How come Gauss beside Bólyai and Lobachewski,

or else The Theorema Egregium.

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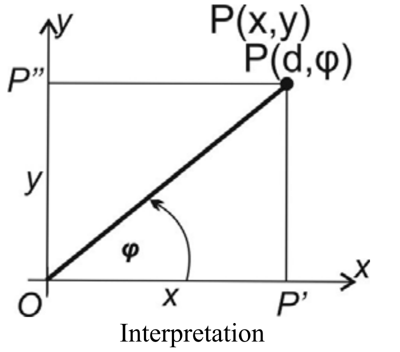
Abstract: To understand Gauss's theory of the non-Euclidean geometry we have to reestablish some definitions of the coordinate system, and also introduce the so called Gaussian coordinates. We show here that the two points distance as a postulate can establish a metric geometry. If we able to show the validity of this postulate on any surface than its have his own geometry, not necessarily Euclidean. Gauss showed in The Theorema Egregium that a surface have such attributes. The different geometries of the regular surfaces written here: Euclidean, spherical, hyperbolic. This theorem has been presented at 1827.
(Based on the lectures of K. Lanczos: Department of Physical Sciences and Applied Mathematics, North Carolina State University, Raleigh, 1968.)

I. ANTECEDENTS.

1. A postulate of the coordinate system establish the metric geometry

The Cartesian coordinate system applicable for the full Euclidean geometry and every points are metric: the distance between any two points can be determine in algebraic way. In other words, the Cartesian coordinate system and all of its correct conversion metric space, and exist such a portion which is fully met with the Euclidean geometry. Gauss showed that the full geometry can be constructed only one postulate.



Interpretation

These is the distance of two points A,*B*: $s^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$.
Gauss showed that the Euclidean geometry can be deduced from this postulate.

We study here only the structure and validity of the coordinate system based on of this postulate:

Axes: Straight lines, which intersect each other at the origin perpendicular in pairs. Such line for example, the number line, the set of real numbers. Two perpendicular axes form a plane, which privileged point is the origin, *O*, than we call respectively, *x axis* or abscissa and *y-axis* or ordinate.

Coordinates: Distances measured from the axes, in geometric terms a perpendicular projection of the point *P* to the axes. The distance is the length of this section.

The point now a pair: $P(x, y) \quad \left| x = \overline{PP''} = \overline{OP'}; y = \overline{PP'} = \overline{OP''}\right.$

The relationship between the pair and the point mutual we can define *x and y* to a point *P*, or vice versa get the *P* point from *x and y*.

Consequences:

1. We may replace all geometric constructions with algebraic operations.
2. We may replace any algebraic operation for an *(x, y)* to geometric construction.

Angle: Is the direction of the straight line through *O* and *P* points, what we usually measure from the *x*-axis. Geometrically the inclination of two lines lay both on origin, i.e. the angle close between of them. Algebraically the ratio of the *P* coordinates: $tg\alpha = \frac{y}{x} \rightarrow \alpha = arctg\frac{y}{x}$ also called *tangent*, whereas *POP '* is a right triangle (the angle at the origin).

Straight line: We can get the straight-line equation if we use $\frac{y}{x} = tg\alpha = m$ for a point *P* lay on a line: $y = mx$, which mean that any solution of this equation – the*(x,y)* pairs - is on the line. If the line not cross the origin then the equation altered to: $y = mx + y_0$, where y_0 the cross point with the *y* axis, or algebraically the equation's solution for *x=0*.

We show here that the algebraic expression $Ax + By + C = 0$ - linear equation with the two unknown quantities - all possible solution's lay on a straight line and also describe all its points. The $y = mx + y_0$ straight line lays on two points: $(0, y_0)$ and $(x_0, 0)$ which are intersects of the line and the axes. So from the two equation $m = tg\frac{y_0}{x_0}$ and $y = \frac{y_0}{x_0}x + y_0$, with common denominator $y = \frac{y_0 x + y_0 x_0}{x_0}$, and reduce to zero we get: $y_0 x - x_0 y + x_0 y_0 = 0$. Use this changes: $A = y_0; B = -x_0; C = x_0 y_0$ (see the meaning of the negative value on the figure: if $y_0 > 0$, then $x_0 < 0$ and vice versa). Thus we showed that the $Ax + By + C = 0$ expression is the coordinate-geometry form of the straight line. Any points on the line are solution of the equation and only those.

Circle: The definition of the circle immediately gives its equation: a geometric location of all points which lay in the same distance from a common point. So, using the distance postulate: $r^2 = (x - x_0)^2 + (y - y_0)^2$, where *r* the radius and (x_0, y_0) is the origin.

Arc: A piece of the circle line. The angle of the full circle is *2π*, and the circumference of a circle is *2rπ*, then proportionally the $\overset{\frown}{AB}$ arc has an angle φ_{AB} and so $\varphi = \frac{\overset{\frown}{AB}}{r}$, and its length $\overset{\frown}{AB} = \varphi r$.

Now we have shown that the basic elements of Euclidean geometry, fully reveal in a Cartesian coordinate system.

Infinitesimal distance: for the sake of generalization of the space concept, we satisfy to study the immediate surroundings of a point, so we interpret the distance in between (x, y) and $(x + \Delta x, y + \Delta y)$ where the Δ can be any small size, i.e. infinitesimal. Then using our only postulate, the distance: $ds^2 = dx^2 + dy^2$, because $x + dx - x = dx$ and $y + dy - y = dy$.

The derivate of a function $y = f(x)$ is: $f'(x) = \frac{dy}{dx} \rightarrow dy = f'(x)dx \rightarrow$ $dy^2 = f'^2(x)dx^2$ then from these $ds^2 = dx^2 + f'^2(x)dx^2$.We get the distance by integration in between the two points: $s = \int_A^B \sqrt{1 + f'^2(x)}dx$ if *s* minimal. Leaving the details of the reduction – requires variation computing - we arrive to the line equation: $Ax + By + C = 0$, which means, the smallest distance between two points is a straight line segment. This refer to our geometric attitude. So we proved that the postulate valid in infinitesimal environment also.

The intersection of two lines: Very interesting task to find the intersection of two lines.
Let's have these two lines:
$e_1 : a_1x + b_1y + c_1 = 0$ and $e_2 : a_2x + b_2y + c_2 = 0$.
The *P* interception point is, obviously, the common solution of the two equations, the *(x, y)* value that satisfies both equations. Leaving the reduction out we get: $x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}$ and $y = \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}$.
Consequently the two lines are always intersect each other if the determinant non-zero: $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$. if it where still zero $a_1b_2 - a_2b_1 = 0$ then $\frac{a_1}{a_2} = \frac{b_1}{b_2}$, which means the tangents are equal because $m_1 = \frac{a_1}{a_2}$ and $m_2 = \frac{b_1}{b_2}$. Thus the two line are parallel i.e. no common points.

2. *Curve line coordinates*

We come to beautiful results and then we continue with our imaginations and we assume that the lines are replaced by arbitrary curves like this: *x = f (t)* and *y = g (t),* where *f (t)* and *g (t)* are continuous function of variable *t*, and ought to be differentiable for infinitesimal use.
There have been examined many curves geometrically – usually each are special case - but we would like to arrive to a general solution on the algebraic way. It is possible as we did not attached any more condition, only the continuity and differentiability. Than in this way, we can determine the 'direction' of a curve and introduce the concept of 'curvature'.

The direction of a curve on any point is the gradient of the tangent line $ds^2 = dx^2 + dy^2 \to 0$ drawn to that point. The direction can be change from point to point: this we call curvature. Now, draw a circle through three points of the curve – the best fitting circle – then the distance from its origin will be proportional with the curvature at that point. If we determine these origins for all points we get another curve with ordinates: $\xi = \varphi(t)$, $\eta = \theta(t)$. This we call *evolute* of the original curve.

3. Gaussian coordinates

As we have seen the geometric problems can be translate to algebraic ones by using orthogonal (x, y) or the polar (r, φ) coordinates and their conversion: $x = r\cos\varphi$; $y = r\sin\varphi$. We also have seen the coordinate-lines dividing a plane to small quadrants.
Let's consider these generally according to Gauss, then we can introduce the following general relations: $x = x(u,v)$; $y = y(u,v)$ and again, they have to be continuous and differentiable in the studying environment and also a non-zero determinant: $\begin{vmatrix} \frac{\delta x}{\delta u} & \frac{\delta x}{\delta v} \\ \frac{\delta y}{\delta u} & \frac{\delta y}{\delta v} \end{vmatrix} \neq 0$. Note that this not met at *r=0*, the origin's environment.

The so introduced *(u, v)* pairs are uniquely define the points of a surface. These are referred to as Gaussian coordinates. The coordinate-lines drawn according to *(u, v)* also divides the plane into small quadrants

Now we have to show that the postulate also valid with the Gauss-coordinates. According to the determinant above: $dx = \frac{\delta x}{\delta u}du + \frac{\delta x}{\delta v}dv$; $dy = \frac{\delta y}{\delta u}du + \frac{\delta y}{\delta v}dv$ and by this the distance i.e. the postulate is

$$ds^2 = dx^2 + dy^2 = \left[\left(\frac{\delta x}{\delta u}\right)^2 + \left(\frac{\delta y}{\delta u}\right)^2\right]du^2 + \left[\left(\frac{\delta x}{\delta v}\right)^2 + \left(\frac{\delta y}{\delta v}\right)^2\right]dv^2 + 2\left(\frac{\delta x}{\delta u}\frac{\delta x}{\delta v} + \frac{\delta y}{\delta u}\frac{\delta y}{\delta v}\right)dudv$$

In case of polar coordinates the expression with the necessary reductions is: $ds^2 = dr^2 + r^2 d\varphi^2$. Now the arc, which is the shortest way between any two points *A, B* gives: $(a\cos\varphi)r + (b\sin\varphi)r + c = 0$. So this is the postulate!

Now we proved that in both, the Cartesian and the Gaussian coordinates - straight line, curve line or orthogonal and non-orthogonal – the distance postulate valid and describes the full geometry.
Now we may expand the here discussed two-dimensional space if the postulate unchangeably valid in the resulting space. We call these spaces *metric-space* according to our modern conceptions, regardless of the number of dimensions.

Many cases, we will be satisfied with, if the conditions apply only in an immediate surroundings of a point in the space. Conversely, if any points in the space have such environment by which a coordinate system can be interpreted, then the space is a *Euclidean topological space*.

4. *Gauss non-Euclidean idea*

Gauss came to an Interesting result, when he had to perform measurements in a hilly area. Provided two sets of curves intersecting each other mutually, like the coordinate-lines. These are the already known *(u, v)* pairs. Now, if we place them into a three-dimensional orthogonal coordinate system, than it is expressed in this way $x = x(u,v)$; $y = y(u,v)$; $z = z(u,v)$, and the arc:

$ds^2 = dx^2 + dy^2 + dz^2$.

Express the former with the latter:

$$dx = \frac{\delta x}{\delta u}du + \frac{\delta x}{\delta v}dv\,;\; dy = \frac{\delta y}{\delta u}du + \frac{\delta y}{\delta v}dv\,;\; dz = \frac{\delta z}{\delta u}du + \frac{\delta z}{\delta v}dv$$

and replace back to the arc expression, than we get $ds^2 = Edu^2 + 2Fdudv + Gdv^2$, where

$$E = \left(\frac{\delta x}{\delta u}\right)^2 + \left(\frac{\delta y}{\delta u}\right)^2 + \left(\frac{\delta z}{\delta u}\right)^2;$$

$$F = \frac{\delta x}{\delta u}\frac{\delta x}{\delta v} + \frac{\delta y}{\delta u}\frac{\delta y}{\delta v} + \frac{\delta z}{\delta u}\frac{\delta z}{\delta v};$$

$$G = \left(\frac{\delta x}{\delta v}\right)^2 + \left(\frac{\delta y}{\delta v}\right)^2 + \left(\frac{\delta z}{\delta v}\right)^2.$$

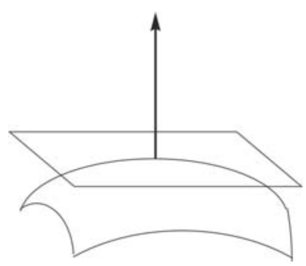
We come to an interest result, that the distance on the surface and the space, may have the same. Rather this points are on a curve on the surface and on a straight line segment in the space. However it can true infinitesimally, as the two points are arbitrarily close to each other. This ds^2 is a limit a common value for the space and the surface. We got a different geometry, the internal geometry of the surface, where this shortest lines are straight. And it shall remain valid as long as we stay on the surface.

Gauss showed that the internal geometry of a curve surface is uncontradictory and do not have be the subject of the Euclidean postulates. If this interface is an ellipsoid for example, than easy to understand that a triangle will incongruent for the move, either the sides or the angles will change. The consequences is that the space changes from point to point.

II. The Theorem

1. *The curvature*

In 1827 Gauss published the Disquisitiones generales circa superficies of the curves, - in English, General studies of the Curved Surfaces - and he signed it outstanding.



This writing defined the curvature as follows:

Let be given a surface, and construct the gradient of the tangent plane for a point *P*. Now we use a plane along the gradient, which will cut a plane-curve from the surface. If we move round this plane along the gradient as an axis, than in each step the cut plane-curve will be different, we get dissimilar radius of curvature. However the curvature will have once the maximum then the minimum value, in the extreme positions of the rotating plane: radius R_1 *and* R_2. Let's call the reciprocal as *curvature* and the extremes as *main curvature*: $k_1 = \frac{1}{R_1}$ and $k_2 = \frac{1}{R_2}$. Of course, this may not be an internal property of the surface, since the gradient is outside from the surface. Therefore, the curvature is available only from a space that contain the surface.

Consider the product of two *main curvature* $k = k_1 k_2 = \frac{1}{R_1 R_2}$.

Gauss came to the surprising conclusion, - which is not explained here, now – that the value of *k* can get from *(E), (F), (G).*
Then, this is notwithstanding an internal property of the surface, regardless of whether we defined externally. The value of k independent from the *(u, v)* coordinates since we constructed it with clear geometry. So *k*, the curvature, invariant in any Gaussian coordinate-system.

2. *The Theorema Egregium*

Now examine the different value of *k.*
In general cases, the value of *k* is constantly changing according to the surface, but let's examine in special cases:
- If *k is zero*, then the surface is a plane, become Euclidean;
- If *k is constant*, then the surface is even and so the forms on it may freely move without changes.
The constant value either positive or negative:
- If the radiuses are on the same side – positive *k* -of the tangent than the surface convex;
- If they are on different side –negative k - then the surface saddle-shaped – in all directions move away from the point *P*.
We showed already the calculations, so we only summarize it here: we are talking about, "even" surface, i.e. the curvature constant. Let's, have a unit size curvature, then we come to the following terms. If

k = 1, then the geometry spherical: $ds^2 = du^2 + dv^2 \sin^2 u$

k = 0, then the geometry Euclidean: $ds^2 = du^2 + dv^2 u^2$

k =-1, then the geometry hyperbolic: $ds^2 = du^2 + dv^2 sh^2 u$

Notice the simple differences between each distances and yet they opens a quite different world. The *Euclidean Geometry* has the multiplier factor u^2, the *Spherical Geometry* has the $\sin^2 u$, and the *Hyperbolic Geometry* has $sh^2 u$.

This is the beauty of mathematics.

3. *Other results*

Gauss in this presentation came to the definition of the non-Euclidean geometries, having evidence about their existence and uncontradictory.

How Gauss's investigations covered this field are little known, but we know his other result which lead to the sum of the angles in a triangle. This definition uses the calculation the area of a triangle, that is, the relationship between the area and the curvature. He came to the following: $\alpha+\beta+\gamma-\pi=\int kd\delta$, where the $d\delta$ is an infinitesimal surface-unit and the integration gives the area of the entire triangle. From the expression using the three constant *k* value we get these:

k=1 $\alpha+\beta+\gamma-\pi=\Delta$

k=0 $\alpha+\beta+\gamma=\pi$

k=-1 $\pi-(\alpha+\beta+\gamma)=\Delta$

This means that the area of a triangle is proportional to the sum of its internal angles. So this emerge that in spherical case *>180°*, while in the hyperbolic case, *<180°,* and we get back the Euclidean case *=180°.*

A further result is that this area calculation can be applied also in general case - in infinitesimal sense -, even when the *k* changes point by point.

III. CONCLUSION

Gauss's work is not the composition of the non-Euclidean geometry, however, these results are undoubtedly deserve the *prominent theory* name. Consequently, Gauss's name suitable to the names of Bólyai and Lobachevsky.

This has special significance for me, because I'm not a fan of the scientific racing, star-making. I rather much believe in the more effective work, make it by one or plenty, anyone. This statement is extremely important these days, when science has also promoted the collaboration. Remember all of the participants and not let just the leaders win the glory and have the recognition.

I know it's not easy, though, only this worthy.

IV. REFERENCES

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