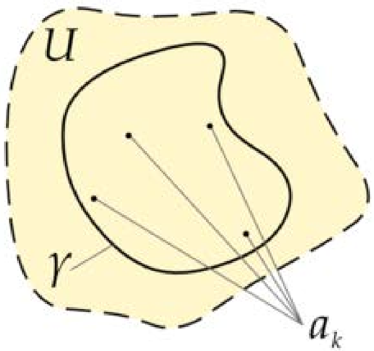
Thanks!!!

Extra material

Cauchy's theorem and prescriptions

Residue theorem
(*from Wikipedia*)

$$\oint_\gamma f(z)\,dz = 2\pi i \sum \mathrm{Res}(f, a_k)$$

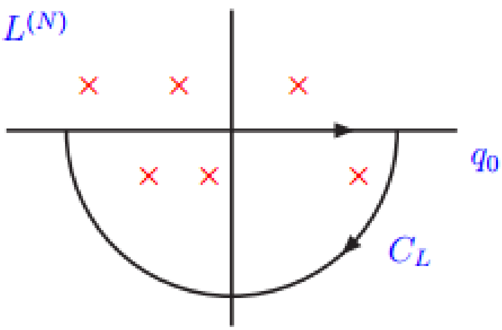


«If f is a holomorphic function in U/{a_i}, and γ a simple positively oriented curve, then the integral is given by the sum of the residues at each singular point a_i»

Residue theorem can be used to compute integrals involving propagators: the prescription and the contour that we choose determine the result!

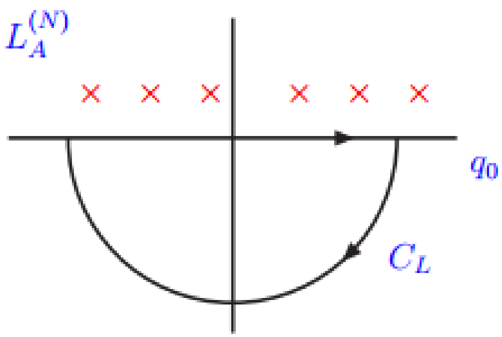
Feynman propagator

$$[G(q)]^{-1} = 0 \quad \Longrightarrow \quad q_0 = \pm\sqrt{\mathbf{q}^2 - i0}$$



Advanced propagator

$$[G_A(q)]^{-1} = 0 \quad \Longrightarrow \quad q_0 \simeq \pm\sqrt{\mathbf{q}^2} + i0$$



NO POLES CLOSED BY C_L!

Feynman tree theorem

Derivation

- **Idea:** *«Sum over all possible m-cuts»*

$$L_A^{(N)}(p_1, p_2, \dots, p_N) = \int_q \prod_{i=1}^{N} G_A(q_i) = 0$$

Residue theorem (using a proper integration path)

$$G_A(q) = G(q) + \tilde{\delta}(q)$$

Using PV prescription

$$L_A^{(N)}(p_1, p_2, \dots, p_N) = \int_q \prod_{i=1}^{N} \left[G(q_i) + \tilde{\delta}(q_i) \right]$$

$$= L^{(N)}(p_1, p_2, \dots, p_N) + L^{(N)}_{1-\mathrm{cut}}(p_1, p_2, \dots, p_N) + \cdots + L^{(N)}_{\mathrm{N-cut}}(p_1, p_2, \dots, p_N)$$

m-cut definition: $L^{(N)}_{\mathrm{m-cut}}(p_1, p_2, \dots, p_N) = \int_q \left\{ \tilde{\delta}(q_1) \dots \tilde{\delta}(q_m)\, G(q_{m+1}) \dots G(q_N) + \text{uneq. perms.} \right\}$

$$\boxed{L^{(N)}(p_1, p_2, \dots, p_N) = - \left[L^{(N)}_{1-\mathrm{cut}}(p_1, p_2, \dots, p_N) + \cdots + L^{(N)}_{\mathrm{N-cut}}(p_1, p_2, \dots, p_N) \right]}$$

Feynman tree theorem

Derivation

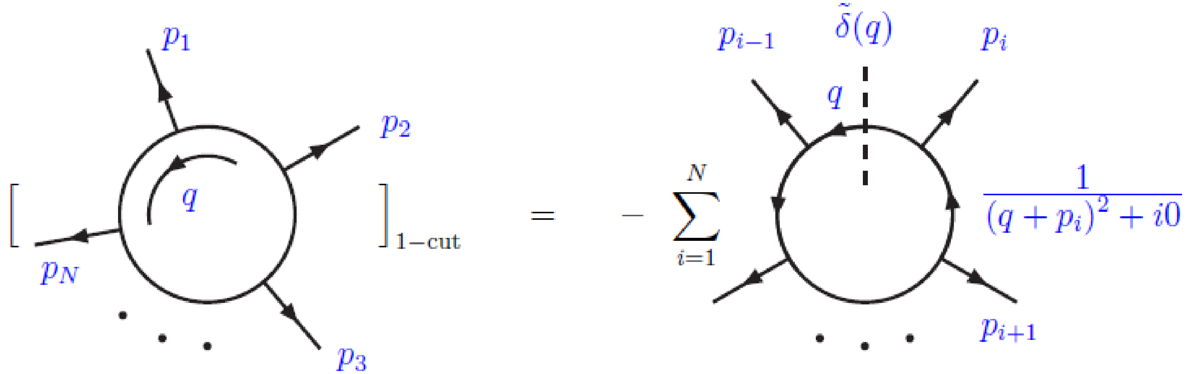
- Some remarks:
  - Making m-cuts decomposes the original **one-loop** diagram into **m-tree level terms**, all of them using the **same prescription**
  - *1-cut = sum over «tree level» terms*

$$L^{(N)}_{1-\mathrm{cut}}(p_1, p_2, \ldots, p_N) = \int_q \sum_{i=1}^{N} \widetilde{\delta}(q_i) \prod_{\substack{j=1 \\ j \neq i}}^{N} G(q_j)$$

$$I^{(n)}_{1-\mathrm{cut}}(k_1, k_2, \ldots, k_n) = \int_q \widetilde{\delta}(q) \prod_{j=1}^{n} G(q + k_j) = \int_q \widetilde{\delta}(q) \prod_{j=1}^{n} \frac{1}{2qk_j + k_j^2 + i0}$$

Basic 1-cut integral (shift in loop momentum)

Loop



Sum of multiple cuts

IR singularities within LTD

Compactness of IR singular regions (massless triangle)

- From the previous plots, we define three contributions:

IR-divergent contributions (ξ_0<1+w)

- *Originated in a* ***finite region*** *of the loop three-momentum*
- *All the IR singularities of the original loop integral*

$$
\begin{aligned}
I^{\rm IR} &= I_1^{(\rm s)} + I_1^{(\rm c)} + I_2^{(\rm c)} = \frac{c_\Gamma}{s_{12}} \left(\frac{-s_{12} - \imath 0}{\mu^2} \right)^{-\epsilon} \\
&\times \left[\frac{1}{\epsilon^2} + \left(\ln(2) \ln(w) - \frac{\pi^2}{3} - 2{\rm Li}_2\left(-\frac{1}{w}\right) + \imath\pi \ln(2) \right) \right] + \mathcal{O}(\epsilon)
\end{aligned}
$$

Forward integrals (v<1/2,ξ_0>1)

- *Free of IR/UV poles*
- *Integrable in 4-dimensions!*

$$
I^{(\rm f)} = \sum_{i=1}^{3} I_i^{(\rm f)} = c_\Gamma \frac{1}{s_{12}} \left[\frac{\pi^2}{3} - \imath\pi \; \log(2) \right] + \mathcal{O}(\epsilon)
$$

Backward integrals (v>1/2,ξ_0>1+w)

- *Free of IR/UV poles*
- *Integrable in 4-dimensions!*

$$
I^{(\rm b)} = c_\Gamma \frac{1}{s_{12}} \left[2{\rm Li}_2\left(-\frac{1}{w}\right) - \ln(2)\ln(w) \right] + \mathcal{O}(\epsilon)
$$

Rodrigo et al, JHEP02(2016)044; arXiv:1506.04617, 1510.01079, 1601.04634, 1604.06699

IR singularities within LTD

Technical details

- Let's stop and make some remarks about the structure of these expressions:
  - Introduction of an **arbitrary cut** w to **include threshold regions.**
  - Forward and backward integrals can be performed in 4D because the sum does not contain poles.
  - Presence of extra Log's in (F) and (B) integrals. They are originated from the expansion of the measure in DREG, i.e.

$$\xi_r^{-1-2\epsilon} = -\frac{Q_S^{-2\epsilon}}{2\epsilon}\delta(\xi_r) + \left(\frac{1}{\xi_r}\right)_C - 2\epsilon\left(\frac{\ln(\xi_r)}{\xi_r}\right)_C + \mathcal{O}(\epsilon^2)$$

for both v and ξ (keep finite terms only). **Unify coordinate system to avoid them!**

  - IR-poles isolated in I^{IR}! ⇒ **IR divergences originated in compact region of the three-loop momentum!!!**

$$\underbrace{L^{(1)}(p_1, p_2, -p_3) = I^{\rm IR}}_{\textit{Explicit poles still present...}} + \underbrace{I^{(\rm b)} + I^{(\rm f)}}_{\textit{Can be done in 4D!}}$$

Rodrigo et al, JHEP02(2016)044; arXiv:1506.04617, 1510.01079, 1601.04634, 1604.06699

UV regularized bubble with LTD

Cancellation of UV singularities

- Using the standard parametrization we define

Regularized two-point function

$$L^{(1)}(p,-p) - I_{\rm UV}^{\rm cnt} = c_\Gamma \left[-\log\left(-\frac{p^2}{\mu_{\rm UV}^2} - i0 \right) + 2 \right] + \mathcal{O}(\epsilon)$$

- Since it is finite, we can express the regularized two-point function in terms of 4-dimensional quantities (i.e. no epsilon required!!)

- ***Physical interpretation of renormalization scale***: Separation between on-shell hyperboloids in UV-counterterm is $2\mu_{\rm UV}$. To avoid intersections with forward light-cones associated with l_1 and l_2, the renormalization scale has to be larger or of the order of the hard scale. So, the minimal choice that fulfills this agrees with the standard choice (i.e. ½ of *the hard scale*).

Rodrigo et al, JHEP02(2016)044; JHEP08(2016)160; JHEP10(2016)162

γ>qqbar@NLO: 4D formulae

- Integration regions: $$\mathcal{R}_1(\xi_0, v) = \theta(1-2v_1)\,\theta\left(\frac{1-2v_1}{1-v_1} - \xi_{1,0}\right)\Big|_{\{\xi_{1,0},v_1\}\to\{\xi_{3,0},v_3\}=\{\xi_0,v\}}$$ $$\mathcal{R}_2(\xi_0, v) = \theta\left(\frac{1}{1+\sqrt{1-v}} - \xi_0\right)$$
- Four-dimensional cross-sections:

$$\tilde{\sigma}_1^{(1)} = \sigma^{(0)}\,\frac{\alpha_S}{4\pi}\,C_F \int_0^1 d\xi_{1,0} \int_0^{1/2} dv_1\, 4\,\mathcal{R}_1(\xi_{1,0}, v_1) \left[2\left(\xi_{1,0} - (1-v_1)^{-1}\right) - \frac{\xi_{1,0}(1-\xi_{1,0})}{(1-(1-v_1)\,\xi_{1,0})^2}\right]$$

$$\tilde{\sigma}_2^{(1)} = \sigma^{(0)}\,\frac{\alpha_S}{4\pi}\,C_F \int_0^1 d\xi_{2,0} \int_0^1 dv_2\, 2\,\mathcal{R}_2(\xi_{2,0}, v_2)\,(1-v_2)^{-1}\left[\frac{2\,v_2\,\xi_{2,0}\,(\xi_{2,0}(1-v_2)-1)}{1-\xi_{2,0}}\right.$$

$$\begin{aligned}
\tilde{\sigma}_V^{(1)} = \sigma^{(0)}\,\frac{\alpha_S}{4\pi}\,C_F \int_0^\infty d\xi \int_0^1 dv\, \Bigg\{ &-2\,(1-\mathcal{R}_1(\xi,v))\; v^{-1}(1-v)^{-1}\,\frac{\xi^2(1-2v)^2+1}{\sqrt{(1+\xi)^2-4v\,\xi}} \\
&+ 2\,(1-\mathcal{R}_2(\xi,v))\,(1-v)^{-1}\left[2\,v\,\xi\,(\xi(1-v)-1)\left(\frac{1}{1-\xi+\imath 0} + \imath\pi\delta(1-\xi)\right) - 1 + v\,\xi\right] \\
&+ 2\,v^{-1}\left(\frac{\xi(1-v)(\xi(1-2v)-1)}{1+\xi} + 1\right) - \frac{(1-2v)\,\xi^3\,(12-7m_{UV}^2-4\xi^2)}{(\xi^2+m_{UV}^2)^{5/2}} \\
&- \frac{2\,\xi^2(m_{UV}^2+4\xi^2(1-6v(1-v)))}{(\xi^2+m_{UV}^2)^{5/2}}\Bigg\}
\end{aligned}$$