Entropic Distance for Nonlinear Master Equation

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Entropic Distance and Divergence

Properties of metric distance:

- $\rho(P, Q) \geq 0$ for a pair of points P and Q.
- $\rho(P, Q) = 0 \Leftrightarrow P = Q$ (then and only then)
- $\rho(P, Q) = \rho(Q, P)$ symmetric measure
- $\rho(P, Q) \leq \rho(P, R) + \rho(R, Q)$ triangle inequality in elliptic spaces

Entropic divergence:

- $\rho(P, Q) \geq 0$ for a pair of distributions P_n and Q_n.
- $\rho(P, Q) = 0 \Leftrightarrow P = Q$ (then and only then)
- $\frac{\mathrm{d}}{\mathrm{d}t}\rho(P, Q) \leq 0$ if Q_n is the stationary distribution

Symmetrized Entropic Divergence

inherited properties

Entropic divergence $\rho(P,Q) = \sum_n \sigma(\xi_n) Q_n$ with $\xi_n = P_n/Q_n$.

Symmetrized kernel function:

$$\mathfrak{s}(\xi) \;:=\; \sigma(\xi) + \xi\,\sigma(1/\xi). \tag{1}$$

Jensen inequality tells for $\sigma'' > 0$:

$$\sum_n \sigma(\xi_n) Q_n \geq \sigma\left(\sum_n \xi_n Q_n\right) = \sigma\left(\sum_n P_n\right) = \sigma(1). \tag{2}$$

For property 1 and 2 one sets: $\sigma(1) = 0$.

it follows $\mathfrak{s}(1) = 0$ and $\mathfrak{s}'' > 0$.

Symmetrized Entropic Divergence

emergent new properties

Derivatives:

$$\begin{aligned}
\mathfrak{s}(\xi) &= \sigma(\xi) + \xi\,\sigma(1/\xi) \\
\mathfrak{s}'(\xi) &= \sigma'(\xi) + \sigma(1/\xi) - \frac{1}{\xi}\sigma'(1/\xi) \\
\mathfrak{s}''(\xi) &= \sigma''(\xi) - \cancel{\frac{1}{\xi^2}\sigma'(1/\xi)} + \cancel{\frac{1}{\xi^2}\sigma'(1/\xi)} + \frac{1}{\xi^3}\sigma''(1/\xi).
\end{aligned} \tag{3}$$

Consequences:

1. $\mathfrak{s}(1) = 2\sigma(1) = 0$
2. $\mathfrak{s}'(1) = \sigma(1) = 0$
3. $\mathfrak{s}'' > 0 \quad \Rightarrow \quad \xi_m = 1$ is a minimum
4. $\mathfrak{s}(\xi) \geq 0$.

Entropic distance evolution

General P-Linear Discrete Markovian

Consider $\rho(P,Q) = \sum_n Q_n \, \mathfrak{s}\left(\frac{P_n}{Q_n}\right)$ and $\dot{P}_n = \sum_m (w_{nm} P_m - w_{mn} P_n)$.

Using $\xi_n = P_n / Q_n$ we obtain

$$\dot{\rho} \; = \; \sum_n \mathfrak{s}'(\xi_n) \, \dot{P}_n \; = \; \sum_{n,m} \mathfrak{s}'(\xi_n) \, \left(w_{nm} \, \xi_m Q_m - w_{mn} \, \xi_n Q_n \right) . \tag{4}$$

Apply $\xi_m = \xi_n + (\xi_m - \xi_n)$ to get

$$\begin{aligned} \dot{\rho} \quad &= \quad \sum_n \mathfrak{s}'(\xi_n) \, \xi_n \cancel{\sum_m \left(w_{nm} \, Q_m - w_{mn} \, Q_n \right)} \\ &+ \quad \sum_{n,m} \mathfrak{s}'(\xi_n)(\xi_m - \xi_n) w_{nm} \, Q_m . \end{aligned} \tag{5}$$

Entropic distance evolution

Taylor series remainder theorem in Lagrange form

Recall the Taylor expansion of the kernel function $\mathfrak{s}(\xi)$,

$$\mathfrak{s}(\xi_m) = \mathfrak{s}(\xi_n) + \boxed{\mathfrak{s}'(\xi_n)(\xi_m - \xi_n)} + \frac{1}{2}\mathfrak{s}''(c_{mn})(\xi_n - \xi_m)^2, \tag{6}$$

with $c_{mn} \in [\xi_m, \xi_n]$.

It delivers

$$\dot{\rho} = \cancel{\sum_{n,m} \left[\mathfrak{s}(\xi_m) - \mathfrak{s}(\xi_n)\right]} w_{nm} Q_m - \frac{1}{2}\sum_{n,m} \mathfrak{s}''(c_{mn})\left(\xi_m - \xi_n\right)^2 w_{nm} Q_m. \tag{7}$$

With positive transition rates, $w_{nm} > 0$ the approach to stationary distribution,

$\dot{\rho} \leq 0$ is hence proven for all $\mathfrak{s}'' > 0$.

Without detailed balance

Example: Kullback–Leibler divergence

In case of $\mathfrak{s}(\xi) = -\ln\xi$, we have $\mathfrak{s}' = -1/\xi$ and $\mathfrak{s}''(\xi) = 1/\xi^2 > 0$.

The integrated entropic divergence formula (no symmetrization) in this case is given as

Kullback–Leibler divergence ☺

$$\rho(P,Q) \;=\; \sum_n Q_n \ln \frac{Q_n}{P_n}. \tag{8}$$

For $P_n^{(12)} = P_n^{(1)}P_n^{(2)}$ also $Q_n^{(12)} = Q_n^{(1)}Q_n^{(2)}$ therefore we have $\xi_n^{(12)} = \xi_n^{(1)}\xi_n^{(2)}$. Aiming at $\mathfrak{s}(\xi^{(12)}) = \mathfrak{s}(\xi^{(1)}) + \mathfrak{s}(\xi^{(2)})$, the solution is $\mathfrak{s}(\xi) = \alpha\ln\xi$. For $\mathfrak{s}'' > 0$ it must be $\alpha < 0$, so o.B.d.A. $\alpha = -1$.

Entropic divergence as entropy difference

Example: logarithm

Entropic divergence from the uniform distribution $U_n = 1/W, n = 1, 2, \ldots W$:

Kullback–Leibler divergence ☺

$$\rho(U,Q) \;=\; \sum_{n=1}^{W} Q_n \ln(WQ_n) \;=\; \ln W - \sum_n Q_n \ln Q_n \;=\; S_{BG}[U] - S_{BG}[Q] \quad (9)$$

with S_{BG} being the Boltzmann–Gibbs–Planck–Shannon entropy formula.

From the Jensen inequality it follows $\rho(U,Q) \geq 0$, so $S_{BG}[U] \geq S_{BG}[Q]$.

Entropic evolution

More general dynamics: P-nonlinear Markovian

Dynamical equation

$$\dot{P}_n = \sum_m \left[w_{nm}\, a(P_m) - w_{mn}\, a(P_n) \right]. \tag{10}$$

Stationarity condition

$$0 = \sum_m \left[w_{nm}\, a(Q_m) - w_{mn}\, a(Q_n) \right]. \tag{11}$$

Entropic distance formula

$$\rho(P,Q) = \sum_n \sigma(P_n, Q_n) \tag{12}$$

the dependence on Q_n can be fixed from $\rho(Q,Q) = 0$.

Without detailed balance

Change of entropic distance

$$\dot{\rho} = \sum_{m,n} \frac{\partial \sigma}{\partial P_n} \left[w_{nm}\, a(Q_m) \xi_m - w_{mn}\, a(Q_n) \xi_n \right] \tag{13}$$

with $\xi_n := a(P_n)/a(Q_n)$.

We put $\xi_m = \xi_n + (\xi_m - \xi_n)$ in the first summand:

$$\dot{\rho} = \sum_n \frac{\partial \sigma}{\partial P_n} \xi_n \cancel{\sum_m \left[w_{nm}\, a(Q_m) - w_{mn}\, a(Q_n) \right]} + \sum_{n,m} \frac{\partial \sigma}{\partial P_n} w_{nm}\, a(Q_m) \left(\xi_m - \xi_n \right) \tag{14}$$

In order to use the remainder theorem one has to identify ☺

$$\frac{\partial \sigma}{\partial P_n} = \mathfrak{s}'(\xi_n) = \mathfrak{s}' \left(\frac{a(P_n)}{a(Q_n)} \right). \tag{15}$$

then $\dot{\rho} < 0$ for $\mathfrak{s}'' > 0$ and $P \neq Q$.

Without detailed balance

Example: q–Kullback–Leibler divergence

In case of $\mathfrak{s}(\xi) = -\ln \xi$, we have $\mathfrak{s}''(\xi) = 1/\xi^2 > 0$.

Now having a fractal nonlinear stohastic dynamics, $a(P) = P^\lambda$.

The integrated entropic divergence formula (no symmetrization):

Tsallis divergence, ☺

$$\frac{\partial \sigma}{\partial P_n} = -\frac{Q_n^\lambda}{P_n^\lambda}, \qquad \rho(P,Q) = \sum_n Q_n \ln_\lambda \frac{Q_n}{P_n}. \tag{16}$$

with

$$\ln_\lambda(x) = \frac{1-x^{\lambda-1}}{1-\lambda}. \tag{17}$$

Without detailed balance

Example: q–Kullback–Leibler divergence

In case of $\mathfrak{s}(x) = -\ln_\nu(x)$, we have $\mathfrak{s}'(x) = -x^{-\nu}$, $\mathfrak{s}''(x) = \nu x^{-\nu-1} > 0$.

Also having a fractal nonlinear stohastic dynamics, $a(P) = P^\lambda$.

The integrated entropic divergence formula (no symmetrization) becomes

Tsallis divergence, $q = \lambda\nu$ ☺

$$\rho(P,Q) = \sum_n \frac{Q_n}{1-q}\left[1 - \left(\frac{P_n}{Q_n}\right)^{1-q}\right] = \sum_n Q_n \ln_q \frac{Q_n}{P_n}. \tag{18}$$

Entropic divergence as entropy difference

Example: q-logarithm

Entropic divergence from the uniform distribution $U_n = 1/W, n = 1, 2, \ldots W$:

$$\rho(U,Q) \;=\; \sum_{n=1}^{W} \frac{Q_n}{1-q} \left[1 - (WQ_n)^{q-1} \right] \;=\; W^{q-1} \left(S_T[U] - S_T[Q] \right). \qquad (19)$$

with S_T being the Tsallis entropy formula:

Tsallis entropy, $q = \lambda\nu$ ☺

$$\boldsymbol{S_T[Q] \;=\; \frac{1}{1-q} \sum_n (Q_n^q - Q_n) \;=\; -\sum_n Q_n \ln_q(Q_n).} \qquad (20)$$

From the Jensen inequality it follows $\rho(U,Q) \geq 0$, so $S_T[U] \geq S_T[Q]$. The factor W^{q-1} signifies non-extensivity.

Schemes of Master Equations

Balanced vs One-Sided Growth

Symmetric Short Jumps: Drift + Diffusion

$$w_{nm} = \lambda_m \delta_{n+1,m} + \mu_m \delta_{n-1,m}$$

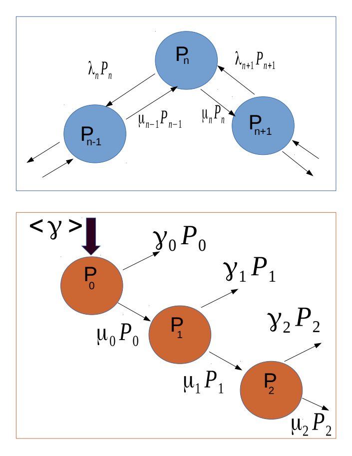
$$\begin{aligned} \dot{P}_n &= [(\lambda P)_{n+1} - (\lambda P)_n] \\ &- [(\mu P)_n - (\mu P)_{n-1}] \end{aligned} \tag{21}$$

Unidirectional + Resetting

$$w_{nm} = \mu_m \delta_{n-1,m} + \gamma_m \delta_{n,0}$$

$\dot{P}_0 = \langle \gamma \rangle - (\gamma_0 + \mu_0) P_0$ and ☺

$$\dot{P}_n = \mu_{n-1} P_{n-1} - (\mu_n + \gamma_n)\, P_n \tag{22}$$



Unidirectional

sources of step-up, μ_n rates

- Propagation (hopping) on a chain: $\mu_n = \text{const.}$
- Rich gets richer: $\mu_n \propto (n + b)$
- Cumulative effect: $\mu_n \propto \sum_{i=1}^{n} \mathrm{const} \propto n$
- Cancer growth: $\mu_n \propto \exp(n)$.

Resetting

sources of γ_n rates

- Loosing all money when having n: $\gamma_n = \text{const.}$
- Exponential dilution of sample space (cf. citations):

$$\frac{\mathrm{d}N_n}{\mathrm{d}t} = \mu_{n-1}N_{n-1} - \mu_n N_n;$$

with $P_n(t) = N_n(t)/N(t)$, $N = \sum_n N_n$ and $\gamma_n = \dot{N}/N =$ const.

- Independent decay rate of all units: $\gamma_n \propto n$
- Evolutionary resets in number of species due to catastrophes: $\gamma \to 0^+$.

Short step-up + long hops to zero:

stationary distribution

Stationary limit: $P_n(t) \to Q_n$, from $\dot{Q}_n = 0$ one obtains

$Q_0 = \langle\gamma\rangle_Q / (\gamma_0 + \mu_0)$ and stationary ☺

$$Q_n = \frac{\mu_{n-1}}{\mu_n + \gamma_n} Q_{n-1} = \cdots = \frac{\mu_0 Q_0}{\mu_n} \prod_{j=1}^{n} \left(1 + \frac{\gamma_j}{\mu_j}\right)^{-1}. \quad (23)$$

Constant rates

$\rightarrow$ exponential distribution

Assume $\mu_j = \sigma$, attachment rate independent of number of links.

$$Q_n = Q_0 \prod_{j=1}^{n} \frac{\sigma}{\sigma + \gamma} = Q_0 \left(1 + \gamma/\sigma\right)^{-n}. \tag{24}$$

Geometrical sum for normalization. We obtain

Boltzmann–Gibbs exponential ☺

$$\boldsymbol{Q_n = \frac{1}{1 + \sigma/\gamma} e^{-n \cdot \ln(1+\gamma/\sigma)}}. \tag{25}$$

Linear preference, constant loss rate

$\rightarrow$ Waring distribution

Linear preference in attachment: $\mu_j = \sigma(j+b) \quad (b>0)$.

$$Q_n = Q_0 \prod_{j=1}^{n} \frac{j-1+b}{j+b+\gamma/\sigma} = Q_0 \frac{(b)_n}{(c)_n}. \qquad (26)$$

with $c = b+1+\gamma/\sigma$. Norm:
$\sum_n Q_n = Q_0\,(c-1)/(c-1-b) = 1$.

Pochhammer ratio (Waring) ☺

$$Q_n = \frac{c-1-b}{c-1} \frac{(b)_n}{(c)_n} \qquad (27)$$

Matthias principle: tail of Waring

$\rightarrow$ power-law!

The above result in the $n \to \infty$ limit:
Since

$$\lim_{n\to\infty} n^{c-b} \frac{\Gamma(n+b)}{\Gamma(n+c)} = 1, \tag{28}$$

we obtain

Pochhammer in $n \to \infty$ limit: power-law! ☺

$$Q_n \to \frac{\gamma}{\gamma + b\sigma} \frac{\Gamma(c)}{\Gamma(b)} n^{-1-\gamma/\sigma}. \tag{29}$$

Avalanche dynamics in the large n limit!

continuous variable: $x = n \cdot \Delta x$

- $P_n(t) = \Delta x \cdot P(n \cdot \Delta x, t)$ ensures $\sum_{n=0}^{\infty} P_n(t) = \int_0^{\infty} P(x,t)dx$.
- $\mu_n = \frac{1}{\Delta x} \cdot \mu(n \cdot \Delta x)$ and $\gamma_n = \gamma(n \cdot \Delta x)$ lead to

Continuum Master:

$$\frac{\partial}{\partial t} P(x,t) = -\frac{\partial}{\partial x}\left(\mu(x)\, P(x,t)\right) - \gamma(x) P(x,t). \tag{30}$$

with the stationary distribution

$$Q(x) = \frac{K}{\mu(x)}\, e^{-\int_0^x \frac{\gamma(u)}{\mu(u)}\, du}. \tag{31}$$

Particular continuous stationary distributions

with constant $\gamma(x) = \gamma$.

For constant rate $\mu(x) = \sigma$ **exponential**:

$$Q(x) = \frac{\gamma}{\sigma} \mathrm{e}^{-\frac{\gamma}{\sigma} x}. \tag{32}$$

For linear preference $\mu(x) = \sigma(x+b)$ **Tsallis–Pareto**:

$$Q(x) = \frac{\gamma}{\sigma \, b} \left(1 + \frac{x}{b}\right)^{-1-\gamma/\sigma}. \tag{33}$$

For exponential dispreference $\mu(x) = \sigma \mathrm{e}^{-ax}$ **Gompertz**

$$Q(x) = \frac{\gamma}{\sigma} \mathrm{e}^{ax + \frac{\gamma}{a\sigma}(1 - \mathrm{e}^{ax})}. \tag{34}$$

Fluctuation – Dissipation

Conituous

Knowing / observing $Q(x)$ and $\gamma(x)$ one obtains

$$\mu(x) = \frac{1}{Q(x)} \int_x^\infty \gamma(u) Q(u)\, du = \langle \gamma \rangle_{\rm cut} \,. \tag{35}$$

Analogy: multiplicative noise

Langevin: $\dot{p} + (\gamma p - \xi) = 0$; stochastic properties: $\langle \gamma p - \xi \rangle = K_1(p)$ and $\langle (\gamma p - \xi)(\gamma p - \xi)' \rangle = K_2(p)$.

Then the Fokker-Planck, $\frac{\partial f}{\partial t} = \frac{\partial}{\partial p}(K_1 f) + \frac{\partial^2}{\partial p^2}(K_2 f) = 0$ has the detailed balance distribution

$$Q(p) = \frac{K}{K_2(p)} \, e^{-\int_0^p \frac{K_1(q)}{K_2(q)} dq} \,. \tag{36}$$

The **Fluctuation–dissipation** theorem has the form

$$K_2(p) = \frac{1}{Q(p)} \int_p^\infty K_1(q) Q(q)\, dq. \tag{37}$$

Fluctuation – Dissipation

Discrete

Summing up the recursion from $n = m + 1$ to ∞ delivers

$$\mu_n = \frac{1}{Q_n} \sum_{m=n+1}^{\infty} \gamma_m Q_m. \tag{38}$$

Kubo formula: apply the above to constant γ and exponential distribution $Q_n = \mathrm{e}^{-\beta\omega n}/Z$.

$$\mu_n = \frac{\gamma}{\mathrm{e}^{\beta\omega} - 1}, \qquad \mu(x) = \frac{\gamma}{\beta\omega} \tag{39}$$

Rate, Survival, Hazard

Connection to failure probability

Fluctuation-Dissipation vs. rate reconstruction vs. hazard

Cumulative hazard	$H(x)$
hazard (rate)	$h(x) = H'(x)$
PDF	$Q(x) = h(x)\, \mathrm{e}^{-H(x)}$
Survival (rate)	$R(x) = \int\limits_x^\infty Q(u)\, du = \mathrm{e}^{-H(x)}$

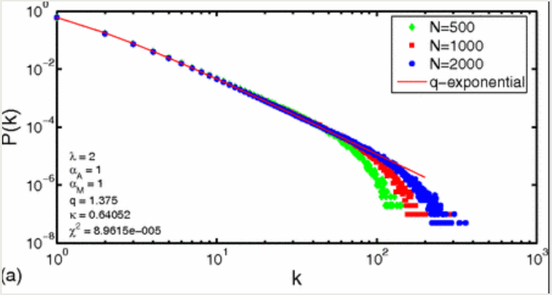
For $\gamma(x) = \gamma$ constant

$$\mu(x) = \gamma \frac{R(x)}{Q(x)} = \frac{\gamma}{h(x)}. \tag{40}$$

Networks: degree distribution

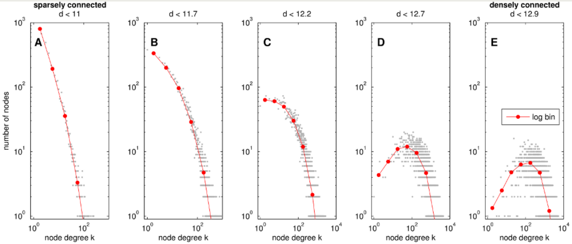
S.Thurner, F.Kyriakopoulos, C.Tsallis, PRE 76 (2007) 036111

- $\mu_n = \sigma(n+b)$ or $\mu(x) = \sigma(x+b)$: Matthew principle for adding the next connection to a node with n
- $\gamma_n = \gamma$: attack success ratio against a node with n connections
- $Q(x) \sim (1+x/b)^{-\gamma/\sigma-1}$: stationary degree distribution (q-exponential)



Networks: degree distribution 2

M.Sholz, J.DataMining & Digital Humanities, 2015



With $\gamma(x) = \gamma \ln(x/a)$, $\mu(x) = \sigma x$ we get **log normal**

$$Q(x)dx \; = \; K e^{-\frac{\gamma}{2\sigma} t^2} \, dt \qquad \text{with} \qquad t = \ln(x/a). \tag{41}$$

Citations

Total number and fraction dynamics

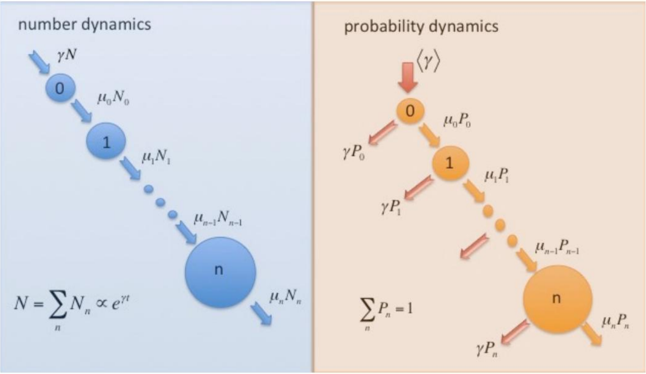


Figure 3. Schematic representation of the coarse-grained random growth model considered in the model. The panel on the left side indicates the growth process in the number of elements with n quanta: N_n. Due to the fact that the total number of elements is exponentially increasing, the probability P_n that an

Citations

Exponential growth

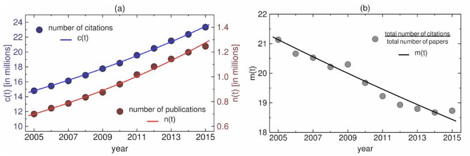


Figure 2. Results for the MEDLINE/PubMed database. **Figure 2a** illustrates the time evolution of the yearly indexed papers, $n(t)$, and the total number of citations, $c(t)$, introduced by them for each year in the 2005-2015 time interval. The trend $n(t)$ can be nicely fitted (red curve) with an exponential curve with $\gamma = 0.06$ using $t_0 = 2005$ and $n_0 = 699915$. Using $t_0 = 2005$, $n_0 = 699915$, $c_0 = 14792864$, $g = 1.4$ and $\gamma = 0.06$ ($\sigma = \gamma / g \approx 0.043$) the trend for $c(t)$ given by equation (2.3) can be fitted by choosing $b \approx 1.6$. **Figure 2b** illustrates the time evolution for the yearly incoming total number of citations divided by the total number of new papers, $m(t)$. Using the parameters from $n(t)$ and $c(t)$ the $m(t)$ trend given by equation (2.1) is plotted by the black curve.

Citations

Fraction of *n* times cited: Facebook and Web of Science

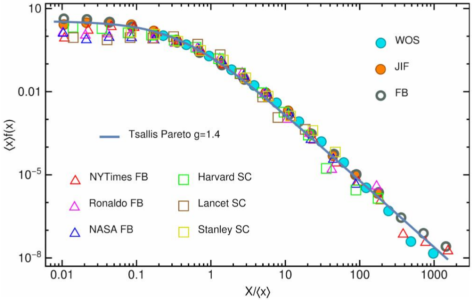


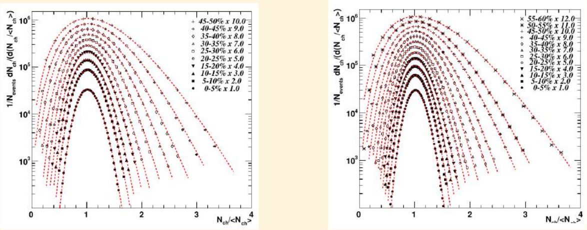
Figure 1. Rescaled distribution of the citation (share) numbers. $f(x)$ is the probability density (PDF) for one paper (post) to have x citations/shares. We present the $\langle x\rangle \cdot f(x)$ value as a function of $x/\langle x\rangle$ ($\langle x\rangle$ the mean value, or first moment of the PDF). For high citation number a clear power-law trend is visible. Different symbols are for different datasets as illustrated in the legend. The considered datasets are described in the Methods section. For high $x/\langle x\rangle$ a clear power-law trend is visible. The entire curve can be well-fitted with a TP distribution (1) with $g \approx 1.4$.

Hadronization

From QGP to *n* hadrons: NBD

PHENIX, PRC 78 (2008) 044902

Au + Au collisons at $\sqrt{s_{NN}} = 62$ (left) and 200 GeV (right). Total charged multiplicities.

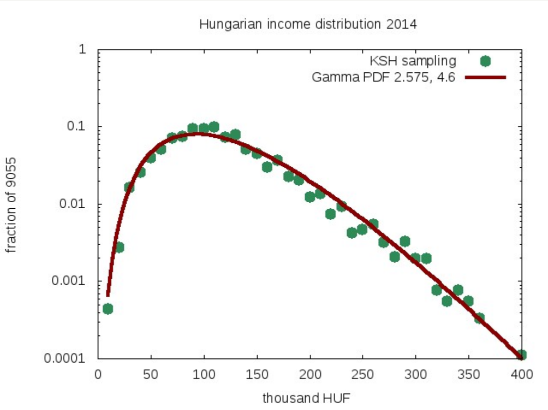


$$\gamma_n = \sigma(n - kf),\ \mu_n = \sigma f(n+k); \qquad Q_n = \binom{n+k-1}{n} f^n (1+f)^{-n-k}.$$

Economy

Income distribution

KSH data



$$\gamma(x) = \sigma(ax - c), \quad \mu(x) = \sigma x \qquad Q(x) = \frac{a^c}{\Gamma(c)} x^{c-1} e^{-ax}.$$

History

Medieval Servant Distribution G.Hegyi, Z.Néda, M.A.Santos, Physica A 380 (2007) 271

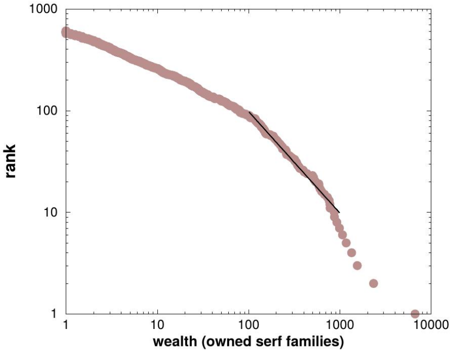


Fig. 2. The rank of noble families and institutions as a function of their estimated total wealth on a log-log scale. Data for the Hungarian noble society between the years 1767-1773. The total wealth of a family is estimated in the number of owned serf families. The power-law fit suggests a Pareto index $\alpha = 0.99$.

Summary of Rates and PDF-s

$\mu(x) = \gamma / h(x)$ at constant aging γ

$\gamma_n,\ \gamma(x)$	$\mu_n,\ \mu(x)$	$Q_n,\ Q(x)$
const	const	geometrical $\rightarrow$ exponential
const	linear	Waring $\rightarrow$ Tsallis/Pareto
const	sublinear power	Weibull
const	quadratic polynomial	Pearson
const	exp	Gompertz
$\ln(x/a)$	σx	Log-Normal
linear	const	Gauss
$\sigma(ax - c)$	σx	Gamma

Deviation shrinks and moves as a soliton: $\dot{x}_c = \mu(x_c)$!

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