Scattering amplitudes over finite fields and multivariate functional reconstruction

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based on: T.P., JHEP **1612** (2016) 030 [arXiv:1608.01902 [hep-ph]].

Outline

1. Introduction and motivation
2. Finite fields and multivariate reconstruction
3. Applications to scattering amplitudes
4. Summary & Outlook

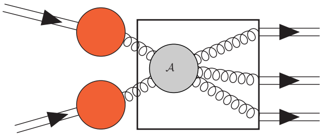
Introduction and motivation

Introduction and motivation

- Experiments at the Large Hadron Collider (LHC)
  - high-accuracy experimental data (up to % level)
  - high c.o.m. energy ⇒ multi-particle final states
  - large SM background (could hide new/interesting physics)

We need scattering amplitudes for theoretical predictions with

- high accuracy
- multi-particle interactions



Scattering amplitudes

- LO not reliable ⇒ need at least NLO
- NNLO needed for high precision

⇒ need to compute loop integrals

Perturbative calculation of scattering amplitudes

- Tree-level
  - solved for virtually any process/theory
    - Feynman diagrams, Berends-Giele/BCFW recursion
  - highly automated
  - focus on performance (especially for real emissions)
- One loop
  - essentially solved for any process/theory
    - tensor/integrand reduction, generalized unitarity, known MIs
  - automated, even for multi-leg processes
  - focus on performance and numerical stability
- Two loops
  - many calculations in recent years
    - diagrams, IBPs, differential equations, etc...
  - it's gradually becoming the new standard...
  - ...but still essentially restricted to $2 \to 2$ processes

Computation of scattering amplitudes

- Decomposed as

$$\mathcal{A} = \sum_k c_k I_k$$

  - I_k integrals (special functions)
  - c_k rational functions of kinematic invariants ($\Leftarrow$ focus of this talk)
  - c_k are a bottleneck at high multiplicity
- One loop $\rightarrow$ mostly numerical
  - many automated codes and toolchains
- Higher loops $\rightarrow$ mostly analytic
  - faster/more stable numerical evaluation
  - some techniques become more convenient (e.g. IBPs)
  - allows many checks/manipulations/studies (singularities, limits, ...)
  - can be used to infer general analytic/algebraic properties

  $\Rightarrow$ more control

Analytic calculation of scattering amplitudes

- Main bottleneck: large intermediate expressions
  - they can be orders of magnitude larger than the final result
  - not constrained by properties and symmetries of the result
- Tools for mitigating the problem
  - computer algebra systems specialised in handling large expressions (e.g. FORM [Vermaseren et al.])
  - generalized unitarity $\Rightarrow$ intermediate steps are gauge invariant

The main idea of this talk

- reconstruct analytic result from “numerical” evaluations
- no large intermediate expression (just numbers!)

Reconstruction of rational functions

- Which kind of "numerical" evaluation is good?
  - floating-point evaluation
    - ✓ very fast
    - ✗ affected by numerical instabilities
  - evaluation over the rational field $\mathcal{Q}$
    - ✓ exact
    - ✗ intermediate results have large numerators/denominators
      $\Rightarrow$ requires slow multi-precision arithmetic
  - evaluation over finite-fields
    - ✓ a finite-number of elements, which can be represented by machine-size integers
    - ✓ fast
    - ✓ exact
    - ✗ some information is lost and must be recovered by repeating the reconstruction over several finite fields

Functional reconstruction over finite fields

- Finite fields
  - used under-the-hood by computer algebra systems (e.g. in polynomial factorization/GCD)
  - proposed for IBPs (univariate applications)
    [von Manteuffel, Schabinger (2014–2016)]

- Efficient algorithm for functional reconstruction [T.P. (2016)]
  - works on (dense) multivariate polynomials and rational functions
  - implemented in C++ code (proof of concept)
  - the input is a numerical procedure computing a function
  - the output is its analytic expression
- Applications
  - spinor-helicity
  - tree-level recursion
  - multi-loop integrand reduction and generalized unitarity

Finite fields and multivariate reconstruction

Finite fields

- In this talk we consider finite fields $\mathcal{Z}_p$, with p prime
- We define

$$\mathcal{Z}_n = \{0, \dots, n-1\}$$

  - addition, subtraction, and multiplication via modular arithmetic

$$4+5\Big|_{\mathcal{Z}_7} = (4+5) \text{ mod } 7 = 2$$

  - if $a \in \mathcal{Z}_n$ and a, n are coprime, we can define an inverse

$$b = a^{-1} \in \mathcal{Z}_n, \qquad a \times b \text{ mod } n = 1$$

  - if $n = p$ prime, an inverse exists for every $a \in \mathcal{Z}_p \Rightarrow \mathcal{Z}_p$ is a field
  - every rational operation is well defined in $\mathcal{Z}_p$

Polynomials and rational functions

- multi-index notation: variables $\mathbf{z} = (z_1, \ldots, z_n)$ and integer list $\alpha = (\alpha_1, \ldots, \alpha_n)$

$$\mathbf{z}^\alpha \equiv \prod_{i=1}^{n} z_i^{\alpha_i}, \qquad |a| = \sum_i \alpha_i$$

- Given a generic field $\mathcal{F}$
  - $\mathcal{F}[\mathbf{z}]$ is the ring of polynomials in $\mathbf{z}$ with coefficients in $\mathcal{F}$

$$f(\mathbf{z}) = \sum_\alpha c_\alpha \, \mathbf{z}^\alpha.$$

  - $\mathcal{F}(\mathbf{z})$ is the field of rational functions in $\mathbf{z}$ with coefficients in $\mathcal{F}$

$$f(\mathbf{z}) = \frac{p(\mathbf{z})}{q(\mathbf{z})} = \frac{\sum_\alpha n_\alpha \, \mathbf{z}^\alpha}{\sum_\beta d_\beta \, \mathbf{z}^\beta},$$

    - technicality: set $d_{\text{min}\beta} = 1$ to make the representation unique.

Rational reconstruction

Functional reconstruction

Reconstruct the monomials z^α and their coefficients from numerical evaluations of the function (over finite fields)

- from $\mathcal{Q}$ to $\mathcal{Z}_p$

$$q = a/b \in \mathcal{Q} \quad \longrightarrow \quad q \bmod p \equiv a \times (b^{-1} \bmod p) \bmod p$$

- how to go back from $\mathcal{Z}_p$ to $\mathcal{Q}$?
- rational reconstruction algorithm: given $c \in \mathcal{Z}_n$ find its pre-image $q = a/b \in \mathcal{Q}$ with "small" a, b [Wang (1981)]
  - it's correct when $a, b \lesssim \sqrt{n}$
- make n large enough using Chinese reminder theorem
  - solution in $\mathcal{Z}_{p_1}, \mathcal{Z}_{p_2} \ldots \Rightarrow$ solution in $\mathcal{Z}_{p_1 p_2 \ldots}$

The black-box interpolation problem

Given a polynomial or rational function f in the variables $\mathbf{z} = (z_1, \ldots, z_n)$

- reconstruct analytic form of f, given a numerical procedure

$$\mathbf{z} \longrightarrow \boxed{\; f \;} \longrightarrow f(\mathbf{z}),$$

- modified black-box interpolation problem, for usage with finite fields

$$(\mathbf{z}, p) \longrightarrow \boxed{\; f \;} \longrightarrow f(\mathbf{z}) \bmod p.$$

  - the two are equivalent because of Chinese reminder theorem
- no further assumptions on f

Multivariate functional reconstruction (summary)

T.P. (2016)

- Univariate polynomials
  - based on Newton's interpolation formula
- Univariate rational functions
  - based on Thiele's (1838–1910) interpolation formula
- Multivariate polynomials
  - recursive application of Newton's interpolation
- Multivariate rational functions
  - use ideas proposed for sparse interpolation [A. Cuyt, W. Lee (2011)]
  - combined with Newton and Thiele's interpolation for dense case

- Notes:
  - all implemented in C++
  - results automatically come out GCD-simplified

Applications to scattering amplitudes

Four-dimensional spinor-helicity formalism

Mangano, Parke

- tree-level amplitudes are rational functions of spinor variables $|p\rangle$ and $|p]$
- satisfying the Dirac equation (in Weyl components)

$$p^\mu \, \sigma_\mu |p\rangle = p^\mu \, \sigma_\mu |p] = 0$$

- momenta and polarization vectors

$$p^\mu = \frac{1}{2} \langle p| \, \sigma^\mu \, |p], \quad \epsilon_+^\mu(p) = \frac{\langle \eta|\sigma^\mu|p]}{\sqrt{2}\,\langle \eta \, p\rangle}, \quad \epsilon_-^\mu(p) = \frac{\langle p|\sigma^\mu|\eta]}{\sqrt{2}\,[p\,\eta]}$$

- helicity amplitudes are combinations of spinor products, e.g.

$$\mathcal{A}_{5g}(1^+, 2^-, 3^+, 4^-, 5^+) = i\, g_s^3 \, \frac{\langle 2\,4\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,5\rangle\langle 5\,1\rangle}$$

- redundancy: spinor components are not all independent

A brief digression on spinor phases

- under a little group tranformation (complex redefinition of phase)

$$|i\rangle \to t_i\, |i\rangle, \qquad |i] \to \frac{1}{t_i}\, |i],$$

  an n-point amplitude $\mathcal{A}(1,\ldots,n)$ transforms as

$$\mathcal{A}(1,\ldots,n) \to \left(\prod_{i=1}^{n} t_i^{-2\,h_i}\right) \mathcal{A}(1,\ldots,n),$$

  where h_i is the helicity of the i-th particle (e.g. $\pm 1/2$ for fermions and ± 1 for gluons)

- extract from the amplitude an overall factor $\mathcal{A}^{(\text{phase})}(1,\ldots,n)$ which transform as the amplitude
- consider $\tilde{A}$ such that

$$\mathcal{A} = \underbrace{\mathcal{A}^{(\text{phase})}}_{\text{only depends on helicities}} \times \underbrace{\tilde{\mathcal{A}}}_{\text{phase-free}}$$

Choice of kinematic variables

Hodges (2009), Badger, Frellesvig, Zhang (2013), Badger (2016)

- phase-free rational parametrization of the n-point phase-space and the spinor components using $3n-10$ momentum-twistor variables
- 5-point example → 5 variables $\{x_1,\ldots,x_5\}$

$$|1\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \qquad |1] = \begin{pmatrix}1\\ \frac{x_4-x_5}{x_4}\end{pmatrix}, \qquad x_k = x_k(s_{ij}, \mathrm{tr}(\sigma_5\, 1\,2\,3\,4))$$

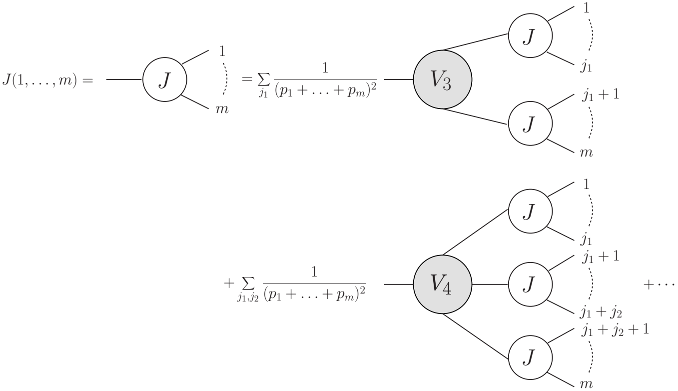
$$|2\rangle = \begin{pmatrix}0\\1\end{pmatrix}, \qquad |2] = \begin{pmatrix}0\\ x_1\end{pmatrix}, \qquad p_i^\mu = \frac{\langle i|\,\sigma^\mu\,|i]}{2}$$

$$|3\rangle = \begin{pmatrix}\frac{1}{x_1}\\1\end{pmatrix}, \qquad |3] = \begin{pmatrix}x_1\,x_4\\ -x_1\end{pmatrix},$$

$$|4\rangle = \begin{pmatrix}\frac{1}{x_1}+\frac{1}{x_1\,x_2}\\1\end{pmatrix}, \qquad |4] = \begin{pmatrix}x_1\,(x_2\,x_3 - x_3\,x_4 - x_4)\\ -\frac{x_1\,x_2\,x_3\,x_5}{x_4}\end{pmatrix},$$

$$|5\rangle = \begin{pmatrix}\frac{1}{x_1}+\frac{1}{x_1\,x_2}+\frac{1}{x_1\,x_2\,x_3}\\1\end{pmatrix}, \qquad |5] = \begin{pmatrix}x_1\,x_3\,(x_4-x_2)\\ \frac{x_1\,x_2\,x_3\,x_5}{x_4}\end{pmatrix}.$$

Tree-level amplitudes via Berends-Giele recursion



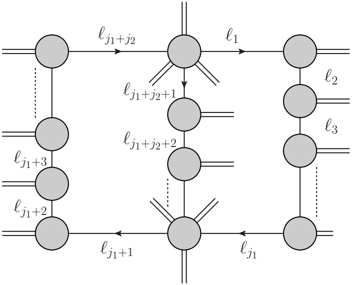
- very efficient for numerical calculations
- functional reconstruction techniques can exploit this for obtaining analytic results

Integrand reduction and generalized unitarity

Britto, Cachazo, Feng (2004), Giele, Kunszt, Melnikov (2008), Bern, Dixon, Kosower et al. (2008)

- Generalized unitarity
  - build irreducible integrands from multiple cuts
  - multiple-cuts $\Rightarrow$ loop propagators go on-shell, $\ell_i^2 = 0$
  - integrand factorizes as product of trees (summed over internal helicities)
  - multiple cuts $\Rightarrow$ unitarity cuts

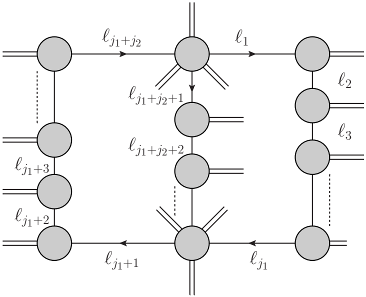
- # unitarity cuts $\ll$ # diagrams
  - lower complexity
- Every intermediate step is gauge invariant
  - no ghosts
  - more compact expressions



Generalized unitarity over finite fields

T.P. (2016)

- Amplitudes over finite fields
  - momentum-twistor variables
  - loop states: embed in 6-dim.
  - spinor-helicity in 4 and 6 dim.
  - tree-level recursion
  - two-loop d-dim. unitarity cuts

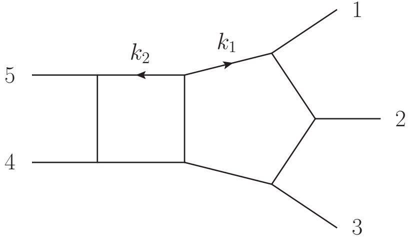


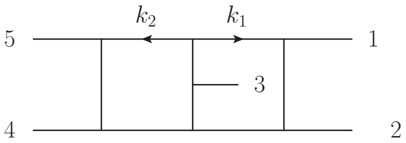
Finite-field implementation

- explicit six-dim. representation of loop states
- efficient numerical techniques for analytic calculations
- two-loop unitarity cuts from Berends-Giele off-shell currents

Finite fields and functional reconstruction: examples

- five-gluon on-shell integrands of maximal cuts ($\equiv$ top-level topology) for





(for a complete set of helicities)

Finite fields and functional reconstruction

- penta-box

Helicity	Non-vanishing coeff.	Max. terms	Max. degree	Avg. non-zero terms
$(1^+,2^+,3^+,4^+,5^+)$	14	19	8	15.00
$(1^-,2^+,3^+,4^+,5^+)$	27	443	19	152.96
$(1^+,2^-,3^+,4^+,5^+)$	37	1977	24	674.97
$(1^+,2^+,3^+,4^-,5^+)$	61	474	18	184.05
$(1^-,2^-,3^+,4^+,5^+)$	35	1511	24	278.77
$(1^-,2^+,3^+,4^+,5^-)$	79	7027	34	1112.82
$(1^+,2^+,3^+,4^-,5^-)$	18	19	8	15.00
$(1^-,2^+,3^-,4^+,5^+)$	41	2412	22	368.41
$(1^+,2^-,3^+,4^-,5^+)$	85	18960	42	3934.96
$(1^-,2^+,3^+,4^-,5^+)$	85	10386	37	1803.52

- double-pentagon

Helicity	Non-vanishing coeff.	Max. terms	Max. degree	Avg. non-zero terms
$(1^+,2^+,3^+,4^+,5^+)$	104	1937	26	626.39
$(1^-,2^+,3^+,4^+,5^+)$	104	1449	27	601.43
$(1^+,2^+,3^-,4^+,5^+)$	104	1554	23	642.90
$(1^-,2^-,3^+,4^+,5^+)$	99	1751	26	739.05
$(1^+,2^-,3^-,4^+,5^+)$	104	2524	24	923.71
$(1^-,2^+,3^+,4^+,5^-)$	104	1838	27	823.00
$(1^-,2^+,3^+,4^-,5^+)$	104	1307	24	630.48

Summary & Outlook

Summary

- Finite-fields and functional reconstruction techniques
  - can be use to solve complex algebraic problems
  - any function which can be implemented as a sequence of rational operations is suited for these algorithms
- Applications to scattering amplitudes
  - spinor-helicity in four and six dimensions
  - tree-level calculations
  - integrand reduction via generalized unitarity

Outlook

- complete five-point two-loop calculations
- apply the algorithm to other techniques (e.g. diagrammatic techniques, tensor reduction, IBPs,. . .)

THANKS!

BACKUP SLIDES

Extended euclidean algorithm

- given integers a, b, find s, t such that

$$a\, s + b\, t = \gcd(a, b).$$

- algorithm: generate sequences of integers $\{r_i\}$, $\{s_i\}$, $\{t_i\}$ and the integer quotients $\{q_i\}$ as follows

$$\begin{aligned} r_0 &= a & \cdots &= \cdots \\ s_0 &= 1 & q_i &= \lfloor r_{i-2}/r_{i-1} \rfloor \\ t_0 &= 0 & r_i &= r_{i-2} - q_i\, r_{i-1} \\ r_1 &= b & s_i &= s_{i-2} - q_i\, s_{i-1} \\ s_1 &= 0 & t_i &= t_{i-2} - q_i\, t_{i-1} \\ t_1 &= 1 & & \end{aligned}$$

- stop when $r_k = 1 \Rightarrow t = t_{k-1}$, $s = s_{k-1}$, $\gcd(a, b) = r_{k-1}$
- multiplicative inverse: if $b = n$ and $\gcd(a, n) = 1 \Rightarrow s = a^{-1}$.

Chinese reminder theorem

- given $a_1 \in \mathcal{Z}_{n_1}$, $a_2 \in \mathcal{Z}_{n_2}$ (n_1, n_2 co-prime) find $a \in \mathcal{Z}_{n_1 n_2}$ such that

$$a \bmod n_1 = a_1, \qquad a \bmod n_2 = a_2.$$

- rational reconstruction over $\mathcal{Q}$
  - reconstruct a function f over several finite fields $\mathcal{Z}_{p_1}, \mathcal{Z}_{p_2}, \ldots$
  - recursively combine the result in $\mathcal{Z}_{p_1 p_2 \cdots}$ using the Chinese reminder
  - use the rational reconstruction algorithm on the combined result over $\mathcal{Z}_{p_1 p_2 \cdots}$ to obtain a guess over $\mathcal{Q}$
  - when $\prod_i p_i$ is large enough the reconstruction is successful
  - the termination criterion is consistency over several finite fields
- we can choose the primes p_i small enough to use machine-size integers
- multi-precision arithmetic only required for Chinese reminder
- 1 or 2 primes are usually sufficient

Rational reconstruction: example

- Reconstruct $q = -611520/341$ from its images over finite fields
- $\mathcal{Z}_{p_1}$, with $p_1 = 897473$

$$a_1 = q \text{ mod } p_1 = 13998,$$
$$\text{first guess: } a_1 \xrightarrow{\text{rational rec. over } \mathcal{Z}_{p_1}} g_1 = -411/577$$

- $\mathcal{Z}_{p_2}$, with $p_2 = 909683$

$$a_2 = q \text{ mod } p_2 = 835862$$
$$g_1 \text{ mod } p_2 = 807205 \quad \Rightarrow \quad \text{guess } g_1 \text{ is wrong}$$

- Chinese reminder: $a_1, a_2 \longrightarrow a_{12} \in \mathcal{Z}_{p_1 p_2}$, with $p_1 p_2 = 816415931059$

$$a_{12} \equiv q \text{ mod } p_1 p_2 = 629669763217 \xrightarrow{\text{rational rec. over } \mathcal{Z}_{p_1 p_2}} g_2 = -611520/341$$

- calculation over other fields $\mathcal{Z}_{p_3}, \ldots$ confirm the guess g_2

Univariate polynomials

- Newton' interpolation formula, form a sequence $\{y_0, y_1, \ldots\}$

$$f(z) = \sum_{r=0}^{R} a_r \prod_{i=0}^{r-1} (z - y_i)$$

$$= a_0 + (z - y_0)\Big(a_1 + (z - y_1)\Big(a_2 + (z - y_2)\big(\cdots + (z - y_{r-1})\, a_r\big)\Big)\Big)$$

- each coefficient a_i can be determined by evaluations $f(y_j)$ with $j \leq i$
  - good when degree is not known
- conversion into canonical form

$$f(z) = \sum_{r=0}^{R} c_r\, z^r .$$

  - addition of univariate polynomials,
  - multiplication of a univ. polynomial by a linear univ. polynomial

Univariate rational functions

- Thiele's (1838–1910) interpolation formula

$$f(z) = a_0 + \cfrac{z - y_0}{a_1 + \cfrac{z - y_1}{a_2 + \cfrac{z - y_3}{\cdots + \cfrac{z - y_{r-1}}{a_N}}}}$$

$$= a_0 + (z - y_0) \left(a_1 + (z - y_1) \left(a_2 + (z - y_2) \left(\cdots + \frac{z - y_{N-1}}{a_N} \right)^{-1} \right)^{-1} \right)^{-1},$$

- analogous to Newton's for rational functions
  - good when degrees of numerator/denominator are not known
- if degrees are known and $d_0 = 1$ (see later), just solve the system

$$f(z) = \frac{\sum_{r=0}^{R} n_r\, z^r}{\sum_{r'=0}^{R'} d_{r'}\, z^{r'}} \quad \Rightarrow \quad \sum_{r=0}^{R} n_r\, y_i^r - \sum_{r'=1}^{R'} d_{r'}\, y_i^{r'} f(y_i) = f(y_i)$$

Multivariate polynomials

- recursive Newton's formula

$$f(z_1,\ldots,z_n) = \sum_{r=0}^{R} a_r(z_2,\ldots,z_n) \prod_{i=0}^{r-1}(z_1 - y_i),$$

- like univariate with

$$f(y_j) \longrightarrow f(y_j, z_2, \ldots, z_n), \qquad a_j \longrightarrow a_j(z_2,\ldots,z_n).$$

- convert it back to canonical representation using
  - addition of multivariate polynomials,
  - multiplication of a multiv. polynomial by a linear univ. polynomial.
- very efficient, even for large polynomials

Multivariate rational functions

- dense algorithm, adapted from sparse one by A. Cuyt, W. Lee (2011)
- overall normalization
  - assume non-vanishing constant term in denominator ($d_{(0,\ldots,0)} = 1$)
  - if not the case, shift args. by appropriate vector $\mathbf{s}$, using $f_{\mathbf{s}} = f(\mathbf{z} + \mathbf{s})$
- define new function $h \in \mathcal{F}(t, \mathbf{z})$ as

$$h(t, \mathbf{z}) \equiv f(t\,\mathbf{z}) = f(t\,z_1, \ldots, t\,z_n) = \frac{\sum_{r=0}^{R} p_r(\mathbf{z})\, t^r}{1 + \sum_{r'=1}^{R'} q_{r'}(\mathbf{z})\, t^{r'}}$$

where

$$p_r(\mathbf{z}) \equiv \sum_{|\alpha|=r} n_\alpha\, \mathbf{z}^\alpha, \qquad q_{r'}(\mathbf{z}) \equiv \sum_{|\beta|=r'} d_\beta\, \mathbf{z}^\beta.$$

$\Rightarrow$ univ. rational fun. in t with (homogeneous) multiv. polynomial coefficients