Cosmology with the compact phase space of matter

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September 26, 2017

*J. Mielczarek & T.T., Phys. Rev. D **96**, 043522 (2017)

Outline:

1. Homogeneous cosmological model
   - Spherical phase space for the field
   - Classical dynamics of the model

2. Perturbative quantum inhomogeneities
   - Quantization of the linearized model
   - First order corrections to the standard case

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Context and the existing work

- Momentum spaces, or phase spaces, with nontrivial geometry that appear in quantum gravity
  - Born reciprocity and three-dimensional gravity
  - Quantum gravity phenomenology, relative locality framework
  - Group field theory, loop quantum cosmology etc.
- Target manifolds for field values of non-linear sigma models and the Tseytlin string action
- The principle of finiteness of physical quantities, at the base of the Born-Infeld theory
- Potential connections between quantum gravity, cosmology and condensed matter physics

Nonlinear Field Space Theory:

J. Mielczarek and T.T., Phys. Lett. B **759**, 424 (2016)
J. Mielczarek, Universe **3**, 29 (2017)
T.T., Acta Phys. Pol. B Proc. Suppl. **10**, 329 (2017)
J. Bilski, S. Brahma, A. Marcianò and J. Mielczarek, arXiv:1708.03207 [hep-th]

Phase space variables

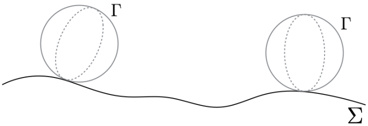
Phase space $\Gamma = \mathbb{R}^2$ is formed by values of a scalar field φ and its conjugate momentum π_φ at every point of space Σ. We assume that Γ is actually a sphere, parametrized in terms of usual angles ϕ and θ or the spin-like vector $\mathbf{S} = (S_x, S_y, S_z)$, so that

$$S_x := S \sin\theta \cos\phi = S \cos\frac{\pi_\varphi}{R_2} \cos\frac{\varphi}{R_1}\,, \tag{1}$$

$$S_y := S \sin\theta \sin\phi = S \cos\frac{\pi_\varphi}{R_2} \sin\frac{\varphi}{R_1}\,, \tag{2}$$

$$S_z := S \cos\theta = S \sin\frac{\pi_\varphi}{R_2}\,, \tag{3}$$

where R_1, R_2 are certain dimensionful constants and $\varphi/R_1 \in [-\pi, \pi)$, $\pi_\varphi/R_2 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. For a field defined on the Minkowski background we have the limiting condition $R_1 R_2 = S$.



Phase space algebra

In the Minkowski case the symplectic form is given by the area 2-form

$$\omega_{\mathrm{M}} = S \sin\theta \, d\phi \wedge d\theta = \cos\frac{\pi_\varphi}{R_{2\mathrm{M}}} \, d\pi_\varphi \wedge d\varphi \,, \quad (4)$$

satisfying $\int_{S^2} \omega_{\mathrm{M}} = 4\pi S$. For the FRW background we introduce the gravitational field variable $q \equiv V_0 a^3$ (here a is the scale factor and V_0 a fiducial spatial volume) and its conjugate momentum p. Then we assume that the generalized symplectic total form is

$$\omega := dp \wedge dq + \cos\frac{\pi_\varphi}{R_2(q)} \, d\pi_\varphi \wedge d\varphi \,. \quad (5)$$

However, for ω to be a closed form we need $R_2(q) = R_2$. The corresponding Poisson bracket has the form

$$\{\cdot,\cdot\} = \left[\frac{\partial\cdot}{\partial q}\frac{\partial\cdot}{\partial p} - \frac{\partial\cdot}{\partial p}\frac{\partial\cdot}{\partial q}\right] + \frac{1}{\cos\frac{\pi_\varphi}{R_2}}\left[\frac{\partial\cdot}{\partial \varphi}\frac{\partial\cdot}{\partial \pi_\varphi} - \frac{\partial\cdot}{\partial \pi_\varphi}\frac{\partial\cdot}{\partial \varphi}\right]. \quad (6)$$

Hamiltonian from the Heisenberg model

The Hamiltonian of the continuous XXZ Heisenberg model coupled to a magnetic field **B** (for convenience, $\mathbf{B} := (B_x, 0, 0)$) has the form

$$H_{XXZ} = -\int d^3x \left(\tilde{J}\left((\nabla S_x)^2 + (\nabla S_y)^2 + \Delta(\nabla S_z)^2\right) + \tilde{\mu}\,\mathbf{B}\cdot\mathbf{S}\right), \quad (7)$$

where $\tilde{J}$, $\tilde{\mu}$ are coupling constants and Δ the anisotropy parameter. The homogeneous field corresponds to the term $\propto$ **B**, which we adapt to the FRW background multiplying the measure d^3x by Na^3, obtaining

$$\begin{aligned} H_{\mathbf{S}\mathrm{matmat}} &= -Nq\,\tilde{\mu}B_x S_x \quad (8)\\ &= Nq\left(-\tilde{\mu}B_x S + \frac{\tilde{\mu}B_x S}{2R_2^2}\pi_\varphi^2 + \frac{\tilde{\mu}B_x S}{2R_1^2}\varphi^2 + \mathcal{O}(\varphi^{4-n}\pi_\varphi^n)\right). \end{aligned}$$

The ordinary scalar field is recovered (up to a shift $\propto S$) in the limit $S \to \infty$ for the following identification of the model's parameters:

$$\tilde{\mu}B_x \equiv \frac{q_0 m}{q^2}, \qquad R_1 \equiv \frac{1}{q}\sqrt{\frac{Sq_0}{m}}, \qquad R_2 \equiv \sqrt{Sq_0 m}. \quad (9)$$

Total Hamiltonian

As the result, the matter Hamiltonian acquires the form

$$H_{\mathbf{S}\mathrm{mat}} = -Nm\frac{q_0}{q}S_x = Nq\left(-Sm\frac{q_0}{q^2} + \frac{\pi_\varphi^2}{2q^2} + \frac{1}{2}m^2\varphi^2 + \mathcal{O}(4)\right) . \quad (10)$$

The first term in the expansion will lead to a cosmic bounce, while the negative energy density that occurs for $S_x > 0$ should be balanced by some additional matter content. Nevertheless, in what follows we will use the positive-definite Hamiltonian

$$H_{\mathrm{mat}} := Nm\frac{q_0}{q}(S - S_x) \quad (11)$$

and then the total Hamiltonian is

$$H_{\mathrm{tot}} = H_{\mathrm{FRW}} + H_{\mathrm{mat}} , \quad H_{\mathrm{FRW}} = -\frac{3\kappa}{4}Nqp^2 , \quad (12)$$

where $\kappa \equiv 8\pi G$. It generates the constraint $\frac{\partial}{\partial N}H_{\mathrm{tot}} = 0$, equivalent to

$$m\frac{q_0}{q^2}(S - S_x) = \frac{3\kappa}{4}p^2 . \quad (13)$$

Friedmann equation

Introducing the Hubble factor $h \equiv \dot{q}/(3q)$, we now find that the Friedmann equation is given by (for the gauge $N = 1$)

$$h^2 = m\frac{q_0}{q^2}\,(S - S_x) \equiv \frac{\kappa}{3}\rho\,, \tag{14}$$

where ρ denotes the matter energy density. If we express it in terms of the energy density and pressure of an ordinary scalar field

$$\rho_\varphi := \frac{\pi_\varphi^2}{2q^2} + \frac{1}{2}m^2\varphi^2\,, \qquad P_\varphi := \frac{\pi_\varphi^2}{2q^2} - \frac{1}{2}m^2\varphi^2\,, \tag{15}$$

we may obtain corrections to the usual Friedmann equation

$$h^2 = \frac{\kappa}{3}\rho_\varphi - \frac{\kappa}{9}\frac{q^2}{Sq_0 m}\left(\rho_\varphi^2 - \frac{1}{2}P_\varphi^2\right) + \mathcal{O}(1/S^2)\,. \tag{16}$$

They become relevant for large q and then can trigger a recollapse.

Evolution equations

Our Hamiltonian $H_{\rm tot}$ also leads to the following equations for gravity

$$\dot{q} = -\frac{3\kappa}{2} Nqp\,, \qquad \dot{p} = \frac{3\kappa}{4} Np^2 + Nm\frac{q_0}{q^2}\left(S - S_x - S_y \arctan\frac{S_y}{S_x}\right) \tag{17}$$

and for the field

$$\dot{S}_x = \frac{3\kappa}{2} Np\, S_y \arctan\frac{S_y}{S_x}, \tag{18}$$

$$\dot{S}_y = Nm\, S_z - \frac{3\kappa}{2} Np\, S_x \arctan\frac{S_y}{S_x}, \qquad \dot{S}_z = -Nm\, S_y\,. \tag{19}$$

Alternatively, on the hemisphere $\varphi/R_1 \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $\pi_\varphi/R_2 \in (-\frac{\pi}{2}, \frac{\pi}{2})$ one may use the angular-like variables, which are governed by

$$\dot{\varphi} = \frac{NR_2}{q} \tan\frac{\pi_\varphi}{R_2} \cos\frac{\varphi}{R_1}\,, \qquad \dot{\pi}_\varphi = -NR_1\, qm^2 \sin\frac{\varphi}{R_1}\,. \tag{20}$$

Hamiltonian for the inhomogeneous field

In this case let us restrict to the regime of $S \to \infty$ but with $\Delta \neq 0$. Then our matter field Hamiltonian becomes (we also choose $N = a$)

$$\begin{aligned} H_\varphi &= \int d^3x\, \mathcal{H}_\varphi \\ &= \int d^3x\, a^4 \left[\frac{\pi_\varphi^2}{2a^6} + \frac{(\nabla\varphi)^2}{2a^2} + \frac{1}{2} m^2 \varphi^2 + \frac{\Delta}{2m^2 a^8} (\nabla \pi_\varphi)^2 \right] . \end{aligned} \tag{21}$$

Changing the variables to $v := a\,\varphi$ and $\pi_v := \partial\mathcal{L}_v / \partial v'$ we can derive

$$\begin{aligned} \mathcal{H}_v = \frac{\pi_v^2}{2} + \frac{(\nabla v)^2}{2} + \frac{1}{2} m_{\text{eff}}^2 v^2 + \frac{\Delta}{2m^2 a^2} (\nabla \pi_v - \mathfrak{h}\, \nabla v)^2 \\ + \mathcal{O}(\Delta^2)\,, \end{aligned} \tag{22}$$

where $m_{\text{eff}}^2 \equiv m^2 a^2 - a''/a$ is the effective mass, $\mathfrak{h} \equiv a'/a$ the conformal Hubble factor and we make an expansion around $\Delta = 0$.

Quantum field operators

Since in the considered limit $S \to \infty$ we simply have

$$\{\varphi(\mathbf{x}), \pi_\varphi(\mathbf{y})\} = \frac{\delta^{(3)}(\mathbf{x}-\mathbf{y})}{\cos(\pi_\varphi(\mathbf{x})/R_2)} \longrightarrow \delta^{(3)}(\mathbf{x}-\mathbf{y})\,, \tag{23}$$

the standard quantization can be applied, leading to

$$[\hat{v}(\mathbf{x}), \hat{\pi}_v(\mathbf{y})] = i\delta^{(3)}(\mathbf{x}-\mathbf{y})\,\hat{\mathbb{I}}\,. \tag{24}$$

Furthermore, we Fourier expand the field operators

$$\hat{v}(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^{3/2}}\, e^{i\mathbf{k}\cdot\mathbf{x}}\,\hat{v}_{\mathbf{k}}\,, \qquad \hat{\pi}_v(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^{3/2}}\, e^{i\mathbf{k}\cdot\mathbf{x}}\hat{\pi}_{v\mathbf{k}} \tag{25}$$

and decompose their modes in the basis of creation and annihilation operators, satisfying $[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{q}}] = \delta^{(3)}(\mathbf{k}-\mathbf{q})$, ($\tau$ is the conformal time)

$$\hat{v}_{\mathbf{k}}(\tau) = f_k(\tau)\,\hat{a}_{\mathbf{k}} + f_k^*(\tau)\,\hat{a}^\dagger_{-\mathbf{k}}\,, \tag{26}$$

$$\hat{\pi}_{v\mathbf{k}}(\tau) = g_k(\tau)\,\hat{a}_{\mathbf{k}} + g_k^*(\tau)\,\hat{a}^\dagger_{-\mathbf{k}}\,. \tag{27}$$

Dynamics of mode functions

Next we may write down the (symmetrized) quantum Hamiltonian

$$\hat{H}_v = \mathcal{O}(\Delta^2) + \frac{1}{4}\int d^3k \left(1 + \frac{\Delta k^2}{m^2 a^2}\right)\left(\hat{\pi}_{v\mathbf{k}}\hat{\pi}^\dagger_{v\mathbf{k}} + \hat{\pi}^\dagger_{v\mathbf{k}}\hat{\pi}_{v\mathbf{k}}\right)$$
$$+ \frac{1}{4}\int d^3k \left(\omega_k^2 + \frac{\Delta k^2}{m^2 a^2}\mathfrak{h}^2\right)\left(\hat{v}_{\mathbf{k}}\hat{v}^\dagger_{\mathbf{k}} + \hat{v}^\dagger_{\mathbf{k}}\hat{v}_{\mathbf{k}}\right)$$
$$- \frac{1}{4}\int d^3k \frac{\Delta k^2}{m^2 a^2}\mathfrak{h}\left(\hat{v}_{\mathbf{k}}\hat{\pi}^\dagger_{v\mathbf{k}} + \hat{v}^\dagger_{\mathbf{k}}\hat{\pi}_{v\mathbf{k}} + \hat{\pi}_{v\mathbf{k}}\hat{v}^\dagger_{\mathbf{k}} + \hat{\pi}^\dagger_{v\mathbf{k}}\hat{v}_{\mathbf{k}}\right), \quad (28)$$

with $\omega_k^2 \equiv k^2 + m_{\rm eff}^2$. It determines the evolution equations of $\hat{v}_{\mathbf{k}}$ and $\hat{\pi}_{v\mathbf{k}}$, which together give us equations for mode functions:

$$f_k'' + 2\mathfrak{h}\frac{\Delta k^2}{m^2 a^2} f_k' + \left[\omega_k^2 + \frac{\Delta k^2}{m^2 a^2}\left(k^2 + m^2 a^2 - 2\mathfrak{h}^2\right)\right] f_k = 0\,, \quad (29)$$

as well as

$$g_k = f_k' + \frac{\Delta k^2}{m^2 a^2}(\mathfrak{h}\, f_k - f_k') + \mathcal{O}(\Delta^2)\,. \quad (30)$$

Vacuum state normalization

Therefore, the Wronskian condition also becomes modified

$$f_k (f_k^*)' - f_k^* f_k' = i\left(1 + \frac{\Delta k^2}{m^2 a^2}\right) + \mathcal{O}(\Delta^2)\,. \qquad (31)$$

We now calculate that energy of the initial ground state is given by

$$\langle 0|\hat{H}_v|0\rangle = \frac{1}{2}\delta^{(3)}(0)\int d^3k\, E_k\,,$$

$$E_k \equiv \left(1 + \frac{\Delta k^2}{m^2 a^2}\right)|g_k|^2 + \left(\omega_k^2 + \frac{\Delta k^2}{m^2 a^2}\mathfrak{h}^2\right)|f_k|^2 + \frac{2\Delta k^2}{m^2 a^2}\mathfrak{h}\, f_k g_k^*\,. \quad (32)$$

The form of f_k can be found by applying the decomposition $f_k = r_k e^{i\alpha_k}$ and looking for a minimum of E_k. In particular, for such k that $k^2 \gg m^2_{\rm eff}$ and $\frac{\Delta k^2}{m^2 a^2} \ll 1$ we obtain the corrected Bunch-Davies vacuum

$$f_k = \frac{e^{-ik\tau}}{\sqrt{2k}}\left[1 + \frac{\Delta k^2}{m^2 a^2}\left(\frac{1}{4} - ika^2\int\frac{d\tau}{a^2}\right)\right] + \mathcal{O}(\Delta^2)\,. \qquad (33)$$

Spectrum of perturbations

Simple quantum correlations are captured by a two-point function

$$\langle 0|\hat{\varphi}(\mathbf{x},\tau)\hat{\varphi}(\mathbf{y},\tau)|0\rangle = \int_0^\infty dk\, \frac{\sin(k|\mathbf{x}-\mathbf{y}|)}{k^2|\mathbf{x}-\mathbf{y}|}\mathcal{P}_\varphi(k,\eta)\,, \tag{34}$$

where $\mathcal{P}_\varphi(k,\tau) \equiv \frac{1}{2\pi^2}k^3|f_k(\tau)/a(\tau)|^2$ is the power spectrum. In particular, in the de Sitter regime (i.e. for $a = -(h\tau)^{-1}$) it simplifies to

$$\mathcal{P}_\varphi(k,\tau) = \left(\frac{h}{2\pi}\right)^2 x^2\left(1+\frac{\Delta}{6\eta}x^2\right) + \mathcal{O}(\Delta^2), \tag{35}$$

with $x \equiv -k\tau$. Consequently, the spectral index is found to be

$$n_S := \frac{d\ln\mathcal{P}_\varphi(x=1)}{d\ln k} = 0\,. \tag{36}$$

Work in progress and outlook

- Joint treatment for the field's background and perturbations
- Calculation of the tensor to scalar ratio
- Analysis of the phase space trajectories
- Quantum theory in the case of the finite size of phase space
- Investigation of relations with condensed matter physics
- ...