Einstein-Weyl Spaces and Near Horizon Geometry

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MD, Jan Gutowski, Wafic Sabra, CQG 2017, arXiv:1610.08953 .

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- Galloway–Schoen 2006: Horizon cross–section admits a metric of positive scalar curvature. $D=4: S^3$ (or quotient), $S^2 \times S^1$, connected sums.

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  - $g=2du\Big(dr+rh-\frac{1}{2}r^2\Delta du\Big)+\gamma,\ \ A=r\Phi du+B$, where

$$\begin{aligned}\gamma &= \gamma_{ij}(r,y)dy^idy^j,\ h=h_i(r,y)dy^i,\ B=B_i(r,y)dy^i,\\ \Delta &= \Delta(r,y),\ \Phi=\Phi(r,y)\ \text{ are all real–analytic in } r.\end{aligned}$$

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- All theorems assume supersymmetry.
- Unexpected spin-off: conformal invariance and integrability on Σ.

Minimal supergravity in 5 dimensions

- $\mathcal{R}$ Ricci scalar of g. Maxwell field $H = dA$

$$S = \int_M \mathcal{R}\,\mathsf{vol}_M - \frac{3}{2} H \wedge *_5 H - H \wedge H \wedge A.$$

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$$dH = 0, \quad d *_5 H + H \wedge H = 0,$$
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$$*_3(d\Phi + h\Phi) = dh, \quad \text{(Maxwell)}$$
$$d *_3 h = 0, \quad \text{(Einstein } ur\text{)}$$
$$R_{ij} + \nabla_{(i} h_{j)} + h_i h_j = \Big(\frac{1}{2}\Phi^2 + h^k h_k\Big)\gamma_{ij} \quad \text{(Einstein } ij\text{)}$$

- Let $\dim(\Sigma) = 3$. A Weyl structure $(\Sigma, [\gamma], D)$
  - Riemannian conformal structure $[\gamma] = \{e^{2\Omega}\gamma, \Omega : \Sigma \to \mathbb{R}\}$.
  - Torision–free connection D on $T\Sigma$.
  - Compatibility $D_i\gamma_{jk} = 2h_i\gamma_{jk}$ for some $h \in \Lambda^1(\Sigma)$.

Hyper–CR Einstein–Weyl geometry

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- An Einstein–Weyl space is Hyper–CR iff there exists $\Phi : \Sigma \rightarrow \mathbb{R}$ s. t.

$$*_3(d\Phi + h\Phi) = dh, \quad W = \frac{3}{2}\Phi^2, \quad W = \text{Ricci scalar of } D.$$

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- Gauduchon–Tod 1999. The only compact EW Hyper–CR examples
  1. Product metric on $S^2 \times S^1$ with $dh = 0$.
  2. Flat torus with $h = 0$.
  3. Berger sphere

$$\gamma = (\sigma_1)^2 + (\sigma_2)^2 + a^2(\sigma_3)^2, \quad h = a\sqrt{(}1 - a^2)\sigma_3$$

where $d\sigma_1 + \sigma_2 \wedge \sigma_3 = 0$.

Main Theorem

Let (γ, h) be a hyper–CR Einstein–Weyl structure on Σ and let $\Omega : \Sigma \to \mathbb{R}^+$ satisfy $d *_3 \left(de^{\Omega}\right) + d *_3 \left(e^{\Omega} h\right) = 0$. Then

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\begin{aligned}
g &= e^{2\Omega}(2du(dr + rh - \frac{1}{3}r^2 W du) + \gamma + 6rdud\Omega) \\
A &= \sqrt{\frac{2}{3}} e^{\Omega} r \sqrt{W} du + \alpha \quad (\star)
\end{aligned}
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is a solution to EMCS. Here $\alpha \in \Lambda^1(\Sigma)$ is s.t. $d\alpha = -e^{\Omega} *_3 (h + d\Omega)$.

- All near–horizon geometries for 5D SUSY back holes/rings/strings are locally of the form $(\star)$.
- If Σ is compact then γ is a metric on the Berger sphere, a product metric on $S^1 \times S^2$ or a flat metric on T^3.

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