ON CLASSIFICATION OF SOME NON-SINGULAR MANIFOLDS IN THE SPACE M(1,3)xR(u)

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Introduction

In many cases the mathematical models of various processes can be described by means of differential equations (linear or nonlinear) in the space M(1,3)xR(u).

Here, and in what follows:

- **M(1,3) is the four-dimensional Minkowsky space;**
- **R(u) is the number axis of the dependent variable u.**

Introduction

In many cases the mathematical models of various processes can be described by means of differential equations (linear or nonlinear) in the space M(1,3)xR(u).

It is well known that the majority of those differential equations have non-trivial symmetry groups.

Some differential equations with non-trivial symmetry groups in the space M(1,3)xR(u)

The Klein-Gordon-Fock equation

$$\left(p^{\mu} p_{\mu} - m^2\right)\psi = 0, \quad \mu = 0, 1, 2, 3.$$

$$p_0 = p^0 = i\frac{\partial}{\partial x_0}, \; p_a = p^a = -i\frac{\partial}{\partial x_a}, a = 1, 2, 3.$$

Some differential equations with non-trivial symmetry groups in the space M(1,3)xR(u)

The Dirac equation

$$\left(\gamma_k P^k - m\right)\psi(x) = 0,$$

$$x = (x_0, x_1, x_2, x_3), P_k = i\frac{\partial}{\partial x_k}, k = 0,1,2,3.$$

γ_k are (4x4) Dirac matrices.

Some differential equations with non-trivial symmetry groups in the space M(1,3)xR(u)

These equations:

The Klein-Gordon-Fock equation

$$\left(p^{\mu} p_{\mu} - m^{2}\right)\psi = 0,$$

The Dirac equation

$$\left(\gamma_{k} P^{k} - m\right)\psi(x) = 0,$$

are invariant under 10-parametrical Poincaré group $P(1,3)$.

Some differential equations with non-trivial symmetry groups in the space M(1,3)xR(u)

The eikonal equation

$$u^{\mu}u_{\mu} \equiv (u_0)^2 - (u_1)^2 - (u_2)^2 - (u_3)^2 = 1$$

where

$$u = u(x), x = (x_0, x_1, x_2, x_3) \in M(1,3),$$

$$u_{\mu} \equiv \frac{\partial u}{\partial x_{\mu}}, u^{\mu} = g^{\mu\nu}u_{\nu}, \mu, \nu = 0,1,2,3.$$

Some differential equations with non-trivial symmetry groups in the space M(1,3)xR(u)

The Euler-Lagrange-Born-Infeld equation

$$\Box u(1-u_\nu u^\nu)+u^\mu u^\nu u_{\mu\nu}=0,$$

where

$$u=u(x), x=(x_0,x_1,x_2,x_3)\in M(1,3),$$

$$u_\mu \equiv \frac{\partial u}{\partial x^\mu}, u_{\mu\nu}\equiv \frac{\partial^2 u}{\partial x^\mu \partial x^\nu}, u^\mu = g^{\mu\nu}u_\nu$$

$$g_{\mu\nu}=(1,-1,-1,-1)\delta_{\mu\nu}, \mu,\nu=0,1,2,3.$$

Some differential equations with non-trivial symmetry groups in the space M(1,3)xR(u)

The homogeneous Monge-Ampère equation

$$\det(u_{\mu\nu}) = 0,$$

where

$$u = u(x), x = (x_0, x_1, x_2, x_3) \in M(1,3),$$

$$u_{\mu\nu} \equiv \frac{\partial^2 u}{\partial x^\mu \partial x^\nu}, \qquad \mu, \nu = 0,1,2,3.$$

Some differential equations with non-trivial symmetry groups in the space M(1,3)xR(u)

The inhomogeneous Monge-Ampère equation

$$\det(u_{\mu\nu}) = \lambda(1 - u_\nu u^\nu)^3, \lambda \neq 0,$$

where

$$u = u(x), x = (x_0, x_1, x_2, x_3) \in M(1,3),$$

$$u_\alpha \equiv \frac{\partial u}{\partial x_\alpha}, u_{\mu\nu} \equiv \frac{\partial^2 u}{\partial x_\mu \partial x_\nu}, u^\nu = g^{\nu\alpha} u_\alpha$$

$$g_{\mu\nu} = (1,-1,-1,-1)\delta_{\mu\nu}, \mu, \nu, \alpha = 0,1,2,3.$$

Some differential equations with non-trivial symmetry groups in the space M(1,3)xR(u)

These equations

- **The eikonal equation**
- **The Euler-Lagrange-Born-Infeld equation**
- **The homogeneous Monge-Ampère equation**
- **The inhomogeneous Monge-Ampère equation**

are invariant under 15-parametrical generalized Poincaré group P(1,4).

Some differential equations with non-trivial symmetry groups in the space M(1,3)xR(u)

See for example

- Fushchych W.I. and Shtelen W.M., The symmetry and some exact solutions of the relativistic eikonal equation, Lett. Nuovo Cim., 1982, V. 34, N16, 498-502.
- Fushchych W.I. and Serov N.I., The symmetry and some exact solutions of the multidimensional Monge-Ampere equation, Dokl. Akad. Nauk SSSR, 1983, V. 283, N3, 543-546.
- Fushchych W.I. and Serov N.I., On some exact solutions of the multidimensional nonlinear Euler-Lagrange equation, Dokl. Akad. Nauk SSSR, 1984, V. 278, N 4, 847-851.

The generalized Poincaré group P(1,4)

The group P(1,4) is a group of rotations and translations of the five-dimensional Minkowski space M(1,4).

The generalized Poincaré group P(1,4)

It is the smallest group, which contains, as subgroups,

- **the extended Galilei group $\tilde{G}(1,3)$ (the symmetry group of classical physics) and**
- **the Poincaré group P(1,3) (the symmetry group of relativistic physics).**

Fushchich W.I., Nikitin A.G. Reduction of the representations of the generalized Poincare algebra by the Galilei algebra, J. Phys. A: Math. And Gen. 1980, 13 (7), 2319-2330.

The Lie algebra of the group P(1,4)

The Lie algebra of the group $P(1,4)$ is given by the 15 basis elements $M_{\mu\nu} = -M_{\nu\mu}$, $\mu, \nu = 0, 1, \dots, 4$ and P_μ, $\mu = 0, 1, \dots, 4$, satisfying the commutation relations

$$[P_\mu, P_\nu] = 0, \qquad [M_{\mu\nu}, P_\sigma] = g_{\nu\sigma} P_\mu - g_{\mu\sigma} P_\nu,$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = g_{\mu\sigma} M_{\nu\rho} + g_{\nu\rho} M_{\mu\sigma} - g_{\mu\rho} M_{\nu\sigma} - g_{\nu\sigma} M_{\mu\rho},$$

where $g_{00} = -g_{11} = -g_{22} = -g_{33} = -g_{44} = 1$, $g_{\mu\nu} = 0$, if $\mu \neq \nu$.

The Lie algebra of the group P(1,4)

Let us consider the following representation of the Lie algebra of the group P(1,4):

$$P_0 = \frac{\partial}{\partial x_0}, P_1 = -\frac{\partial}{\partial x_1}, P_2 = -\frac{\partial}{\partial x_2}, P_3 = -\frac{\partial}{\partial x_3},$$

$$P_4 = -\frac{\partial}{\partial u}, M_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu, x_4 \equiv u,$$

$$\mu, \nu = 0,1,2,3.$$

The Lie algebra of the group P(1,4)

Further, we will use the following basis elements:

$$G = M_{04}, \quad L_1 = M_{23}, \quad L_2 = -M_{13}, \quad L_3 = M_{12},$$

$$P_a = M_{a4} - M_{0a}, \quad C_a = M_{a4} + M_{0a}, \quad (a = 1, 2, 3),$$

$$X_0 = \frac{1}{2}(P_0 - P_4), \quad X_k = P_k \quad (k = 1, 2, 3), \quad X_4 = \frac{1}{2}(P_0 + P_4).$$

The Lie algebra of the group P(1,4)

Continuous subgroups of the group P(1,4) have been described in

- **V.M. Fedorchuk**, Ukr. Mat. Zh., **31**, No. 6, 717-722 (1979).
- **V.M. Fedorchuk**, Ukr. Mat. Zh., **33**, No. 5, 696-700 (1981).
- **W.I. Fushchich, A.F. Barannik, L.F. Barannik and V.M. Fedorchuk**, J. Phys. A: Math. Gen., **18**, No.14, 2893-2899 (1985).

The Lie algebra of the Galilei group

Lie algebra of the extended Galilei group $\tilde{G}(1,3)$ is generated by the following bases elements:

$$L_1, L_2, L_3, P_1, P_2, P_3, X_0, X_1, X_2, X_3, X_4$$

Some differential equations with non-trivial symmetry groups in the space M(1,3)xR(u)

In many cases the investigation of the mathematical models reduces to the construction of exact solutions of the corresponding differential equations.

Differential equations with non-trivial symmetry groups

In 1895, Lie considered solutions invariant with respect to groups admitted by higher-order partial differential equations.

Lie S. Zur allgemeinen Theorie der partiellen Differentialgleichungen beliebiger Ordnung, *Leipz. Berichte*, I. 53. (Reprinted in Lie, S., *Gesammelte Abhandlungen*, Vol. 4, Paper IX.), 1895.

Some differential equations with non-trivial symmetry groups in the space M(1,3)xR(u)

It is well known that each invariant solution of the above mentioned P(1,4)-invariant PDEs is a non-singular manifold in the space M(1,3)xR(u).

Some differential equations with non-trivial symmetry groups in the space M(1,3)xR(u)

To classify invariant solutions of the above mentioned P(1,4)-invariant PDEs we have to classify corresponding non-singular manifolds in the space M(1,3)xR(u).

Classification of non-singular manifolds

The present report is devoted to the classification of non-singular manifolds in the space M(1,3)xR(u) invariant under one-, two- and three- dimensional non-conjugate subalgebras of the Lie algebra of the Poincaré group P(1,4).

Some differential equations with non-trivial symmetry groups in the space M(1,3)xR(u)

To date, we have studied the relationship between structural properties of low-dimensional (dim L = 1,2,3) nonconjugate subalgebras of the same rank of the Lie algebra of the group P(1,4) and the properties of the corresponding non-singular manifolds.

Classification of non-singular manifolds

The non-singular manifolds in the space M(1,3)×R(u) invariant under non-conjugate subalgebras of the Lie algebra of the group P(1,4) can be written in the form

$$\boldsymbol{F}\left(\boldsymbol{J_1}, \boldsymbol{J_2}, \ldots, \boldsymbol{J_{t_0}}\right) = \mathbf{0}$$

where *F* is a smooth function of its arguments and $\{J_1, J_2, \ldots, J_{t_0}\}$ is a functional bases of invariants of the non-conjugate subalgebras of the Lie algebra of the group P(1,4).

Classification of non-singular manifolds

Below, we present a short review of the results obtained.

Classification of non-singular manifolds

To classify those manifolds we have used the classification of one-dimensional subalgebras of the Lie algebra of the group P(1,4), which has performed in

- **Fedorchuk V.M., Fedorchuk V.I.** Proceedings of Institute of Mathematics of NAS of Ukraine, 2006, V.3, N2, 302-308.

Classification of non-singular manifolds

- According to this classification, all one-dimensional non-conjugate subalgebras of the Lie algebra of the group P(1,4) belong to the same class, which is denoted by A_1.

Proposition 1.

There exist **20** non-conjugate subalgebras of the type A_1 of the Lie algebra of the group P(1,4).

Classification of non-singular manifolds

Consequently, there exist **20** non-singular manifolds in the space M(1,3)xR(u) invariant under one-dimensional non-conjugate subalgebras of the type A_1 of the Lie algebra of the group P(1,4).

Classification of non-singular manifolds

Some examples:

$\langle G \rangle$

$$F\left(x_1, x_2, x_3, (x_0^2 - x_4^2)^{1/2}, u\right) = 0$$

Classification of non-singular manifolds

Some examples:

$$\langle L_3 + \alpha X_3 \rangle, \alpha > 0$$

$$F\left(x_0, x_4, (x_1^2 + x_2^2)^{1/2}, \alpha arctan\frac{x_1}{x_2} + x_3, u\right) = 0$$

Classification of non-singular manifolds

From the invariants of one subalgebra **it is impossible to construct an ansatz**, which reduces the above mentioned equations.

Let's present bases elements of the **subalgebra** and corresponding **functional basis of invariants**.

$$\langle X_4 - X_0 \rangle: x_0, x_1, x_2, x_3.$$

Classification of non-singular manifolds

Now, we consider the non-singular manifolds in the space $M(1,3)\times R(u)$ invariant under two-dimensional non-conjugate subalgebras of the Lie algebra of the group $P(1,4)$.

Classification of non-singular manifolds

To classify these manifolds we have used the classification of two-dimensional subalgebras of the Lie algebra of the group P(1,4), which has performed in

Fedorchuk V.M., Fedorchuk V.I. Proceedings of Institute of Mathematics of NAS of Ukraine, 2006, V.3, N2, 302-308.

Classification of non-singular manifolds

According to this classification all two-dimensional non-conjugate subalgebras of the Lie algebra of the group P(1,4) belong to two different (non-isomorphic) classes, which are denoted by $2A_1$ and A_2 .

Classification of non-singular manifolds

First, we present the non-singular manifolds in the space M(1,3)×R(u) invariant under two-dimensional non-conjugate subalgebras of the Lie algebra of the group P(1,4) of the type $2A_1$ (Abelian subalgebras).

Classification of non-singular manifolds

Proposition 2.

There exist **37** non-conjugate subalgebras of the type $2A_1$ of the Lie algebra of the group P(1,4).

- **Fedorchuk V.M., Fedorchuk V.I.** Proceedings of Institute of Mathematics of NAS of Ukraine, 2006, V.3, N2, 302-308.

Classification of non-singular manifolds

Consequently, there exist **37** non-singular manifolds in the space M(1,3)xR(u) invariant under two-dimensional non-conjugate subalgebras of the type $2A_1$ of the Lie algebra of the group P(1,4).

Classification of non-singular manifolds

Some examples:

$$\langle P_1, P_2 \rangle$$

$$F\left(x_3, x_0 + x_4, (x_0^2 - x_1^2 - x_2^2 - x_4^2)^{1/2}, u\right) = 0$$

Classification of non-singular manifolds

Some examples:

$\langle G, L_3 \rangle$

$$F\left(x_3, (x_0^2 - x_4^2)^{1/2}, (x_1^2 + x_2^2)^{1/2}, u\right) = 0$$

Classification of non-singular manifolds

From the invariants of 5 subalgebras it is impossible to constract ansatzes which reduce the above mentioned equations.

Let's present bases elements of some subalgebra and corresponding functional basis of invariants.

$$\langle X_1 \rangle \oplus \langle X_4 - X_0 \rangle: x_0, x_2, x_3.$$

Classification of non-singular manifolds

Now, we present the non-singular manifolds in the space M(1,3)×R(u) invariant under two-dimensional non-conjugate subalgebras of the type A_2 of the Lie algebra of the group P(1,4) (non-Abelian subalgebras).

Proposition 3.

There exist **7** non-conjugate subalgebras of the type A_2 of the Lie algebra of the group P(1,4).

Classification of non-singular manifolds

Consequently, there exist **7** non-singular manifolds in the space M(1,3)xR(u) invariant under two-dimensional non-conjugate subalgebras of the type A_2 of the Lie algebra of the group P(1,4).

Classification of non-singular manifolds

Some examples:

$\langle -G, P_3 \rangle$

$$F\left(x_1, x_2, (x_0^2 - x_3^2 - x_4^2)^{1/2}, u\right) = 0$$

Classification of non-singular manifolds

Some examples:

$$\langle -G - \alpha X_1, P_3 \rangle, \alpha > 0$$

$$F\left(x_1 - \alpha ln(x_0 + x_4), x_2, (x_0^2 - x_3^2 - x_4^2)^{1/2}, u\right) = 0$$

Classification of non-singular manifolds

From the invariants of one subalgebra it is impossible to construct ansatz which reduces the eikonal equation.

Classification of non-singular manifolds

Let's present bases elements of the subalgebra and corresponding functional basis of invariants.

$$\langle -G, X_4 \rangle : x_1, x_2, x_3.$$

Some applications of the results obtained

The obtained manifolds can be used for the classification of symmetry reductions as well as corresponding invariant solutions for some differential equations in the space M(1,3)xR(u) invariant under the group P(1,4) or its non-conjugate subgroups.

Some applications of the results obtained

The classification of the non-singular manifolds in the space M(1,3)xR(u) as well as corresponding invariant solitons invariant under three-dimensional nonconjugate subalgebras of the Lie algebra of the group P(1,4) can be found in

Fedorchuk V., Fedorchuk V. On Classification of Symmetry Reductions for the Eikonal Equation. Symmetry 2016, **8** (6), Art. 51, 32pp; doi: 10.3390/sym8060051.

Thank you for your attention!