New applications of symplectic structures in quantum information theory

Zimborás Zoltán

Theoretical Physics Department, Wigner Research Centre for Physics

Bolyai-Gauss-Lobachevsky Conference
Károly Róbert Campus, Eszterházy Károly University, Gyöngyös

The symplectic group

$$MJ_{2n}M^t = J_{2n}.$$

$$J_{2n} = \begin{bmatrix} 0 & -1 & & & & & & \\ 1 & 0 & & & & & & \\ & & 0 & -1 & & & & \\ & & 1 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & \ddots & & \\ & & & & & & 0 & -1 \\ & & & & & & 1 & 0 \end{bmatrix} = \begin{bmatrix} J_2 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_2 \end{bmatrix}.$$

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9.4

Canonical transformations in a symplectic form

$$\dot{\zeta} = \mathbf{A}\mathbf{J}\widetilde{\mathbf{A}}\frac{\partial H}{\partial \zeta}$$

- **For the transformations to be canonical:** $\dot{\zeta} = \mathbf{J}\frac{\partial H}{\partial \zeta}$
- **Hence, the canonicity criterion is:**

$$\boxed{\mathbf{A}\mathbf{J}\widetilde{\mathbf{A}} = \mathbf{J}}$$

- **For the case M = 1, it is reduced to (check yourself)**

$$\frac{\partial P}{\partial p} = \frac{\partial q}{\partial Q};\ \frac{\partial Q}{\partial q} = \frac{\partial p}{\partial P};\ \frac{\partial P}{\partial q} = -\frac{\partial p}{\partial Q};\ \frac{\partial Q}{\partial p} = -\frac{\partial q}{\partial P}$$

Elementary unitary quantum control theory I

- Assume that we can implement interactions from a given set $\mathcal{I} = \{iH_1, iH_2, \ldots\}$ of Hamiltonians with tunable control parameters: $H(t) = \sum_j \alpha_j(t) H_j$. This generates a unitary of the form

$$U = \mathcal{T} \int_{t=0}^{1} \exp\left[\sum_{j=1}^{m} i\alpha_j(t) H_j\right]$$

- Two basic questions:
  - Which are the gates (unitaries) that we can generate?
  - How can we achieve a given gate in the most efficient way?

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- Using the Lie -Trotter formulas, we have

$$e^{[iH_k, iH_l]} = \lim_{n\to\infty} \left(e^{iH_k/\sqrt{n}} e^{iH_l/\sqrt{n}} e^{-iH_k/\sqrt{n}} e^{-iH_l/\sqrt{n}} \right)^n,$$

$$e^{-i(\alpha H_k + \beta H_l)} = \lim_{n\to\infty} \left(e^{-i(\alpha H_k/n)} e^{-i(\beta H_l/n)} \right)^n,$$

shows that one can obtain exponential of all commutators $[iH_k, iH_l]$, $[[iH_k, iH_l], iH_m], \ldots$(and their linear combinations), i.e., we end up with the full Lie algebra generated by $\mathcal{I}$:

$$i\tilde{H} \in \langle iH_1, iH_2, \ldots, iH_n \rangle_{Lie}$$

we can obtain $U = e^{i\tilde{H}}$.

Elementary unitary quantum control theory II

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Unitary controllability, pure state controllability

- Full unitary controllability. Any unitary gate can be reached iff

$$\langle iH_1, iH_2, \ldots \rangle = \mathfrak{su}(d).$$

- Pure-state controllability:

$$\langle iH_1, iH_2, \ldots \rangle = \mathfrak{su}(d) \quad \text{when } d \text{ is odd,}$$
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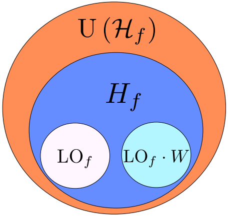
Extensions of LO_f for the case of half-filling $N = d/2$.



- A When N - even, then $H_f = \langle \mathrm{SO}(H_f), \exp(i\phi)\mathbf{1} \rangle$ and we have **no transitivity for pure states**;
- When N - odd, then $H_f = \langle USp(H_f), \exp(i\phi)\mathbf{1} \rangle$ and we have **transitivity for pure states**;
- An extra Hamiltonian with correlated hopping terms $H'_{in} = \sum_j i \left(a_j n_{j+1} a^\dagger_{j+2} - \mathrm{h.c} \right)$ promotes LO_f to H_f.

Werner states

- From the beginning of entanglement theory it has been useful to consider examples of entangled states with high symmetry, e.g. the Werner states $\rho \in \mathcal{B}(\mathbb{C}^d \otimes \mathbb{C}^d)$

$$[\rho, U \otimes U] = 0, \ \forall U \in U(d), \Rightarrow \rho = \alpha \mathbf{1} + \beta F$$

R.F Werner, Phys. Rev. A, 40 (1989) 4277

- They are invariant with respect to unitary twirling

$$\rho \to \int_{U(d)} \mathrm{d}U \, (U \otimes U) \, \rho \, (U^\dagger \otimes U^\dagger)$$

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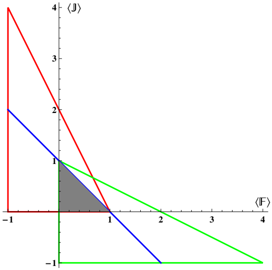

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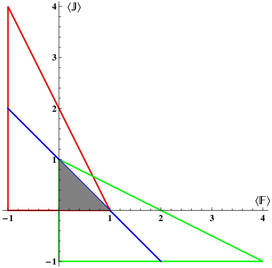

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