Quasi-local Hamiltonian – four decades of Kijowski's formula

Jacek Jezierski, Uniwersytet Warszawski

Jerzy Kijowski, *On a New Variational Principle in General Relativity and the Energy of the Gravitational Field*, General Relativity and Gravitation, Vol. 9, (1978), pp. 857-877

The paper GRG (1978) contains formulation of variational principle leading to the formula firstly presented at MG meeting (1990) and then full analysis is given in GRG (1998).
The metod has been applied to several topics like

The paper GRG (1978) contains formulation of variational principle leading to the formula firstly presented at MG meeting (1990) and then full analysis is given in GRG (1998).
The metod has been applied to several topics like

- Energy positivity theorem
- Bondi mass
- Thermodynamics of black holes

The paper GRG (1978) contains formulation of variational principle leading to the formula firstly presented at MG meeting (1990) and then full analysis is given in GRG (1998).
The metod has been applied to several topics like

- Energy positivity theorem
- Bondi mass
- Thermodynamics of black holes
- aAdS symplectic structure on scri
- Linearized gravity
- Initial-boundary problem in GR

Simple examples – other theories

scalar field

$$0 = \int_V (\dot{\pi}\delta\varphi - \dot{\varphi}\delta\pi) + \delta\mathcal{H}(\varphi, \pi) + \int_{\partial V} p^{\perp}\delta\varphi \, ,$$

Simple examples – other theories

scalar field

$$0 = \int_V (\dot{\pi}\delta\varphi - \dot{\varphi}\delta\pi) + \delta\mathcal{H}(\varphi,\pi) + \int_{\partial V} p^{\perp}\delta\varphi\,,$$

Maxwell field

$$0 = \int_V \left(\dot{\mathcal{D}}^k \delta A_k - \dot{A}_k \delta\mathcal{D}^k\right) + \delta\overline{\mathcal{H}} + \int_{\partial V} A_0 \delta\mathcal{D}^3 + \int_{\partial V} \mathcal{F}^{3B}\delta A_B\,,$$

where

$$\overline{\mathcal{H}} \;=\; \int_V \left(A_0 \partial_k \mathcal{D}^k + \mathcal{D}^k (A_{k0} - A_{0k}) - L\right) = \frac{1}{2}\int_V (\mathbf{D}^2 + \mathbf{B}^2)$$

Homogeneous formula

Time-like vector-field $X = \partial_0$, variational formula proved by Kijowski in dimension $3+1$, for a Lorentzian metric interacting with a matter field φ:

$$\begin{aligned} 0 &= \frac{1}{2\gamma}\int_V \left(\dot{P}^{kl}\delta g_{kl} - \dot{g}_{kl}\delta P^{kl}\right) + \frac{1}{\gamma}\int_{\partial V}(\dot{\lambda}\delta\alpha - \dot{\alpha}\delta\lambda) \\ &\quad + \frac{1}{2\gamma}\int_{\partial V}(2\nu\delta\mathbf{Q} - 2\nu^A\delta\mathbf{Q}_A + \nu\mathbf{Q}^{AB}\delta g_{AB}) \\ &\quad + \int_V (\dot{p}\delta\varphi - \dot{\varphi}\delta p) + \int_{\partial V} p^\perp\delta\varphi \end{aligned} \tag{1}$$

$\gamma = 8\pi$, g_{kl} and P^{kl} describe gravitational Cauchy data on V, φ and p represent the matter fields, if any. $\mathbf{Q}$'s describe various components of the extrinsic curvature of the world tube $\partial\Omega$, whereas ν, ν^A, and g_{AB} encode the metric induced on $\partial\Omega$ using an $(2+1)$ decomposition. $\lambda = \sqrt{\det g_{AB}}$ is the volume form on ∂V whereas α denotes the hyperbolic angle between V and $\partial\Omega$.

In *Differential geometry and its applications (Opava, 2001)* the formula was generalized to the case when X is spacelike. Next, in *Phys. Rev. D* **70** *(2004), 124010, arXiv:gr-qc/0412042* the field was allowed to be null-like. Finally, it has been pointed out in *Phys. Rev.* **D87** *(2013), 124015, arXiv:1305.1014 [gr-qc]* that the formula remains true for vacuum metrics, possibly with a cosmological constant, in any space-dimension $n \geq 2$, with a constant γ which depends upon dimension.

Full generality

The above equation appears in the most general form in *Phys. Rev.* **D92** *(2015), 084030, arXiv:1507.03868v1 [gr-qc]* one obtains the variational formula for a general vector field $X = X^0\partial_0 + Y$:

$$0 = \int_V \left(\mathcal{L}_X g_{kl}\delta P^{kl} - \mathcal{L}_X P^{kl}\delta g_{kl}\right)$$
$$+2\int_{\partial V} (\mathcal{L}_{X^0\partial_0}\alpha\delta\lambda - \mathcal{L}_{X^0\partial_0}\lambda\delta\alpha)$$
$$-\int_{\partial V} X^0(2\nu\delta\mathbf{Q} - 2\nu^A\delta\mathbf{Q}_A + \nu\mathbf{Q}^{AB}\delta g_{AB})$$
$$-\int_{\partial V} \left(2Y^k\delta P^n{}_k - Y^n P^{kl}\delta g_{kl}\right)$$

Theorem

Field dynamics in a four-dimensional region $\mathcal{O}$ is equivalent to

$$\delta \int_{\mathcal{O}} L = -\frac{1}{16\pi} \int_{\partial\mathcal{O}} g_{kl}\, \delta\Pi^{kl} \ ,$$

where g_{kl} is the three-dimensional metric induced on the boundary $\partial\mathcal{O}$ by $g_{\mu\nu}$ and Π is the extrinsic curvature (in A.D.M. densitized form) of $\partial\mathcal{O}$.

shows the universality of the symplectic structure:

$$\int_{\partial\mathcal{O}} \delta\Pi^{kl} \wedge \delta g_{kl} \ .$$

AdS

In the case of AdS metric

$$\tilde{g}_{\mathrm{AdS}} = \frac{l^2}{z^2}\left[\mathrm{d}z^2 - \left(\frac{1+z^2}{2}\right)^2 \mathrm{d}\bar{t}^2 + \left(\frac{1-z^2}{2}\right)^2 \mathrm{d}\Omega_2\right] \qquad (2)$$

we have

$$g_{\mathrm{AdS}} = \Omega^2 \tilde{g}_{\mathrm{AdS}}, \quad \text{where} \quad \Omega := \frac{z}{l}\,.$$

Cosmological constant $\Lambda = -3/l^2$

Our four-dimensional asymptotic AdS spacetime metric $\tilde{g}$ assumes in canonical coordinates[1] the following form:

$$\tilde{g} = \tilde{g}_{\mu\nu}\mathrm{d}z^{\mu} \otimes \mathrm{d}z^{\nu} = \frac{l^2}{z^2}\left(\mathrm{d}z \otimes \mathrm{d}z + h_{ab}\mathrm{d}z^{a} \otimes \mathrm{d}z^{b}\right) \qquad (3)$$

and the three-metric h obeys the following asymptotic condition:

$$h = h_{ab}\mathrm{d}z^{a} \otimes \mathrm{d}z^{b} = \overset{(0)}{h} + z^2\overset{(2)}{h} + z^3\chi + O(z^4)\,. \qquad (4)$$

The term χ vanishes for the pure AdS given by (2). Moreover, the terms $\overset{(0)}{h}$ and $\overset{(2)}{h}$ have the standard form

$$\overset{(0)}{h} = \frac{1}{4}(\mathrm{d}\Omega_2 - \mathrm{d}\bar{t}^2) \qquad \overset{(2)}{h} = -\frac{1}{2}(\mathrm{d}\Omega_2 + \mathrm{d}\bar{t}^2)\,.$$

[1] Sometimes it is called Fefferman-Graham coordinate system.

AdS

In our case a boundary data on S consists of the three-metric $\tilde{h}_{ab}$ and canonical A.D.M. momentum $\tilde{Q}^{ab}$ which is related with extrinsic curvature $\tilde{K}_{ab}$ in the usual way:

$$\tilde{Q}_{ab} = \sqrt{-\tilde{h}} \left(\tilde{K}_{ab} - \tilde{h}^{cd} \tilde{K}_{cd} \tilde{h}_{ab} \right) .$$

Conformal rescaling of the three-metric and extrinsic curvature

$$\tilde{h}_{ab} = \Omega^{-2} h_{ab} \, , \quad \tilde{K}_{ab} = -\frac{\tilde{\Gamma}^3_{ab}}{\sqrt{\tilde{h}^{33}}} = \Omega^{-1} \left(K_{ab} - \frac{1}{z} h_{ab} \right)$$

enables one to analyze the symplectic structure as follows:

$$\begin{aligned} \tilde{h}_{ab} \delta \tilde{Q}^{ab} &= \delta \left(\tilde{h}_{ab} \tilde{Q}^{ab} \right) - \tilde{Q}^{ab} \delta \tilde{h}_{ab} \\ &= \delta \left(\tilde{h}^{ab} \tilde{Q}_{ab} \right) + \tilde{Q}_{ab} \delta \tilde{h}^{ab} \, . \end{aligned} \quad (5)$$

In particular, (5) implies

$$\int_S \delta\tilde{h}_{ab}\wedge\delta\tilde{Q}^{ab} = \int_S \delta\tilde{Q}_{ab}\wedge\delta\tilde{h}^{ab}\,.$$

Moreover,

$$\begin{aligned}\tilde{Q}_{ab}\delta\tilde{h}^{ab} &= \Omega^{-2}\sqrt{-h}\left(K_{ab}-\frac{2}{z}h_{ab}-K^c{}_c h_{ab}\right)\delta h^{ab} \qquad (6)\\ &= 4\Omega^{-3}\frac{1}{l}\delta\sqrt{-h}+\Omega^{-2}\sqrt{-h}\left(K_{ab}-h^{cd}K_{cd}h_{ab}\right)\delta h^{ab}\,.\end{aligned}$$

With the help of standard variational identitites:

$$\delta\sqrt{-h}=\frac{1}{2}\sqrt{-h}h^{ab}\delta h_{ab}\,,\quad \delta R_{ab}(h)=\delta\Gamma^c{}_{ab|c}-\delta\Gamma^c{}_{ac|b}\,,$$

$$\begin{aligned}\delta\left(\sqrt{-h}R(h)\right) &= \sqrt{-h}\left(R_{ab}-\frac{1}{2}h_{ab}R\right)\delta h^{ab} \qquad (7)\\ &\quad +\partial_c\left[\sqrt{-h}\left(h^{ab}\delta\Gamma^c{}_{ab}-h^{ac}\delta\Gamma^b{}_{ab}\right)\right]\,,\end{aligned}$$

we analyze the singular part of (6) as follows:

$$\begin{aligned} \operatorname{sing}(\tilde{Q}_{ab}\delta\tilde{h}^{ab}) &= \delta\left(4\Omega^{-3}\frac{1}{l}\sqrt{-h}\right) + \sqrt{-h}\,\Omega^{-2}z\left(\mathcal{R}_{ab} - \frac{1}{2}h_{ab}h^{cd}\mathcal{R}_{cd}\right)\delta h^{ab} \\ &= \frac{1}{l}\delta\left\{4\Omega^{-3}\sqrt{-h} + \Omega^{-2}\sqrt{-h}\mathcal{R}\right\} + \text{full divergence} \qquad (8) \end{aligned}$$

which is a full variation up to boundary terms.

$$\lim_{\epsilon\to 0}\int_{S_\epsilon} \delta\tilde{Q}_{ab}\wedge\delta\tilde{h}^{ab} = 3l^2\int_{\mathscr{I}}\delta\left[\sqrt{-\det \overset{(0)}{h}}\left(\chi_{ab} - \overset{(0)}{h}{}_{ab}\chi^c{}_c\right)\right]\wedge\delta\overset{(0)}{h}{}^{ab}$$

$$= \int_{\mathscr{I}}\delta\pi^{ab}\wedge\delta\overset{(0)}{h}{}_{ab}\,, \qquad (9)$$

where $S_\epsilon := \{z = x^3 = \epsilon\}$ is a tube close to infinity, symplectic structure (9) on conformal boundary consists of the metric $\overset{(0)}{h}{}_{ab}$ and momenta $\pi^{ab} := 3l^2\sqrt{-\det \overset{(0)}{h}}\left(\chi^c{}_c\overset{(0)}{h}{}^{ab} - \chi^{ab}\right)$.

Linearized gravity

The 6 dynamical, second order equations can be written as the system of 12 first order equations:

$$\dot{P}_{kl} = -\Lambda E_{kl} + \frac{\Lambda}{2}\left(h^0{}_{0|kl} - \eta_{kl} h^0{}_0{}^{|m}{}_m\right) \tag{10}$$

$$\dot{h}_{kl} = 2\Lambda^{-1}\left(P_{kl} - \frac{1}{2}\eta_{kl} P\right) + h_{0k|l} + h_{0l|k} \tag{11}$$

$$E_{mk} := \delta\mathcal{R}_{mk} = \eta^{lj}\delta\mathcal{R}_{mlkj} = \frac{1}{2}(h^l_{m|lk} + h^l_{k|lm} - h_{|mk} - h_{mk}{}^{|l}{}_l)$$

$$\sqrt{\det \eta_{mn}} B_{ij} = P_{il|k}\varepsilon^{lk}{}_j$$

Both E_{kl} and B_{kl} are symmetric and traceless.

$$E^{kl}{}_{|l} = 0 \tag{12}$$

$$B^{kl}{}_{|l} = 0 \tag{13}$$

The constraint equations can be written as follows

$$P^{kl}{}_{|l} = 0 \tag{14}$$

$$h^{kl}{}_{|kl} - h^{|k}{}_{k} = 0 \tag{15}$$

Equations (14) and (15) are called the vector constraint and the scalar constraint respectively.

The gauge splits into its time-like component ξ_0:

$$\Lambda^{-1} P_{kl} \rightarrow \Lambda^{-1} P_{kl} - \xi_{0|kl} + \eta_{kl} \xi_0{}^{|m}{}_m$$

and a 3 dimensional gauge ξ_k:

$$h_{kl} \rightarrow h_{kl} + \xi_{l|k} + \xi_{k|l}$$

The Cauchy data (g_{kl}, P^{kl}) in V are equivalent with $(\overline{g}_{kl}, \overline{P}^{kl})$ if they can be related by the gauge transformation with ξ_μ vanishing on ∂V. The evolution of canonical variables P^{kl} and h_{kl} given by equations (10 - 11) is not unique unless the lapse function $(h^0{}_0)$ and the shift vector $(h^0{}_k)$ are specified. The dynamics contains however the gauge-invariant sector which does not depend upon the choice of the lapse and the shift.

$\dot{P}\delta h$

$$\mathbf{x} = 2x^k x^l E_{kl}\,, \quad \mathbf{y} = 2x^k x^l B_{kl}\,.$$

$$\int_{S(r)} \frac{\Lambda}{2} \left(h^0_{0|kl} - \eta_{kl} h^0_{0|mn} \eta^{mn} \right) \delta h^{kl} = \tag{16}$$

$$-\frac{1}{2}\partial_3 \left[\int_{S(r)} \Lambda h^0_{0,3} \delta H + \frac{\Lambda}{r} h^0_0 \delta(\mathbf{x} - \Psi - H) \right]$$

$\dot{P}\delta h$

$$\text{polar part of} \int_{S(r)} \Lambda E_{kl} \delta h^{kl} =$$

$$\frac{1}{2}\delta \int_{S(r)} \frac{\Lambda}{r^2} \left\{ \mathbf{x}(\Delta+2)^{-1}\mathbf{x} - \partial_3(r\mathbf{x})\Delta^{-1}(\Delta+2)^{-1}\partial_3(r\mathbf{x}) \right\} +$$

$$\partial_3 \int_{S(r)} \frac{\Lambda}{r^2} \left[\frac{1}{2} r\mathbf{x}(\Delta+2)^{-1}\delta \underbrace{(\mathbf{x}-\Psi)}_{2K} + r\partial_3(r\mathbf{x})\Delta^{-1}(\Delta+2)^{-1}\delta(r^2\chi^{AB}{}_{||AB}) \right]$$

$$\text{axial part of} \int_{S(r)} \Lambda E_{kl}\delta h^{kl} = \frac{1}{2}\delta \int_{S(r)} \Lambda \partial_0 \mathbf{y} \Delta^{-1}(\Delta+2)^{-1}\partial_0 \mathbf{y}$$

$$+\partial_3 \int_{S(r)} \frac{\Lambda}{r^2} \left[(r^2 \varepsilon^{AC}\delta\chi_A{}^B{}_{||BC})\Delta^{-1}(\Delta+2)^{-1}(r^2\partial_0\mathbf{y}) \right]$$

$\dot{h}\delta P$

$$\int_{S(r)} \dot{h}_{kl}\delta P^{kl} =$$

$$\delta\,\frac{1}{2}\int_{S(r)}\left\{\frac{\Lambda}{r^2}\left[\mathbf{y}(-\Delta)^{-1}\mathbf{y}+\frac{1}{r^2}\partial_3(r^2\mathbf{y})\Delta^{-1}(\Delta+2)^{-1}\partial_3(r^2\mathbf{y})\right]\right.$$

$$\left.+\Lambda\partial_0\mathbf{x}\Delta^{-1}(\Delta+2)^{-1}\partial_0\mathbf{x}\right\}$$

$$+\partial_3\int_{S(r)}\left[\Lambda h_{0A||B}\varepsilon^{AB}(-\Delta)^{-1}\delta\mathbf{y}+\frac{1}{2}\dot{H}r\delta P^3_3\right]$$

$$+\partial_3\int_{S(r)}\left[(\Lambda^{-1}\mathbf{X}-r^2\dot{\chi}^{AB}{}_{||AB})\Delta^{-1}(\Delta+2)^{-1}\delta(rQ)\right]$$

Linearized gravity

Energy functional $\mathcal{H}$ takes the following form in Minkowski spacetime:

$$\begin{aligned}\mathcal{H} &= \frac{1}{32\pi}\int_\Sigma \Big[(r\dot{\mathbf{x}})\Delta^{-1}(\Delta+2)^{-1}(r\dot{\mathbf{x}}) + (r\dot{\mathbf{y}})\Delta^{-1}(\Delta+2)^{-1}(r\dot{\mathbf{y}}) + \\ &\quad + (r\mathbf{x})_{,3}\Delta^{-1}(\Delta+2)^{-1}(r\mathbf{x})_{,3} - \mathbf{x}(\Delta+2)^{-1}\mathbf{x} + \\ &\quad + (r\mathbf{y})_{,3}\Delta^{-1}(\Delta+2)^{-1}(r\mathbf{y})_{,3} - \mathbf{y}(\Delta+2)^{-1}\mathbf{y}\Big] \mathrm{d}r \sin\theta \mathrm{d}\theta \mathrm{d}\varphi \end{aligned} \tag{17}$$

$$0 = \frac{1}{16\pi}\int_\Sigma \left(\dot{P}^{kl}\delta h_{kl} - \dot{h}_{kl}\delta P^{kl}\right) + \delta\mathcal{H}$$

The following formula describes the true energy of linearized gravitational field:

$$
\begin{aligned}
16\pi\overline{\mathcal{H}} &= \int_{\Sigma}\left(E^{ab}(-\triangle)^{-1}E_{ab} + B^{ab}(-\triangle)^{-1}B_{ab}\right) \\
&= \iint_{\Sigma\times\Sigma}\left[\frac{E^{ab}\left(\vec{r'}\right)E_{ab}\left(\vec{r''}\right)}{4\pi\|\vec{r'}-\vec{r''}\|} + \frac{B^{ab}\left(\vec{r'}\right)B_{ab}\left(\vec{r''}\right)}{4\pi\|\vec{r'}-\vec{r''}\|}\right]\mathrm{d}\vec{r'}\mathrm{d}\vec{r''}
\end{aligned}
$$

It is manifestly covariant with respect to the Euclidean group.

In the future we also plan to incorporate boundary terms because we want to generalize the above formulae to finite region with boundary.

Theorem

For localized data $\mathcal{H} = \overline{\mathcal{H}}$.

Final remarks

- In full gravity the Hamiltonian is always a boundary term

Final remarks

- In full gravity the Hamiltonian is always a boundary term
- In linearized gravity the control data is rigid when the volume is a ball

Final remarks

- In full gravity the Hamiltonian is always a boundary term
- In linearized gravity the control data is rigid when the volume is a ball
- The method should be generalized in the future to generic volume

Final remarks

- In full gravity the Hamiltonian is always a boundary term
- In linearized gravity the control data is rigid when the volume is a ball
- The method should be generalized in the future to generic volume
- How to choose a boundary? The concept of rigid or round sphere.
- Problem of division boundary data into charges (mono-dipole part) and wave data
- Relation between quasi-local formulas in full and linearized gravity: test for quasi-local mass.