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Uniform equivalence of two linear nonautonomous scalar differential equations

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In 1973 a short note by L.M. Lerman, L.P. Shilnikov was published where for nonautonomous vector fields, given on a closed smooth manifold M, a definition of equivalency of two such vector fields was formulated and on this base the structural stability of nonautonomous vector fields was defined. This allowed for the case dim M = 2 to distinguish a class of structurally stable nonautonomous vector fields, for which the invariant determining the uniform equivalence was presented. The Morse type inequalities connecting the topology of M and the set of its integral curves were derived.

The idea itself on the necessity of extending the notion of roughness (structural stability) onto nonautonomous vector fields goes back to A.A. Andronov and was publicized by his wife and collaborator E.A. Leontovich-Andronova in her talk at the III All-Union USSR Mathematical Congress.

Recall some notions. Let M be a C^∞- smooth closed manifold, $V^r(M)$ be the Banach space of C^r -smooth vector fields on M endowed with C^r-norm.

A C^r -smooth nonautonomous vector field on M is an uniformly continuous bounded map **v**: $R \to V^r(M)$. If this map **v** is also uniformly continuous C^s -differentiable map, we call **v** to be a $C^{r,s}$ -smooth nonautonomous vector field.

Every nonautonomous vector field **v** (NVF) generates a foliation *F* of manifold $M \times R$ into its integral curves. We consider the manifold $M \times R$ with its standard uniform structure of the direct product.

Definition 1. A map h: $X \to Y$ of two metric spaces is called **uniformly continuous** if for any $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that for any two points $(x, y) \in X, d_1(x, y) < \delta$ one has $d_2\big(h(x), h(y)\big) < \varepsilon$.
Here d_1, d_2 are metrics in X, Y respectively.

In this talk we discuss some results for a one dimensional situation, that is when M is the circle S^1. This means that we study scalar differential equations of the type

$$\dot{x} = f(t,x), \qquad f(t,x+1) \equiv f(t,x) \tag{1}$$

Suppose some solution $\boldsymbol{\gamma(t)}$ of this equation possesses exponential dichotomy on the whole line R . This means as follows.

Definition 2. One says that the solution satisfies the exponential dichotomy of the **stable** type on the semi-axis $t \geq 0$ $(t \leq 0)$, if there are positive constants **C,** $\boldsymbol{\lambda}$ such that the linearized at this solution of equation $\boldsymbol{\dot{\xi}} = \boldsymbol{a}$**(t)·** $\boldsymbol{\xi}$, $\mathbf{a}(\boldsymbol{t}) = \boldsymbol{f}\big(\boldsymbol{t}, \boldsymbol{x}(\boldsymbol{t})\big)$ satisfies the inequality:

$$exp\,[\int_\tau^t a(s)ds] \leq C \cdot exp\big(-\lambda(t-\tau)\big)$$

for all t, τ, **t** $\geq \boldsymbol{\tau} \geq$ **0** $(\boldsymbol{\tau} \leq \mathbf{0}, \boldsymbol{t} \geq \boldsymbol{\tau})$

Definition 3. One says that this solution satisfies the exponential dichotomy of the **unstable** type on the semi-axis $t \geq 0$ $(t \leq 0)$, if there are positive constants C, $\boldsymbol{\lambda}$ such that the linearized at this solution of equation

$\dot{\xi} = \boldsymbol{a}(\mathbf{t})\cdot\,\xi,\ \boldsymbol{a}(\boldsymbol{t}) = \boldsymbol{f}_{\boldsymbol{x}}\big(\boldsymbol{t}, \boldsymbol{x}(\boldsymbol{t})\big)$ satisfies the inequality:

$$exp\left[\int_t^\tau a(s)ds\right] \leq C \cdot exp[\lambda(t-\tau)]$$

for all $t, \tau: 0 \leq t \leq \tau \ \ (\tau \leq 0, t \leq \tau)$

A particular question arises here: suppose we have two linear homogeneous differential equation and both of them of the same type of dichotomy (stable or unstable); whether these two equations are uniformly equivalent in some their uniform neighborhoods of zero solutions?

To demonstrate the method of constructing an equimorphism, we consider two scalar linear homogeneous differential equations:

$$\dot{x} = a(t) \cdot x(t), \qquad (2)$$

$$\dot{y} = b(t) \cdot y(t). \qquad (3)$$

Denote $u(t,\tau)$ the solution of (2) with the condition $u(\tau,\tau) = 1$, and let $u_1(t,\tau)$ be the similar solution for (3):

$$u(t,\tau) = \exp[\int_{\tau}^{t} a(s)ds], \; u_1(t,\tau) = \exp[\int_{\tau}^{t} b(s)ds].$$

We assume both these equations to satisfy the exponential dichotomy condition:

$$u(t,\tau) = \exp[\int_{\tau}^{t} a(s)ds] \le M\exp[-\lambda(t-\tau)];\ t \ge \tau,\ \lambda > 0,\ M > 0$$

$$u_1(t,\tau) = \exp[\int_{\tau}^{t} b(s)ds] \le M_1\exp[-\lambda_2(t-\tau)],\ t > \tau,\ \lambda_1 > 0,\ M_1 > 0. \tag{4}$$

Define Lyapunov functions S and S_1 for these equations:

$$S(t,x) = \int_t^{\infty} u^2(\xi,t)d\xi \cdot x^2 = s^2(t)x^2;$$

$$s^2(t) = \int_t^{\infty} u^2(\xi,t)d\xi = \int_t^{\infty} \exp[(2\int_t^{\xi} a(u)du)]d\xi$$

$$S_1(t,y) = \int_t^{\infty} u_1^2(\xi,t)d\xi \cdot y^2 = s_1^2(t) \cdot y^2;$$

$$s_1^2(t) = \int_t^{\infty} u_1^2(\xi,t)d\xi = \int_t^{\infty} \exp[(2\int_t^{\xi} b(u)du)]d\xi$$

(5)

These integrals converge and define positive bounded functions $s(t)$, $s_1(t)$. From expressions for u, u_1 the inequalities follow:

$$\frac{1}{2a_0} \leq S^2(t) \leq \frac{M_1}{2\lambda_1}$$

$$\frac{1}{2b_0} \leq S_1^2(t) \leq \frac{M_2}{2\lambda_2} \qquad (6)$$

where $a_0 = \sup_t |a(t)|,\ b_0 = \sup_t |b(t)|.$

Observe that functions $s(t), s_1(t)$ obey the equations:

$$\frac{ds}{dt} = \frac{-2a(t)s^2(t) - 1}{2s(t)}$$

$$\frac{ds_1}{dt} = \frac{-2b(t)s_1^2(t) - 1}{2s_1(t)} \qquad (7)$$

Define the strip D by the boundary curves:

$$x = 0, \quad x = \frac{C^*}{s(\tau)}, \quad C^* > 0 \tag{8}$$

Similar formulae define the strip D_1 with the change:

$$x \to y, \ C^* \to \mathrm{C}_1^*, \ s(\tau) \to s_1(\tau) \tag{9}$$

Our goal in this talk is to prove that the map from the strip D in the plane (x, t) onto the strip D_1 in the plane (y, t) is an equimorphism.

Instead of (x, t) we take as new coordinates (C, t), similarly, we change (y, t) to (C_1, t).

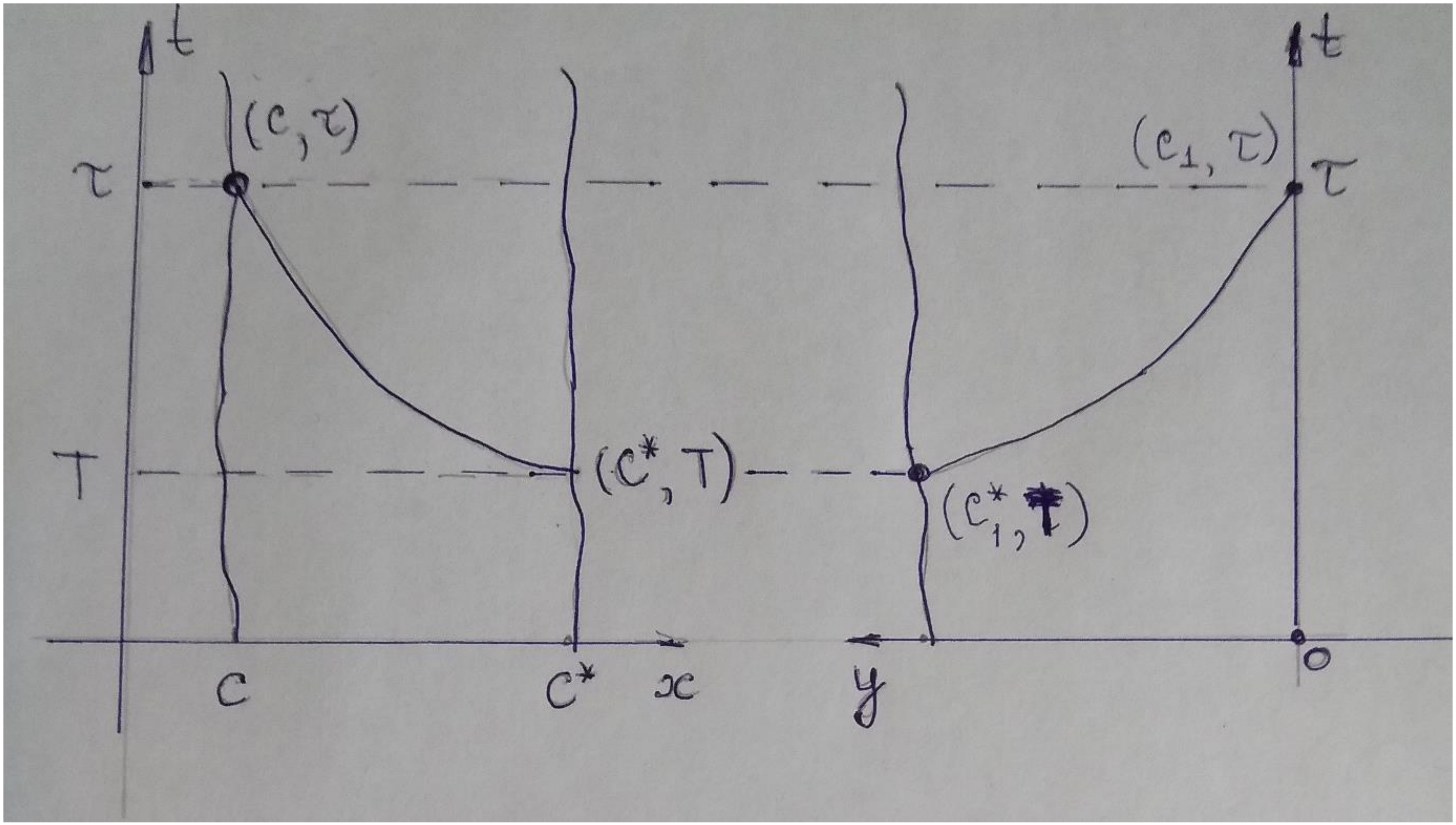
These transformations are smooth in both variables. Values C and C_1 define the respective lines of level.

Then the map $h: D \to D_1$ is given as follows.

Take some point $(C, \tau) \in D$ and consider the integral curve of the first equation through this point. This curve intersects transversely the level line defined by C^* at some time $T(C, \tau) < \tau$. We fix some smooth monotone function $\alpha: R \to R$ with positive uniformly bounded from above and from zero first derivatives: $0 < r_0 \le \alpha' \le r_1$

and put $T_1 = \alpha(T)$. For instance, as a such map one can take the identical map $\alpha(T) = T$.

Then the point on the boundary of D with coordinates $(C^*, T(C, \tau))$ transforms to the point $\left(C_1^*, T(C, \tau)\right)$ on the boundary curve of D_1. On the integral curve $y = y(t)$ of the second equation through this point we choose that point whose t-coordinate is τ. C_1-coordinate of this point is $C_1 = y(\tau) \cdot S_1(\tau)$.


t
(c, τ)
τ
T
(c*, T)
(c1, τ)
(c1*, T)
c
c*
x
y
0

Integral curve through the point $(C = S(\tau) \cdot x, \tau)$ is

$$x(t) = x_0 \exp[\int_{\tau}^{t} a(u)du] = \frac{C}{s(\tau)} \exp[\int_{\tau}^{t} a(u)du], \ x_0 > 0. \tag{10}$$

This curve intersects the boundary level line $S(T)x = C^*$ at the point with coordinates (C^*, T), where T is found from the equation:

$$Cs(T) \exp[\int_{\tau}^{T} a(u)du] - C^* s(\tau) = 0. \tag{11}$$

Denote $R(T, C, \tau)$ the left hand side of this equation. Let function $T(C, \tau)$ be a solution of the equation $R(T, C, \tau) = 0$. Integral curve $(y(t), t)$ of the second equation through the point (C_1^*, T) is given as

$$y(t) = \frac{C_1^* \exp[\int_{T}^{t} b(u)du]}{s_1(T)}. \tag{12}$$

For the point $h(C,\tau)$ the time variable on this curve is $\tau_1 = \tau$.

C_1 -coordinate of this point is found from the equation

$$C_1 = \frac{C_1^* s_1(\tau) \exp[\int_T^\tau b(u)du]}{s_1(T)}. \qquad (13)$$

The right hand side of the equation will be denoted as $g(T,\tau)$.

$$C_1 = g(T(C,\tau),\tau), \ \tau_1 = \tau. \qquad (14)$$

We need to prove that the homeomorphism (14) is in fact equimorphism, that is, it is uniformly continuous along with its inversed one. The problem here is that the homeomorphism is defined for the strip $C > 0$ and when approaching the line $C = 0$, the derivatives of this map tend to infinity.

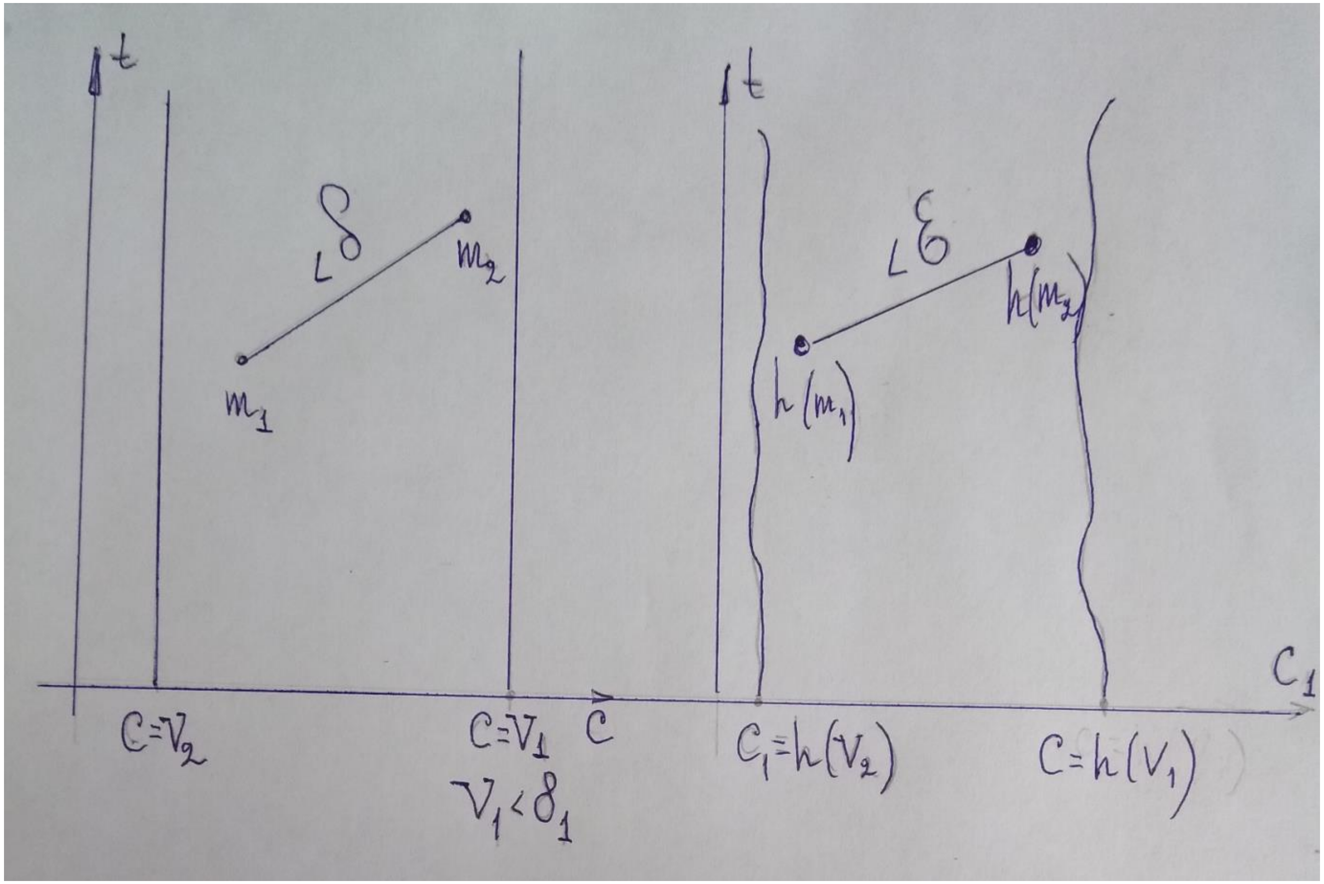
We cannot prove the uniform continuity using estimates for derivatives. This forces us to break the proof into **two** steps. We will prove the uniform continuity for the direct map **h**.

First, given $\varepsilon > 0$ we choose a line $C = v_1$ such that any two points m_1, m_2 within the strip $0 < C < \delta_1$, for which related τ_1 , τ_2 satisfying the estimate

$$|\tau_1 - \tau_2| < \frac{\delta_1}{2}, \quad |\mathrm{c}_1 - \mathrm{c}_2| < \frac{\delta_1}{2} \qquad (15)$$

obey the estimate $d\big(h(m_1), h(m_2)\big) < \varepsilon$.

After that we take the line $C = v_2$, $v_2 < v_1$, and for its image in the second strip prove that this image is a curve given by the graph of the function $C_1 = s(\tau)y$ for which s is bounded away from zero and is uniformly bounded. Thus we get a cover of the second strip by two its sub-strips which boundaries are defined by the *h*-images of lines $C = v_1$ и $C = v_2$.


t
δ
m_2
m_1
$C=V_2$
$C=V_1$
C
$V_1<\delta_1$
t
ε
$h(m_2)$
$h(m_1)$
C_1
$C_1=h(V_2)$
$C=h(V_1)$

It is proved that this cover has a positive Lebesque number. This implies that there is some $\delta(\varepsilon)$ such that any two points m_1, m_2 in the first strip being δ -close, are transformed by ***h*** into ε-close points $h(m_1)$, $h(m_2)$ in the second strip.
In the similar way, the proof of uniform continuity for the inverse map is done.

So, it is proved that the map from the strip D of the plane (x, t) to the strip D_1 of the plane (y, t) is an **equimorphism,** and that two scalar linear homogeneous differential equations are **equivalent.**

Formulas for derivatives are presented. From these formulas it is evidently that derivatives are bounded.

$$\frac{\partial g}{\partial C} = \frac{C_1^* S_1\big(\alpha(\tau)\big) S^2(T) \cdot EI\alpha}{C \cdot S_1^5\big(\alpha(T)\big)}$$

$$EI\alpha = exp\left(\int_{\tau}^{T} a(u)du \right)$$

$$\frac{\partial g}{\partial \tau} = \frac{C_1^* EI\alpha}{2S_1^3\big(\alpha(T)\big) S_1\big(\alpha(\tau)\big) S^2(\tau)} \left[S_1^2\big(\alpha(\tau)\big) \cdot \alpha'(T) \cdot S^2(T) - \alpha'(\tau) \cdot S^2(\tau) \cdot \alpha'(T) \cdot S^2(\tau) \right]$$

Thank you for attention!