Bence Racskó
University of Szeged
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- Shockwaves are ubiquitous in physics and GR (general relativity) is no exception.
- More general case: boundary surfaces.
- Examples: Stellar collapse, stellar boundaries, impulsive electromagnetic or gravitational waves, supernovae, cosmological phase transitions.
- In all examples, a hypersurface partitions spacetime into two disjoint domains.

- Physical conditions can be suitably different in the two domains.
- The local fields in each domain constitute a solution of the EFE.
- Conditions for the total spacetime to be a solution? → Junction conditions.
- Principal work by Lánczos, Darmois and Israel, modern formulation by Israel.

- Analogous situation in Maxwell theory – the well-known jump conditions on the **E**, **D**, **B** and **H** fields:

$$[\mathbf{D}]\cdot\mathbf{n}=\sigma,\ \mathbf{n}\times[\mathbf{H}]=\mathbf{K},\ \mathbf{n}\times[\mathbf{E}]=0 \text{ and } \mathbf{n}\cdot[\mathbf{B}]=0.$$

- Unlike the relatively simple case in ED, the junction formalism in GR suffers from three complications.

- Unlike, for example, ED, in GR the spacetime manifolds themselves need to be matched, not just the field variables.

  This is nontrivial and actually implies the “first” or “preliminary” junction condition: $[h_{ab}] = 0$

- GR, like differential geometry, can and should be formulated in a coordinate-free manner.
- Because GR is physics, all things considered should be resolvable into “calculatable numbers” (a subset of reals), hence, coordinates are needed for direct computation.
- Coordinates are adapted to symmetry – which can differ in the two domains – the preferred coordinates might be discontinous.

- Mismatching coordinates make comparison of tensor fields difficult – even smooth tensor fields appear discontinuous if different representations are compared.
- Therefore, expressing junction conditions with intrinsic hypersurface tensors (in intrinsic hypersurface coordinates) is recommended.

- Israel’s formalism works when the hypersurface is spacelike (sudden phase transitions) or timelike (stellar boundaries, et al).
- It doesn’t work when the surface is null/lightlike.
- The null case is of physical interest – electromagnetic and gravitational wave spacetimes.
- Generalization of Israel’s formalism by Barrabés and Israel.

- Normal vector is also tangential for null surfaces.
- Extrinsic curvature is no longer a carrier of transverse information.
- Induced metric is degenerate and a semi-Riemannian structure doesn’t exist.
- Generalization replaces the normal vector with transversal.
- Considerable gauge freedom – end result is invariant.

- Scalar-tensor theories are modified/extended theories of gravity (ETG).
- Why extend gravity? Answer: Dark energy and inflation models, Kaluza-Klein theory, low-energy limits of string theory compactifications.
- Scalar-tensor theory: In addition to the metric, a scalar field is also present as a configuration variable.

- Solar system tests: GR is highly accurate on a Solar system scale – ETGs should give no different results.
- Ostrogradskij-instability: Lagrangian systems with higher than first derivatives tend to be unstable. Not all higher derivative theories are affected. If EOMs are second order, problems are avoided. ETGs should avoid Ostrogradskij instability.

- Origin: Kaluza-Klein theory, the metric function giving the size of the compactified dimension is a scalar w.r.t. 4D transformations.
- Horndeski theory – Discovered in 1974 by G.W. Horndeski, is the most general scalar-tensor theory in 4D with second order field equations.
- Rediscovered lately as a generalization of galileon theory.

- Horndeski contains 4 arbitrary (smooth) functions of the scalar field.
- Contains known scalar-tensor theories as subcases, such as Brans-Dicke theory, flat galileon theory, covariant galileon theory, k-essence etc.
- Studying Horndeski’s theory allows one to make sweeping statements about a very large family of ETGs.

- Junction conditions for Horndeski has been found by Padilla & Sivanesan for the timelike/spacelike cases.
- This work aims to find junction conditions in the general case, including null surfaces.
- As a test case, we considered generalized Brans-Dicke theory instead of Horndeski, which is less general, but still allow for a considerable amount of cases.

- Initial situation: We are given two spacetimes, M_1 and M_2 with hypersurfaces Σ_1 and Σ_2. The hypersurfaces cut both manifolds into two subdomains, M_{1+}, M_{1-}, M_{2+} and M_{2-}.
- We let $M^+ = M_1^+ \text{ and } M^- = M_2^-$, and attempt to create a unified spacetime M, consisting of M^+ and M^- along a common boundary.
- Immediate requirement: existence of diffeomorphism $i : \Sigma_2 \to \Sigma_1$ identifying the surfaces.

- Glueing along the common hypersurface creates a topological manifold. This allows for the identification of the tangent spaces $T_x\Sigma_2$ and $T_{i(x)}\Sigma_1$.
- To obtain a C^1 structure on M, identification of the full tangent spaces of M^+ and M^- are needed. This process is more or less arbitrary. The C^1 structure induces a unique C^∞ structure on M.

- Clarke & Dray: If the induced metrics on Σ agree, then there is a unique C^1 structure on M, in which the full metric is continuous across Σ.
- Proof is based on canonical forms of the metric (Gauss normal coordinates for timelike/spacelike surfaces).
- Continuity of the full metric carries no more information than the agreement of the induced metrics, as the full metric contains no more DoFs than the induced metrics.

- Let F be a physical quantity. The following notation is used:
- $[F] = (F^{+} - F^{-})\,|_{\Sigma}$ - the “jump” of F;
- $\tilde{F} = F^{+}\Theta(\varphi) + F^{-}\Theta(-\varphi)$ - the “soldering” of F, where Θ is the Heaviside step function and φ is the function whose zeroes generate Σ.
- Distributionally $\partial_\mu \tilde{F} = (\partial_\mu F)^{\sim} + [F]\, n_\mu \delta(\varphi)\alpha$, where $\alpha n_\mu = \varphi_{,\mu}$

- Assume that a coordinate system x^μ is given which is smooth across Σ.
- Assume that the preliminary junction condition $[g_{\mu\nu}] = 0$ holds.
- Express the metric as $g_{\mu\nu} = g^{+}_{\mu\nu}\Theta(\varphi) + g^{-}_{\mu\nu}\Theta(-\varphi)$
- Calculate the EFE from this metric. If terms proportional to the delta function remain, the source must also be singular.

- For timelike/spacelike hypersurfaces,

$$T^{\mu\nu} = \tilde{T}^{\mu\nu} + S^{ab} e^{\mu}_{a} e^{\nu}_{b} \delta(\varphi) \alpha$$

where $S_{ab} = -\frac{1}{8\pi G}\left([K_{ab}] - [K]\, h_{ab}\right)$.

- The equation for S_{ab} is called Lánczos' equation.
- If the EFE is to be regular, the extrinsic curvatures must agree. This condition is violated if an idealized thin layer of stress-energy is given on Σ. The Lánczos eq relates its SEM tensor to the jump in K_{ab}.

- The equations $[h_{ab}] = 0$ and $S_{ab} = -\frac{1}{8\pi G}\left([K_{ab}] - [K]\, h_{ab}\right)$ together provide the junction conditions .
- The junction conditions are expressed with the ξ^a hypersurface coordinates.

- Here we present a derivation of the **general case**.
- Because $[g_{\mu\nu}] = 0$, the linear connexion can be expressed as $\Gamma^{\sigma}{}_{\mu\nu} = \tilde{\Gamma}^{\sigma}{}_{\mu\nu}$.
- Let l^{μ} be a *transverse* vector field to Σ, normalized as $l^{\mu} n_{\mu} = \eta^{-1}$.
- The jump $[\partial_{\sigma} g_{\mu\nu}]$ might be nonzero but must be directed normally: $[\partial_{\sigma} g_{\mu\nu}] = n_{\sigma} c_{\mu\nu}$. This is because the discontinuity must be transversal: $l^{\sigma} [\partial_{\sigma} g_{\mu\nu}] = \eta^{-1} c_{\mu\nu}$ and $e^{\sigma}_{a} [\partial_{\sigma} g_{\mu\nu}] = 0$.

- The jump in the linear connexion is then

$$[\Gamma^{\sigma}{}_{\mu\nu}] = \frac{1}{2}\left(n_\mu c_\nu{}^{\sigma} + n_\nu c_\mu{}^{\sigma} - n^{\sigma} c_{\mu\nu}\right)$$

- The form of the curvature tensor is

$$R^{\rho}{}_{\sigma\mu\nu} = \tilde{R}^{\rho}{}_{\sigma\mu\nu} + \mathcal{R}^{\rho}{}_{\sigma\mu\nu}\delta(\varphi)\alpha$$

where

$$\mathcal{R}^{\rho}{}_{\sigma\mu\nu} = n_{[\mu}c_{\nu]}{}^{\rho} n_\sigma - n_{[\mu}c_{\nu]\sigma}n^{\rho}$$

- Notation: $c_\mu = c_{\mu\nu}n^{\nu},\ c^{\dagger} = c_\mu n^{\mu}$ and $c = c^{\mu}_{\mu}$.

- The Einstein tensor is $G_{\mu\nu} = \tilde{G}_{\mu\nu} + \mathcal{G}_{\mu\nu}\delta(\varphi)\alpha$, where

$$\mathcal{G}_{\mu\nu} = \frac{1}{2}\left(2n_{(\mu}c_{\nu)} - n_\mu n_\nu c - c^\dagger g_{\mu\nu} - \epsilon\left(c_{\mu\nu} - c g_{\mu\nu}\right)\right)$$

  and thus $S_{\mu\nu} = \frac{1}{8\pi G}\mathcal{G}_{\mu\nu}$ is the singular part of the SEM tensor.
- This SEM tensor is gauge-invariant and its contraction with the normal vanishes: $S_{\mu\nu}n^\nu = 0$.

- Because Σ might be null, only the contravariant form of the distributional SEM tensor may be represented as intrinsic Σ-tensor as $S^{\mu\nu} = S^{ab} e_a^\mu e_b^\nu$.
- This form is given by

$$S^{ab} = \frac{\eta}{8\pi G}\left(2n^{(a}h_*^{b)c}n^d - n^a n^b h_*^{cd} - h_*^{ab} n^c n^d - \epsilon\left(h_*^{ac}h_*^{bd} - h_*^{cd}h_*^{ab}\right)\right)[\mathcal{K}_{cd}]$$

where $n^a = \theta_\mu^a n^\mu$, $h_*^{ab} = g^{\mu\nu}\theta_\mu^a\theta_\nu^b$ and θ_μ^a is the dual frame to e_a, satisfying $\theta_\mu^a e_b^\mu = \delta_b^a \text{ and } \theta_\mu^a l^\mu = 0$.

- The SEM tensor is thus presented in intrinsic components.

- The action is given by $S_T\left[g,\phi,\Psi_m\right]=S_{GBD}\left[g,\phi\right]+S_m\left[g,\Psi_m\right]$, where S_m is the matter action (without scalar field) and S_{GBD} is the generalized Brans-Dicke action.
- $$S_{GBD}=\int_{\mathcal{D}}d^n x\sqrt{-g}\left[\frac{1}{2}F\left(\phi\right)R+B\left(\phi\right)X+2G\left(\phi\right)\Box\phi X\right]$$

  where $X=1/2\ \phi_{;\mu}\phi^{;\mu}$ is the kinetic term and F, B and G are arbitrary smooth functions of the scalar field Φ.

- Equations of motion are given by functional differentiation.
- The following notation is utilized:

$$E^{(g)}_{\mu\nu} = \frac{1}{\sqrt{-g}}\frac{\delta S_{GBD}}{\delta g^{\mu\nu}}, \quad E^{(\phi)} = \frac{1}{\sqrt{-g}}\frac{\delta S_{GBD}}{\delta \phi} \quad T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}}$$

- Then, the equations of motion are $E^{(g)}_{\mu\nu} = \frac{1}{2}T_{\mu\nu} \text{ and } E^{(\phi)} = 0$
- Let $E^{(g)}_{\mu\nu} = \mathcal{E}^{(g2)}_{\mu\nu} + \mathcal{E}^{(g3)}_{\mu\nu} + \mathcal{E}^{(g4)}_{\mu\nu}$ and $E^{(\phi)} = \mathcal{E}^{(\phi 2)} + \mathcal{E}^{(\phi 3)} + \mathcal{E}^{(\phi 4)}$

- The concrete form of the EoMs are given by

$$
\begin{aligned}
\mathcal{E}^{(g2)}_{\mu\nu} &= \frac{1}{2}\left(B\phi_{,\mu}\phi_{,\nu} - BXg_{\mu\nu}\right), \\
\mathcal{E}^{(g3)}_{\mu\nu} &= G\Box\phi\phi_{;\mu}\phi_{;\nu} + G_{;\lambda}\phi^{;\lambda}Xg_{\mu\nu} + GX_{;\lambda}\phi^{;\lambda}g_{\mu\nu} \\
&\quad + GXg_{\mu\nu}\Box\phi - 2G_{;(\mu}\phi_{;\nu)}X - 2GX_{;(\mu}\phi_{;\nu)}, \\
\mathcal{E}^{(g4)}_{\mu\nu} &= \frac{1}{2}\left(FG_{\mu\nu} + \Box F g_{\mu\nu} - F_{;\mu\nu}\right), \\
\mathcal{E}^{(\phi2)} &= B_\phi - B_\phi\phi_{,\mu}\phi^{,\mu} - B\Box\phi, \\
\mathcal{E}^{(\phi3)} &= 2\Box GX + 2G^{;\mu}X_{;\mu} - 2G_{;\mu}\phi^{;\mu}\Box\phi - 2G\left(\Box\phi\right)^2 \\
&\quad + G\phi^{;\mu\nu}\phi_{;\mu\nu} + GR_{\mu\nu}\phi^{;\mu}\phi^{;\nu}, \\
\mathcal{E}^{(\phi4)} &= \frac{1}{2}F_\phi R.
\end{aligned}
$$

- We assume the scalar field is continuous: $[\phi] = 0$
- First derivatives are normal directed: $[\phi_{;\mu}] = \zeta n_\mu$
- The form of second derivatives are $\phi_{;\mu\nu} = (\phi_{;\mu\nu})^{\sim} + [\phi_{;\mu}]\, n_\nu \delta(\varphi)\alpha$
- The d'Alembertian of the scalar field is $\Box\phi = (\Box\phi)^{\sim} + \epsilon\zeta\delta(\varphi)\alpha$
- The jump of the kinetic term involves both zeta and the arithmetic mean of the values of the scalar field derivative. For this reason it shall be denoted simply by $[X]$.

- For now we assume $\epsilon = n \cdot n = 0$.
- The term $\mathcal{E}^{(g2)}_{\mu\nu}$ produces no delta function terms.
- The term $\mathcal{E}^{(g3)}_{\mu\nu}$ produces the delta function terms

$$\mathcal{J}^{(g3)}_{\mu\nu} = G\left[X\right] n^{\lambda}\phi_{;\lambda} g_{\mu\nu} - 2G\left[X\right] n_{(\mu}\phi_{;\nu)}$$

- This expression is tangential and gives the intrinsic result

$$\mathcal{J}^{ab}_{(g3)} = G[X]\phi_n h^{ab}_{*} - G[X] n^a \phi^b - G[X] n^b \phi^a$$

- The next term is given by $\mathcal{J}^{(g4)}_{\mu\nu} = \frac{1}{2}(F\mathcal{G}_{\mu\nu} - F_\phi \zeta n_\mu n_\nu)$
- This is also tangential, intrinsic form is

$$\mathcal{J}^{ab}_{(g4)} = \frac{1}{2}(F\mathcal{G}^{ab} - F_\phi \zeta n^a n^b)$$

- The first scalar term $\mathcal{J}^{(\phi 2)}$ is zero.
- The second scalar term is

$$\mathcal{J}^{(\phi 3)} = 2G_{;\mu} n^\mu [X] + 2G\phi_{;\mu\nu} n^\mu n^\nu \zeta + \frac{G}{2}\left(2n_{(\mu} c_{\nu)} - n_\mu n_\nu c\right) \phi^{;\mu} \phi^{;\nu}$$

- The final scalar term is $\mathcal{J}^{(\phi 4)} = \frac{1}{2} F_{\phi} c^{\dagger}$
- The junction conditions are given by $\frac{1}{2} S^{ab} = \mathcal{J}^{ab}_{g2} + \mathcal{J}^{ab}_{g3} + \mathcal{J}^{ab}_{g4}$ and $0 = \mathcal{J}^{\phi 2} + \mathcal{J}^{\phi 3} + \mathcal{J}^{\phi 4}$.

- The $\epsilon = n \cdot n = 0$ condition was taken because otherwise the equations were not well defined. Why? Need to investigate.
- Padilla & Sivanesan derived equations from variational principle. Is it possible to do that in the general case?
- Investigation of analogues of Einstein frame, including disformal couplings.
- General matching conditions to be implemented in full Horndeski