An Algebraic Approach to Hyperbolic Geometry - for Finite Fields

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Universal hyperbolic geometry

Book: *Divine Proportions: Rational Trigonometry to Universal Geometry* (2005 N J Wildberger)

Papers:

- Universal Hyperbolic Geometry I: Trigonometry (*Geometriae Dedicata*, 163, no.1)
- Universal Hyperbolic Geometry II: A pictorial overview (*KoG* **14**, 2010)
- Universal Hyperbolic Geometry III: First steps in projective triangle geometry (*KoG* **15**, 2011).
- (with A. Alkhaldi) Universal Hyperbolic Geometry IV: Sydpoints and Twin Circumcircles (*KoG* **15**, 2011).
- (with S. Blefari) Quadrangle centroids in Universal Hyperbolic Geometry I (*KoG* **21**, 2017, 41-60)

YouTube: The series *UnivHypGeom* at user: *njwildberger.*

Two famous questions

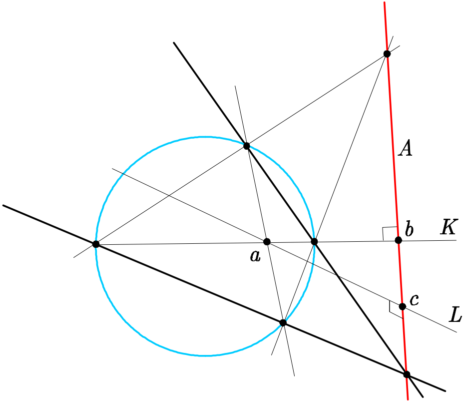
- Q1. What is the physical nature of the "continuum"?
- Q2. What is the mathematical nature of the "continuum"?

The answer to Q1 is, according to modern physics, very difficult and perhaps inaccessible.
The answer to Q2 is, according to modern mathematics, contained in the theory of "real numbers". Unfortunately this theory is very thin logically, unsupported by computation, rarely laid out completely, and often reliant on set theoretic axiomatics. (Eg. What is $\pi + e =$?)

To do geometry, do we need "real numbers"? No! Rational Trigonometry allows us a wider approach, valid over general fields, and Universal Hyperbolic Geometry (UHG) extends that to the projective setting.
YouTube: The series *MathFoundations* at user: *njwildberger*.

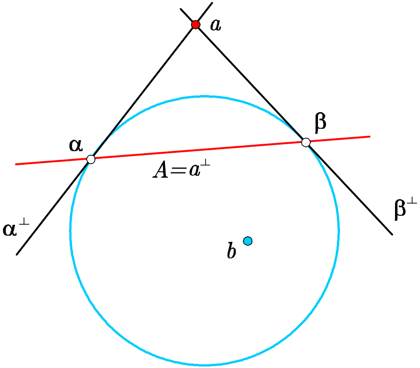
UHG via polarity wrt a circle (Apollonius)

We work in the projective plane, with a fixed conic/circle (in blue). *Polarity* defines *duality*, which defines *perpendicularity*, both of points and lines. Hyperbolic geometry is projective geometry + a fixed conic (Cayley Klein point of view). But with UHG we look at the metrical structure differently!



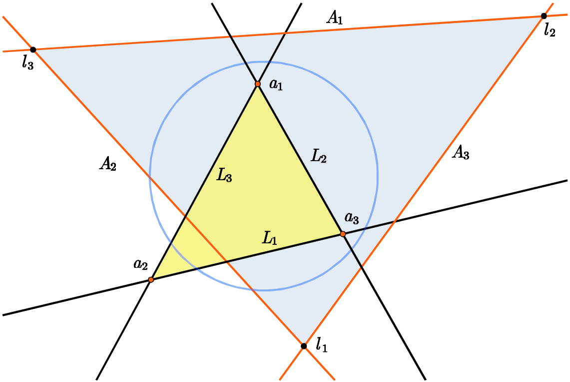
Null points and null lines

Null points are perpendicular to themselves- these are the points on the "absolute conic" in the language of Cayley-Klein. **Null lines** are also perpendicular to themselves.



A triangle a1a2a3 and its dual

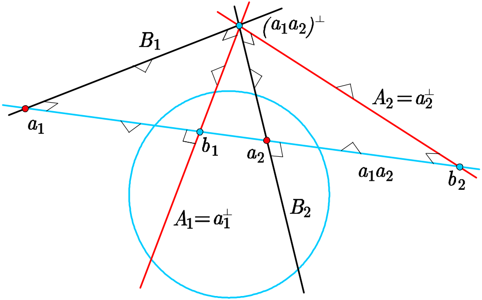
Here we see a triangle $\overline{a_1 a_2 a_3}$ and its dual triangle $\overline{l_1 l_2 l_3}$.



Metrical structure is algebraic!

Define the **quadrance** between points a_1 and a_2 to be the cross-ratio

$$q(a_1, a_2) \equiv (a_1, b_2 : a_2, b_1).$$



Define the **spread** between lines dually, so that $S(A_1, A_2) = q(a_1, a_2)$.

Definitions via a bilinear form

A **(hyperbolic) point**: $a \equiv [x : y : z]$. (Corresponds to the planar point $\left[\frac{x}{z}, \frac{y}{z}\right]$). A **(hyperbolic) line**: $L \equiv (l : m : n)$. (Corresponds to the line $lx + my = n$).
The point $a \equiv [x : y : z]$ **lies on** the line $L \equiv (l : m : n)$, or equivalently L **passes through** a, precisely when

$$lx + my - nz = 0.$$

The point $a \equiv [x : y : z]$ is **dual** to the line $L \equiv (l : m : n)$ precisely when

$$x : y : z = l : m : n.$$

In this case we write $a^\perp = L$ or $L^\perp = a$.
These definitions work over a general field!

Hyperbolic metrical structure

The **quadrance** between points $a_1 \equiv [x_1 : y_1 : z_1]$ and $a_2 \equiv [x_2 : y_2 : z_2]$ is

$$q(a_1, a_2) \equiv 1 - \frac{(x_1 x_2 + y_1 y_2 - z_1 z_2)^2}{(x_1^2 + y_1^2 - z_1^2)(x_2^2 + y_2^2 - z_2^2)}.$$

The **spread** between lines $L_1 \equiv \langle l_1 : m_1 : n_1 \rangle$ and $L_2 \equiv \langle l_2 : m_2 : n_2 \rangle$ is

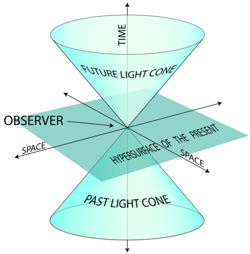
$$S(L_1, L_2) \equiv 1 - \frac{(l_1 l_2 + m_1 m_2 - n_1 n_2)^2}{(l_1^2 + m_1^2 - n_1^2)(l_2^2 + m_2^2 - n_2^2)}.$$

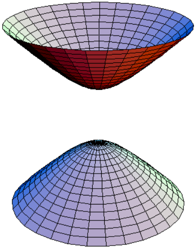
In the Beltrami Klein model, for interior points:

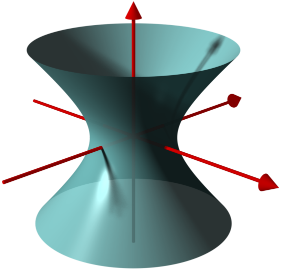
$$\begin{aligned} q(a_1, a_2) &= -\sinh^2(d(a_1, a_2)) \\ S(L_1, L_2) &= \sin^2(\theta(L_1, L_2)). \end{aligned}$$

Connection with relativistic geometry

The usual geometry of SR, in $2+1$ dimensions, when looked at projectively, gives UHG.







Circles

A **hyperbolic circle** with fixed center a and quadrance k is the locus of points x which satisfy $q(a,x) = k$. This is a conic. In Figure 1 we see circles centered at the external point a and their quadrances; these are conics which are tangential to the null conic and include what in the classical literature are called "equi-distant curves".

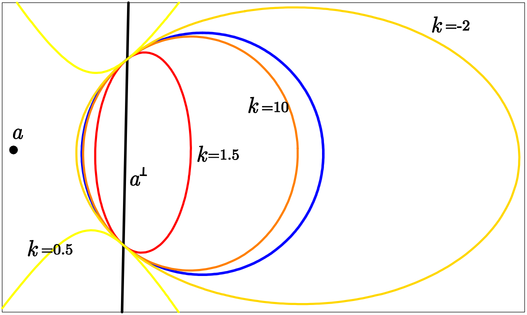


Figure: Hyperbolic circles with center a and different radii.

Midpoints and bilines

A **midpoint** m of a side $\overline{a_1 a_2}$ is a point lying on $a_1 a_2$ which satisfies $q(a_1, m) = q(a_2, m)$. Midpoints exist precisely when $1 - q(a_1, a_2)$ is a square in the field. Here is a construction:

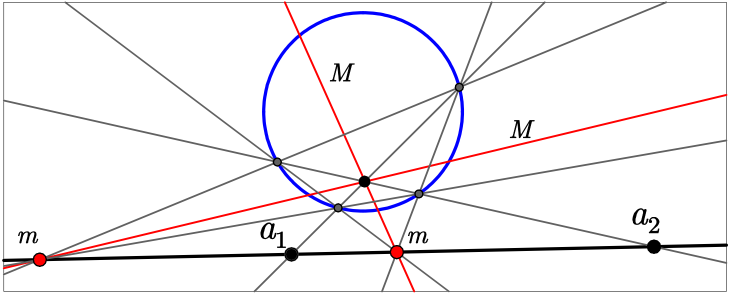


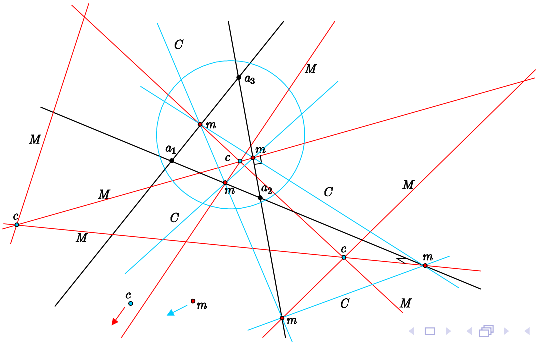
Figure: Midpoints and midlines of a side

There are in general two midpoints if they exist at all, and they are perpendicular.

Midpoints and circumcenters

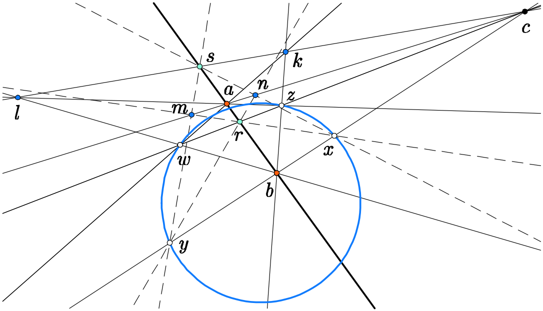
Theorem (Circumcenters)

The six midpoints m of the triangle $\overline{a_1 a_2 a_3}$ are collinear three at a time, lying on four distinct **circumlines** *C. The six midlines M of $\overline{a_1 a_2 a_3}$ are concurrent three at a time, meeting at four distinct* **circumcenters** *c which are dual to the circumlines C. The circumcenters are the centers of the four hyperbolic circles which go through the points of the triangle.*



Sydpoints (with Ali Alkhaldi)

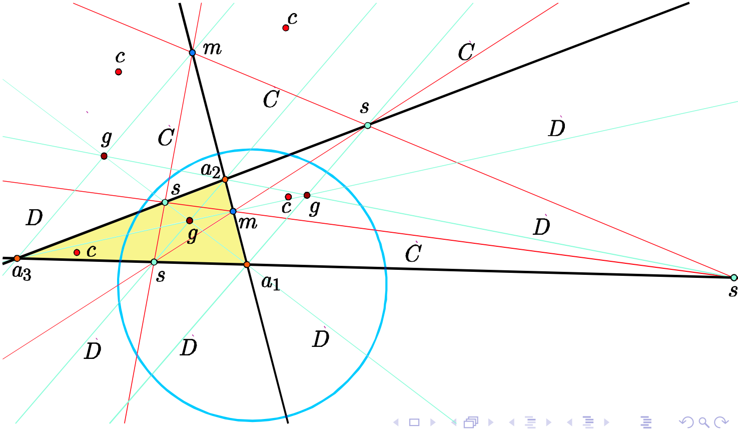
A **sydpoint** of a non-null side $\overline{ab}$ is a point s lying on ab which satisfies $q(a,s) = -q(b,s)$. Sydpoints exist precisely when $q(a,b) - 1$ is a square in the field. There are in general two sydpoints, if they exist at all, but they are not perpendicular. A construction of sydpoints r and s of $\overline{ab}$:



First construct $c = (ab)^{\perp}$, then the midpoints m and n of $\overline{ac}$, and then use the null points x and y lying on bc as shown.

Sydpoints work with midpoints to extend triangle geometry to triangles both inside and outside the null conic. Here we see centroids g and Circumlines C of such a triangle.

Circumlines



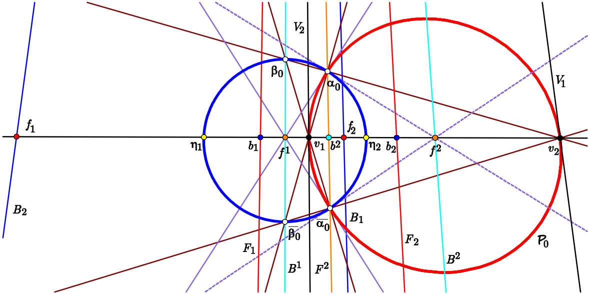
$_1$2.pdf

The parabola (with Ali Alkhaldi)

The **hyperbolic parabola** P_0 is the locus of a point p_0 (actually a conic) satisfying

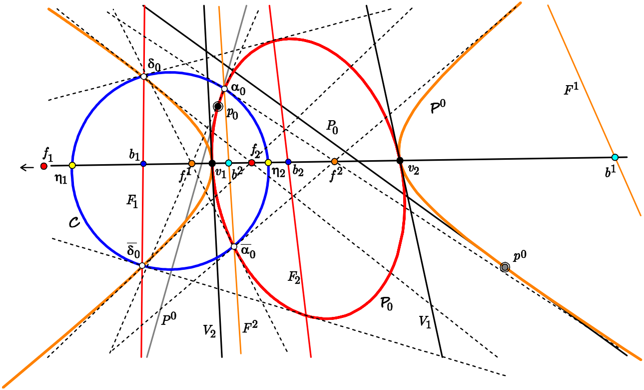
$$q(p_0, f_1) + q(p_0, f_2) = 1$$

or equivalently $q\left(p_0, f_1\right) = q\left(p_0, F_2\right)$ or $q\left(p_0, f_2\right) = q\left(p_0, F_1\right)$, where f_1, f_2 are the **foci**, and $F_1 \equiv f_1^{\perp}$, $F_1 \equiv f_1^{\perp}$ are the **directrices**.

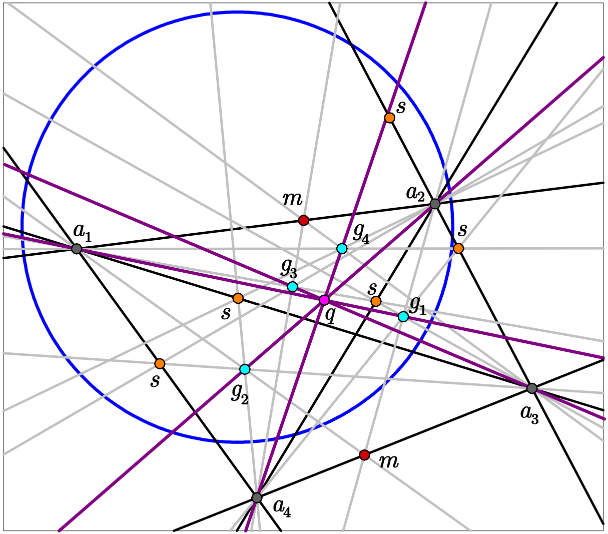


The twin parabola

A parabola P_0 has a dual conic – the **twin parabola** P^0 whose foci f^1, f^2 are the sydpoints of the original foci pair f_1, f_2.



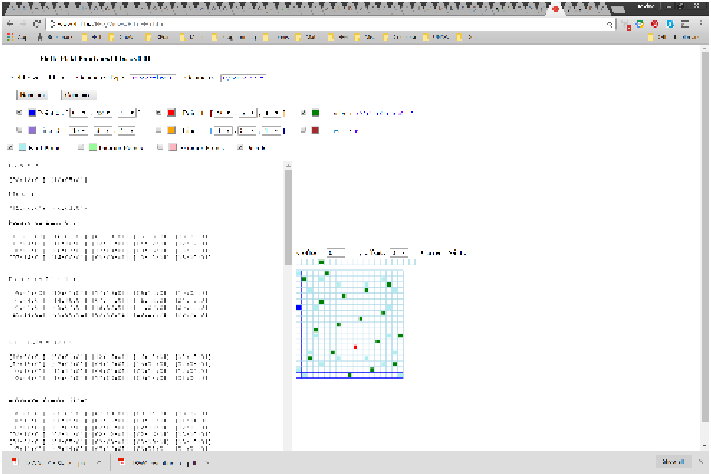
A quadrangle and centroids (with Sebastian Blefari)



Midpoints and sydpoint together?

The conditions that $1 - q(a, b)$ is a square and $q(a, b) - 1$ are a square are not simultaneously satisfied over our usual number system, or over a field F_p where $p \equiv 3 \bmod 4$.
But in the case of F_p where $p \equiv 1 \bmod 4$, then -1 is a square, so both midpoints and sydpoints can exist together! This is an entirely new aspect of hyperbolic geometry that is invisible to us usually. But how can we visualize UHG over a finite field?
Fortunately a new and powerful program being developed by Michael Reynolds at UNSW Sydney accomplishes exactly this!

Reynold's Finite Field Geometry Package



Reynold's program exhibits affine (Euclidean blue, relativistic red and green geometries) planar geometries, and also projective (hyperbolic and elliptic) planar geometries over finite fields.

New developments

- Michael Reynolds' *Finite Field Geometry Program* (beta version) will be appearing online shortly at:

(1) http://www.4ct.biz/Wild/FFG.html
(2) http://www.4ct.biz/FFG/Views/PtLnFFG.html

- *Plimpton 322 is Babylonian Exact Sexagesimal Trigonometry* (with Daniel Mansfield) -published TODAY in Historica Mathematica!
- **YouTube:** The series *Old Babylonian Mathematics* at user: *njwildberger.*
- *Algebraic Calculus One*: a new and better calculus course, to appear January 2018 on Open Learning (videos to appear at *Wild Egg Mathematics Courses* on YouTube)

THANK YOU!