Stabilization of stochastic inflation

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Plan of the talk

- Stochastic wave equation
- The energy-momentum tensor of the interaction with a random environment
- Stochastic equation for slow roll inflation
- The scale factor $a(\phi)$
- The probability distribution

Stochastic wave equation

I consider an infinite set of fields χ^a with masses m_a interacting with the inflaton ϕ by a linear coupling $\lambda_a\chi^a\phi$. Eliminating χ I obtain in an expanding metric

$$\partial_t^2\phi - a^{-2}\triangle\phi + (3H+\gamma^2)\partial_t\phi + m^2\phi + V'(\phi) + \frac{3}{2}\gamma^2 H\phi = \gamma a^{-\frac{3}{2}}\eta. \quad (1)$$

η is the white noise related to Brownian motion

$$\langle dB(t,\mathbf{x})dB(s,\mathbf{y})\rangle = \langle\eta(t,\mathbf{x})\eta(s,\mathbf{y})\rangle dt = \delta(t-s)\mathcal{G}_t(\mathbf{x}-\mathbf{y})dt \quad (2)$$

x-dependence neglected in a homogeneous universe

The energy-momentum

The energy-momentum tensor of the scalar field in the presence of noise is not conserved. We have to compensate the energy-momentum by means of a compensating energy-momentum T_{de} which we associate with the dark sector .
$T^{\mu\nu}_{tot}$ is

$$T^{\mu\nu}_{tot} = T^{\mu\nu} + T^{\mu\nu}_{de}. \tag{3}$$

From the conservation law

$$(T^{\mu\nu}_{de})_{;\mu} = -(T^{\mu\nu})_{;\mu}. \tag{4}$$

The energy-momentum as an ideal fluid

$$T^{\mu\nu}_{de} = (\rho_{de} + p_{de})u^{\mu}u^{\nu} - g^{\mu\nu}p_{de}, \tag{5}$$

where ρ is the energy density and p is the pressure. The velocity u^{μ} satisfies the normalization condition

$$g_{\mu\nu}u^{\mu}u^{\nu} = 1.$$

For the scalar field we have the representation

$$u^{\mu} = \partial^{\mu}\phi(\partial^{\sigma}\phi\partial_{\sigma}\phi)^{-\frac{1}{2}}, \tag{6}$$

We have

$$(T^{\mu\nu})_{;\mu} = \partial^{\nu}\phi(\gamma a^{-\frac{3}{2}}\eta - \gamma^2\partial_t\phi - \frac{3}{2}\gamma^2 H\phi) \tag{9}$$

For $T^{0\nu}$ in a homogeneous metric

$$d\rho + 3(1+w_I)H\rho dt = \gamma\partial_t\phi \circ a^{-\frac{3}{2}}dB - \frac{3}{2}\gamma^2 H\phi\partial_t\phi dt - \gamma^2(\partial_t\phi)^2 dt, \tag{10}$$

For a potential V we have

$$w_I = (\frac{1}{2}(\partial_t\phi)^2 - V)(\frac{1}{2}(\partial_t\phi)^2 + V)^{-1}. \tag{11}$$

The compensating energy density must have the (non)conservation law with an opposite sign

$$d\rho_{de} + 3H(1+w)\rho_{de}dt = \frac{3}{2}\gamma^2 H\phi\partial_t\phi dt + \gamma^2(\partial_t\phi)^2 dt - \gamma\partial_t\phi a^{-\frac{3}{2}} \circ dB, \tag{12}$$

where

$$w = \frac{p_{de}}{\rho_{de}}. \tag{13}$$

Einstein equations are written in the form

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 8\pi G T^{\mu\nu}_{tot}, \tag{14}$$

The Friedman equation in the FRLW flat metric reads

$$H^2 = \tfrac{8\pi G}{3}(\rho + \rho_{de}). \tag{15}$$

Differentiating

$$da = Hadt \tag{16}$$

$$dH^2 = \frac{8\pi G}{3}(d\rho + d\rho_{de}) = -8\pi GH\Big((1+w_l)\rho + (1+w)\rho_{de}\Big)dt \tag{17}$$

To the environmental noise B I add Starobinsky-Vilenkin noise W describing quantum fluctuations of the inflaton. I get a closed consistent system of stochastic equations

$$d\phi = \Pi dt \tag{18}$$

$$d\Pi = -(3H+\gamma^2)\Pi dt - V' dt - \frac{3}{2}\gamma^2 H\phi dt + \gamma a^{-\frac{3}{2}} \circ dB + \frac{1}{2\pi} H^{\frac{3}{2}} \circ dW \tag{19}$$

$$dH = -4\pi G \Pi^2 dt \tag{20}$$

$$da = Hadt \tag{21}$$

The slow roll system

The diffusion (small roll) system reads

$$(3H+\gamma^2)d\phi = -V'dt - \frac{3}{2}\gamma^2 H\phi dt + \gamma a^{-\frac{3}{2}} \circ dB + \frac{1}{2\pi} H^{\frac{3}{2}} \circ dW \quad (22)$$

$$dH = -4\pi G(\partial_t \phi)^2 dt \quad (23)$$

$$da = Hadt \quad (24)$$

The Starobinsky-Vilenkin slow roll (quantum) system corresponds to the limit $\gamma \to 0$ limit

$$\partial_t \phi = -\frac{1}{3H} V' + \frac{1}{2\pi} H^{\frac{3}{2}} \circ \partial_t B \tag{25}$$

together with

$$\partial_t H = -4\pi G (\partial_t \phi)^2 \tag{26}$$

I can get a relation between H and V from the equations of motion

$$\partial_t\left(\frac{1}{2}\Pi^2 + V + \frac{3}{4}\gamma^2 H\phi^2 - \frac{3}{8\pi G}H^2 + \frac{1}{4\pi G}\gamma^2 H + \Lambda\right) + 3\pi G\gamma^2\phi^2\Pi^2$$
$$= \gamma a^{-\frac{3}{2}}\Pi \circ \partial_t B + \frac{3}{2\pi}H^{\frac{5}{2}}\Pi \circ \partial_t W \tag{27}$$

I use the approximation

$$H = \gamma^2(\frac{1}{3} + 4\pi G\phi^2) + \sqrt{\frac{8\pi G}{3}(V + \Lambda) + \frac{1}{2}\Pi^2 + \gamma^4(\frac{1}{3} + 4\pi G\phi^2)^2} \tag{28}$$

There remains to derive $a(\phi)$ from

$$\ln a = \int H(\phi)dt = \int H(\phi)(\partial_t\phi)^{-1}(\phi)d\phi \tag{29}$$

Finally

$$H = \sqrt{\frac{8\pi G}{3}(V + \Lambda)} \tag{30}$$

Then

$$\ln(a) = -8\pi G \int d\phi (V')^{-1}(\Lambda + V) \tag{31}$$

For the slow roll system we have the Fokker-Planck equation for the probability distribution

$$\partial_t P = \partial_\phi \frac{\gamma^2}{18Ha^{\frac{3}{2}}} \partial_\phi \frac{1}{Ha^{\frac{3}{2}}} P + \frac{1}{8\pi^2} \partial_\phi H^{\frac{3}{2}} \partial_\phi H^{\frac{3}{2}} P + \partial_\phi (3H)^{-1} V' P \tag{32}$$

The evolution scale factor $a(\phi)$ in some inflationary models

We know $H(\phi)$ as a function of ϕ. The dependence of a on ϕ is more involved (we need to calculate some integrals). If $V = \frac{m^2}{2}\phi^2$ then

$$a = \exp\Big(- 8\pi G \Lambda m^{-2} \ln|\phi| - 2\pi G \phi^2 \Big) \tag{33}$$

Large ϕ corresponds to small a and small ϕ to large a.

If $V = g\phi^n$ $(n > 2)$ then

$$a = \exp\left(- \frac{8\pi G\Lambda}{(2-n)ng}\phi^{2-n} - 4\pi G n^{-1}\phi^2 \right) \tag{34}$$

If $\phi \to \infty$ then $a \to 0$, if $\phi \to 0$ then $a \to \infty$ (for $\Lambda > 0$).
If $V = g\exp(\lambda\phi)$ then

$$a = \exp\left(\frac{8\pi G\Lambda}{g\lambda^2}\exp(-\lambda\phi) - \frac{8\pi G}{\lambda}\phi \right) \tag{35}$$

If $\phi \to +\infty$ then $a \to 0$, if $\phi \to -\infty$ then $a \to \infty$.
For a flat potential

$$V = \frac{L+\phi^2}{K+\phi^2} \tag{36}$$

we have

$$a = \exp\Bigg(- 8\pi G\Lambda\Big(\tfrac{\Lambda K^2}{2(K-L)} \ln|\phi| + \tfrac{\Lambda K}{2(K-L)}\phi^2 + \tfrac{\Lambda}{8(K-L)}\phi^4 + \tfrac{\Lambda}{8(K-L)}\phi^4 + \tfrac{LK}{2(K-L)} \ln|\phi| + \tfrac{L+K}{4(K-L)}\phi^2 + \tfrac{1}{8(K-L)}\phi^4 \Big)\Bigg) \tag{37}$$

Let $V = g\cos\phi$ then

$$a = \exp\Big(-8\pi G\Big(-g^{-1}\Lambda \ln|\tan(\frac{\phi}{2})| - \ln|\sin\phi)|\Big)\Big) \qquad (38)$$

$a \to 0$ when $\phi \to 0$. When $\phi \to \pi$ then a may goes to ∞ if $g^{-1}\Lambda > 1$ (otherwise $a \to 0$).
The special case of "natural inflation" with $V(\phi) = g(1 - \cos\phi)$ gives

$$a = \exp\Big(8\pi G \ln(2\cos^2(\frac{\phi}{2})\Big)$$

Then, $a \to 0$ when $\phi \to \pi$ (starts from the maximum of the potential reaching the minimum).

There is an interesting case of the double-well potential

$$V(\phi) = \frac{g}{4}\phi^4 - \frac{\mu^2}{2}\phi^2 \tag{39}$$

then

$$a = |\phi|^{\frac{8\pi G\Lambda}{\mu^2}} |g\phi^2 - \mu^2|^{\frac{\pi G\mu^2}{g} - \frac{4\pi G\Lambda}{\mu^2}} \exp(-\pi G\phi^2) \tag{40}$$

If $\phi \to 0$ then $a \to 0$ (for $\Lambda > 0$, if if $\Lambda = 0$ then $a \to const \neq 0$. If $\phi \to \mu g^{-\frac{1}{2}}$ then a goes either to 0 or to infinity depending on the value of Λ . When $\phi \to \infty$ then $a \to 0$.

Probability distribution of universes created in stochastic inflation

The probability distribution determines the probability of an appearance of the universe with given ϕ or $a(\phi)$. Let us consider the simplest cases first

The stationary solution of Fokker-Planck without the Starobinsky-Vilenkin noise is

$$P = \sqrt{V}\exp\Big(-12\pi G\int^{\phi} d\phi'(V')^{-1}(\Lambda + V)\Big)$$
$$\exp\Big(-\frac{6}{\gamma^2}\sqrt{\frac{8\pi G}{3}}\int d\phi V'\sqrt{V+\Lambda}\exp\big(-24\pi G\int^{\phi} d\phi'(V')^{-1}(\Lambda + V)\big)\Big) \tag{41}$$

If we assume that V does not grow faster than exponentially and is an even function of ϕ then for a large $|\phi|$

$$P \simeq Ha^{\frac{3}{2}} = \sqrt{V}\exp\Big(-12\pi G \int^{\phi} d\phi'(V')^{-1}(\Lambda + V)\Big) \qquad (42)$$

If $\gamma = 0$ (the environmental noise is absent) then we obtain the Linde-Starobinsky-Vilenkin-Hartle-Hawking solution

$$P = (V+\Lambda)^{-\frac{3}{4}}\exp(\frac{3}{8G^2}\frac{1}{V+\Lambda}) \qquad (43)$$

This formula fails to express a probability distribution (P is not integrable) if V does not fall quickly enough for large ϕ or if $V+\Lambda = 0$ at a certain ϕ_c (as for the double well potential (50) and ϕ^n with $\Lambda = 0$). I show that an environmental noise allows to avoid this difficulty.

Let us write

$$\tilde{P} = H^{-1} a^{-\frac{3}{2}} P$$

Then, equation for stationary distribution reads

$$\frac{\gamma^2}{18} H^{-1} a^{-\frac{3}{2}} \partial_\phi \tilde{P} + \frac{1}{8\pi^2} H^{\frac{3}{2}} \partial_\phi (H^{\frac{5}{2}} a^{\frac{3}{2}} \tilde{P}) \\ = -\frac{1}{3} V' a^{\frac{3}{2}} \tilde{P} \tag{44}$$

Using the formulas for H and for a I obtain

$$\ln \tilde{P} = -6 \int d\phi H a^3 (\gamma^2 + \frac{9}{4\pi^2} H^5 a^3)^{-1} \\ \Big(V' + \frac{3}{8\pi^2} (\frac{8\pi G}{3})^2 (V + \Lambda)^2 (\frac{5}{4} (V + \Lambda)^{-1} V' - \frac{9}{2} \frac{8\pi G}{3} (V + \Lambda)(V')^{-1}) \Big) \tag{45}$$

When $a \to 0$ for $\phi \to \infty$ then we get for large ϕ that $P \simeq H a^{\frac{3}{2}}$ as in the model without the quantum noise.

For the potential ϕ^n the formula for a gives for a large ϕ (small a)

$$P = |\phi|^{\frac{n}{2}} \exp(-6\pi G n^{-1} \phi^2) \tag{46}$$

This is the well-known gamma distribution in statistics(more precisely the χ^2 distribution).If on the other hand $a^3 H^5$ tends to infinity then

$$\ln \tilde{P} = -\frac{8\pi^2}{3} \int d\phi H^{-4} \Big(V' + \frac{3}{8\pi^2} (\frac{8\pi G}{3})^2 V^2 (\frac{5}{4} V^{-1} V' - \frac{9}{2} \frac{8\pi G}{3} V (V')^{-1}) \Big) \tag{47}$$

This is the Linde-Starobinsky-Vilenkin-Hartle-Hawking solution

If we set

$$\hat{P} = H^{\frac{3}{2}} P$$

then the equation for $\hat{P}$ reads

$$\frac{\gamma^2}{18} H^{-1} a^{-\frac{3}{2}} \partial_\phi (H^{-\frac{5}{2}} a^{-\frac{3}{2}} \hat{P}) + \frac{1}{8\pi^2} H^{\frac{3}{2}} \partial_\phi \hat{P} = -\frac{1}{3} H^{-\frac{5}{2}} V' \hat{P}$$

When we calculate the derivatives of H and a then we obtain

$$\begin{aligned} &\Big(\tfrac{1}{8\pi^2}(\tfrac{8\pi G}{3})^{\frac{3}{4}} a^3 V^{\frac{5}{2}} + \tfrac{\gamma^2}{18}(\tfrac{3}{8\pi G})^{\frac{7}{4}}\Big)\partial_\phi \ln \hat{P} \\ &= -\tfrac{1}{3}(\tfrac{3}{8\pi G})^{\frac{5}{4}} a^3 (V+\Lambda)^{\frac{1}{2}} V' \\ &+ \tfrac{5\gamma^2}{72}(\tfrac{3}{8\pi G})^{\frac{7}{4}} (V+\Lambda)^{-1} V' - \tfrac{\gamma^2}{4}(\tfrac{3}{8\pi G})^{\frac{3}{4}} (V+\Lambda)(V')^{-1} \end{aligned} \tag{48}$$

From this formula we can also see that if $a \to 0$ for a large ϕ then $P \simeq H a^{\frac{3}{2}}$

If a is large (either for large ϕ or small ϕ) so that $a^3 V^{\frac{5}{2}} \to \infty$ then the terms independent of a can be omitted in the formula above and we get the Linde-Starobinsky-Vilenkin-Hartle-Hawking formula

$$P \simeq (V + \Lambda)^{-\frac{3}{4}} \exp(\frac{8}{3G^2(V + \Lambda)}) \tag{49}$$

Conclusions

1) If $a^{\frac{3}{2}}H^5 \to \infty$ then we get Linde-Starobinsky-Vilenkin-Hartle-Hawking formula

$$P \simeq (V + \Lambda)^{-\frac{3}{4}} \exp\left(\frac{8}{3G^2(V + \Lambda)}\right) \tag{50}$$

2) If $a^{\frac{3}{2}}H^5 \to 0$ we get the formula $P \simeq Ha^{\frac{3}{2}}$ as if there were no quantum fluctuations