From hyperheavenly spaces to complex and real, twisting type $[\mathrm{N}] \otimes [\mathrm{N}]$ spaces

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Introduction

Why type $[\mathrm{N}]$ is so interesting in General Theory of Relativity?

- Peeling Theorem and possible relation between type $[\mathrm{N}]$ and gravitational waves

$$C_{abcd} = \frac{[\mathrm{N}]}{\lambda} + \frac{[\mathrm{III}]}{\lambda^2} + \frac{[\mathrm{II}]}{\lambda^3} + \frac{[\mathrm{I}]}{\lambda^4} + O\left(\frac{1}{\lambda^5}\right)$$

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Type $[\mathrm{N}]$ in Lorentzian geometry

Let $C_{ABCD} = C_{(ABCD)}$ be a spinorial image of the SD part of the Weyl tensor. According to the *Penrose Theorem* it can be decomposed into product of 1-index spinors m_A, n_A, r_A and s_A which are called *undotted Penrose spinors*.

$$C_{ABCD} = m_{(A} n_B r_C s_{D)}$$

We say that the spacetime is of the type $[\mathrm{N}]$ if

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In Lorentzian geometry *dotted Penrose spinors* $m_{\dot{A}} = \overline{m_A}$, so

$$C_{\dot{A}\dot{B}\dot{C}\dot{D}} = m_{\dot{A}} m_{\dot{B}} m_{\dot{C}} m_{\dot{D}} = \overline{C_{ABCD}}$$

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Congruences of the null geodesics

Consider the null and geodesic vector field K_a in affine parametrization. The optical properties of such family of the null lines in the null tetrad $(e^1, e^2 = \overline{e^1}, e^3, e^4)$ are described by three parameters

$$\begin{aligned} \text{expansion:}\ \Theta &:= \frac{1}{2}\nabla^a K_a \\ \text{twist:}\ \tau^2 &:= \frac{1}{2}\nabla_{[a}K_{b]}\,\nabla^a K^b \\ \text{shear:}\ \sigma\overline{\sigma} &:= \frac{1}{2}\nabla_{(a}K_{b)}\,\nabla^a K^b - \Theta^2 \end{aligned}$$

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Goldberg - Sachs Theorem

Theorem (*Goldberg-Sachs Theorem*, Goldberg, Sachs, 1962)

In Einstein spaces the following statements are equivalent

- *space admits a shearfree null geodesic congruence*
- *Weyl tensor is algebraically degenerate*

Different classes of the type $[\mathrm{N}]$ metrics

There are 3 vacuum classes of Lorentzian type $[\mathrm{N}]$ metrics

- Kundt class (nontwisting, nonexpanding, pp-waves as a special subclass)
- Robinson - Trautman class (nontwisting, expanding)
- Twisting class. The only known explicit solution is *Houser solution* which is equipped with two symmetries (one Killing vector, one homothetic Killing vector)

Killing equations: $\nabla_{(a}K_{b)} = \chi_0 g_{ab}$
$\chi_0 = 0$ - K_a is Killing vector
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- The results are valid in 4-dimensional spaces with the neutral signature metric $(++--)$

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Complex counterpart of the Lorentzian type [N]

In complex geometry there is no relation between undotted and dotted Penrose spinors, so there exist spaces of the "mixed" types, like $[\mathrm{N}] \otimes [\mathrm{D}]$.

Theorem (*Rózga Theorem*, Rózga, 1977)

Lorentzian slice of the complex space exists only if SD and ASD Weyl spinors are of the same Petrov-Penrose types.

Lorentzian geometry: type $[\mathrm{N}]$
Complex geometry: type $[\mathrm{N}] \otimes [\mathrm{N}]$

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Congruence of the SD null strings

Consider 2-dimensional SD distribution $\mathcal{D} = \{m_A a_{\dot{B}}, m_A b_{\dot{B}}\}$, $a_{\dot{A}} b^{\dot{A}} \neq 0$. It is integrable in the Frobenius sense, if

$$m^A m^B \nabla_{A\dot{M}} m_B = 0 \tag{1}$$

Equations (1) are called *SD null string equations*. The integral manifolds of the distribution $\mathcal{D}$ are 2-dimensional, holomorphic, totally null and geodesics surfaces, called *null strings*. Their family constitutes *the congruence of the SD null strings*.

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Congruence of the SD null strings

From SD null strings equations we find

$$\nabla_{A\dot{M}} m_B = m_B Z_{A\dot{M}} + \in_{AB} M_{\dot{M}}$$

Spinor field $M_{\dot{M}}$ is called *expansion of the congruence*.

- $M_{\dot{M}} = 0$ - *nonexpanding congruence*.
- $M_{\dot{M}} \neq 0$ - *expanding congruence*.

Nonexpanding congruence = distribution $\mathcal{D}$ is parallely propagated:

$$\nabla_V X \in \mathcal{D} \text{ for any vector field } V \text{ and any vector field } X \in \mathcal{D}$$

Spaces which admit nonexpanding congruence of SD null strings are called *Walker spaces*.

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Generalized Goldberg - Sachs Theorem

Theorem (*Generalized Goldberg-Sachs Theorem*, Plebański, Hacyan, 1975)

In complex Einstein spaces the following statements are equivalent

- *space admits a congruence of SD null strings generated by the spinor* m^A
- *SD Weyl spinor is algebraically degenerate and spinor* m^A *is a multiple Penrose spinor*

$$C_{ABCD} = m_{(A} m_B n_C s_{D)}$$

Properties of the intersection of the SD and ASD congruences of the null strings

Consider the space which admits both SD and ASD congruences of the null strings. Then

$$M_{\dot{A}} \quad - \quad \text{expansion of the SD congruence of the null strings}$$
$$M_{A} \quad - \quad \text{expansion of the ASD congruence of the null strings}$$

Intersection of these congruences constitutes the congruence of the complex, null geodesics. It is given by the vector field $K_a \sim m_A m_{\dot{B}}$. Define *expansion* and *twist* by the formulas

$$\theta := \frac{1}{2}\nabla^a K_a \quad \sim \quad m_A M^A + m_{\dot{A}} M^{\dot{A}}$$
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There are three classes of the type $[\mathrm{N}] \otimes [\mathrm{N}]$ spaces

- Type $[\mathrm{N}]^n \otimes [\mathrm{N}]^n$ - then $\theta = \tau = 0$.
- Type $[\mathrm{N}]^n \otimes [\mathrm{N}]^e$ or $[\mathrm{N}]^e \otimes [\mathrm{N}]^n$ - such spaces do not admit real Lorentzian slices.
- Type $[\mathrm{N}]^e \otimes [\mathrm{N}]^e$

Real Lorentzian spaces of the type $[\mathrm{N}]$ with nonzero twist are contained in complex spaces of the type $[\mathrm{N}]^e \otimes [\mathrm{N}]^e$ equipped with expanding SD and ASD congruences of the null strings.

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Hyperheavenly spaces - definition

Definition

Hyperheavenly space ($\mathcal{HH}$-space) is a 4-dimensional complex analytic differential manifold equipped with a holomorphic metric ds^2 which satisfies the vacuum Einstein equations and such that the self-dual part of the Weyl tensor is algebraically degenerate.

Hyperheavenly spaces - the metric

The metric of the Einstein type $[\mathrm{N}] \otimes [\mathrm{any}]$ spaces can be brought to the form [Plebański, Robinson, 1976]

$$\begin{aligned} ds^2 &= 2\phi^{-2}\{(d\eta dw - d\phi dt) - \phi\, W_{\eta\eta}\, dt^2 \\ &\qquad + (2W_\eta - 2\phi\, W_{\eta\phi})\, dwdt + (2W_\phi - \phi\, W_{\phi\phi})\, dw^2\} \end{aligned}$$

where (ϕ, η, w, t) are local coordinates called *Plebański - Robinson - Finley coordinates*, function $W = W(\phi, \eta, w, t)$ is *the key function*, which satisfies *the hyperheavenly equation*

$$\begin{aligned} & W_{\eta\eta}W_{\phi\phi} - W_{\eta\phi}W_{\eta\phi} + 2\phi^{-1}W_\eta W_{\eta\phi} - 2\phi^{-1}W_\phi W_{\eta\eta} \\ & + \phi^{-1}(W_{w\eta} - W_{t\phi}) = \gamma \end{aligned}$$

$\gamma = \gamma(w, t)$ is an arbitrary function such that $\gamma_t \neq 0$.

Hyperheavenly spaces - the ASD curvature

$C_{\dot{A}\dot{B}\dot{C}\dot{D}}$ is of the type $[\mathrm{N}]$ with nonzero twist, if $W_{\phi\phi\phi\phi} \neq 0$, $W_{\eta\eta\eta\eta} \neq 0$ and

$$\begin{aligned} W_{\eta\eta\eta\phi} &= hW_{\eta\eta\eta\eta} \\ W_{\eta\eta\phi\phi} &= hW_{\eta\eta\eta\phi} \\ W_{\eta\phi\phi\phi} &= hW_{\eta\eta\phi\phi} \\ W_{\phi\phi\phi\phi} &= hW_{\eta\phi\phi\phi} \end{aligned} \tag{2}$$

where $h = h(\phi, \eta, w, t)$.

Hyperheavenly spaces - the ASD curvature

Integrability conditions of the set (2) imply

$$h_\phi = hh_\eta$$

with solution

$$\eta + \phi h = f(h, w, t)$$

where $f = f(h, w, t)$ is an arbitrary function.
It suggests coordinate transformation $\eta \to h$.

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Hyperheavenly spaces - the key function for the types $[\mathrm{N}] \otimes [\mathrm{N}]$

The key function for the spaces $[\mathrm{N}]^e \otimes [\mathrm{N}]^e$ in coordinates (ϕ, h, w, t) is the third order polynomial in ϕ. It reads

$$W = -F\phi^3 + \frac{1}{2}(R - 2hS + h^2\Omega)\phi^2 + (B - Ah)\phi + C$$

where $F = F(h, w, t)$ and $f = f(h, w, t)$ are arbitrary functions and

$$\Omega := \int \dot{f}\dddot{F} dh, \quad S := \int \dot{f}(h\dddot{F} - \ddot{F}) dh, \quad R := \int \dot{f}(h^2\dddot{F} - 2h\ddot{F} + 2\dot{F}) dh$$

$$A := \int \dot{f} \int \dot{f}\dddot{F} dhdh, \quad B := \int \dot{f} \int \dot{f}(h\dddot{F} - \ddot{F}) dhdh$$

$$C := \int \dot{f} \int \dot{f} \int \dot{f}\dddot{F} dhdhdh, \quad \dot{f} \equiv \frac{df}{dh}, \quad \textit{etc.}$$

Hyperheavenly equation for the types $[\mathrm{N}] \otimes [\mathrm{N}]$

Putting the key function into the hyperheavenly equation we obtain the following set

$$
\begin{aligned}
&(R + h^2\Omega - 2hS)\ddot{F} + (2S - 2h\Omega)\dot{F} - h\dot{F}_t + 3F_t + \dot{F}_w = 0 \\
&S^2 - \Omega R + 4A\dot{F} - 2hA\ddot{F} + 2B\ddot{F} - R_t + hS_t - f_t(h\ddot{F} - 2\dot{F}) \\
&\qquad +S_w - h\Omega_w + f_w\ddot{F} = \gamma \\
&2SA - 2\Omega B - B_t + f_tS + A_w - f_w\Omega = 0
\end{aligned}
$$

It is overdetermined system of three equations for two functions $F(h, w, t)$ and $f(h, w, t)$ of three variables.

Symmetries in hyperheavenly spaces

It has been proved [Sonnleitner A., Finley J.D. III (1982), A.C (2013)] that in hyperheavenly spaces ten Killing equations can be reduced to the single *master equation*. For the hyperheavenly spaces of the type $[\mathrm{N}] \otimes [\mathrm{any}]$ with $\Lambda = 0$ this equation reads

$$\begin{aligned} K(W) &= -(4\chi_0 + 2a_w - 3b_t)W + \alpha\phi^3 + \frac{1}{2}(\epsilon_w \phi + \epsilon_t \eta) + \beta \\ &\quad + \frac{1}{2}\Big(- b_{ww}\phi^2 - b_{tt}\eta^2 + (a_{ww} - 2b_{tw})\eta\phi \Big) \end{aligned}$$

where vector K has the form

$$\begin{aligned} K &= a\frac{\partial}{\partial w} + b\frac{\partial}{\partial t} + (b_t - 2\chi_0)\phi\frac{\partial}{\partial \phi} \\ &\quad + \Big((2b_t - a_w - 2\chi_0)\eta + b_w\phi - \epsilon \Big)\frac{\partial}{\partial \eta} \end{aligned}$$

where $a = a(w)$, $b = b(w,t)$, $\epsilon = \epsilon(w,t)$, $\beta = \beta(w,t)$, $\alpha = \alpha(w,t)$.

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Killing vector

There are two different types of the Killing vectors in hyperheavenly spaces of the type $[\mathrm{N}] \otimes [\mathrm{N}]$ with $\Lambda = 0$

- ∂_η (in this case congruence of the null complex geodesics is nontwisting)
- ∂_w

Let us equip hyperheavenly space of the type $[\mathrm{N}] \otimes [\mathrm{N}]$ with symmetry

$$K^{(1)} = \frac{\partial}{\partial w}$$

then $F = F(h,t)$, $f = f(h,t)$ and $\gamma = \gamma(t)$.

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then $F = F(h,t)$, $f = f(h,t)$ and $\gamma = \gamma(t)$.

Killing vector

There are two different types of the Killing vectors in hyperheavenly spaces of the type $[\mathrm{N}] \otimes [\mathrm{N}]$ with $\Lambda = 0$

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Homothetic Killing vector

With the symmetry given by ∂_w, the homothetic Killing vector $K^{(2)}$ can be brought to the form

$$K^{(2)} = w \frac{\partial}{\partial w} + t \frac{\partial}{\partial t} + (1 - 2\chi_0)\phi \frac{\partial}{\partial \phi} + (1 - 2\chi_0)\eta \frac{\partial}{\partial \eta}$$

Further steps

The next steps are:

- Solve the master equation for the homothetic Killing vector $K^{(2)}$
- Insert the solution into the set of field equations - we obtain the set of four equations for three functions of one variable
- One of the equations is an identity, so the set of the field equations is not overdetermined anymore

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The metric

Finally we arrive at the metric

$$\begin{aligned} ds^2 &= 2\phi^{-2}\Bigg\{ \left(t^{1-2\chi_0} - \phi\frac{\mathrm{d}h}{\mathrm{d}v}\right)\mathrm{d}v\mathrm{d}w - h\,\mathrm{d}\phi\mathrm{d}w - \mathrm{d}\phi\mathrm{d}t \\ &\quad - \left(\phi t^{-1}\left(\frac{\mathrm{d}T}{\mathrm{d}v} - \frac{1-2\chi_0}{2}\right) - \phi^2 t^{2\chi_0-2}\left(h\frac{\mathrm{d}^2T}{\mathrm{d}v^2} - \frac{\mathrm{d}^2Z}{\mathrm{d}v^2}\right)\right)\mathrm{d}t^2 \\ &\quad +2\Bigg(t^{-2\chi_0}T - \phi h t^{-1}\left(\frac{\mathrm{d}T}{\mathrm{d}v} - \frac{1-2\chi_0}{2}\right) \\ &\qquad + \frac{1}{2}\phi^2 t^{2\chi_0-2}\left(h^2\frac{\mathrm{d}^2T}{\mathrm{d}v^2} - \frac{\mathrm{d}P}{\mathrm{d}v}\right)\Bigg)\mathrm{d}w\mathrm{d}t \\ &\quad +\Bigg(2t^{-2\chi_0}Z + \phi t^{-1}\left(P - 2h\frac{\mathrm{d}Z}{\mathrm{d}v}\right) \\ &\qquad +\phi^2 t^{2\chi_0-2}h\left(h\frac{\mathrm{d}^2Z}{\mathrm{d}v^2} - \frac{\mathrm{d}P}{\mathrm{d}v}\right)\Bigg)\mathrm{d}w^2\Bigg\} \end{aligned}$$

where (ϕ, v, w, t) are local coordinates, $T = T(v)$, $Z = Z(v)$, $P = P(v)$. Moreover, $h = Z'''/T'''$, where $Z' \equiv \frac{dZ}{dv}$, etc.

Equations

Functions $T = T(v)$, $Z = Z(v)$, $P = P(v)$ have to satisfy the set of equations

$$T\frac{\mathrm{d}Z}{\mathrm{d}v} - Z\frac{\mathrm{d}T}{\mathrm{d}v} + \frac{1}{2}Z = 0 \tag{3a}$$

$$2T\frac{\mathrm{d}P}{\mathrm{d}v} - P\left(\frac{\mathrm{d}T}{\mathrm{d}v} + \frac{2\chi_0 - 3}{2}\right) - 2Z\frac{\mathrm{d}^2Z}{\mathrm{d}v^2} + \left(\frac{\mathrm{d}Z}{\mathrm{d}v}\right)^2 = \gamma_0 \tag{3b}$$

$$\left(\frac{\mathrm{d}^3Z}{\mathrm{d}v^3}\right)^2 = \frac{\mathrm{d}^3T}{\mathrm{d}v^3}\frac{\mathrm{d}^2P}{\mathrm{d}v^2} \tag{3c}$$

Solutions of the equations (3a) and (3b) are simple

$$Z(v) = \frac{1}{Q'}\ , \quad T(v) = \frac{1}{2}\frac{Q}{Q'}\ , \quad Q' \equiv \frac{dQ}{dv}$$

$$P(v) = Q^{\chi_0 - 1}Q'^{-\frac{1}{2}}\int Q^{-\chi_0}\left(3Q'^{-\frac{5}{2}}Q''^2 - 2Q'^{-\frac{3}{2}}Q''' + \gamma_0 Q'^{\frac{3}{2}}\right)\mathrm{d}v$$

where $Q = Q(v)$.

Equations

Equation (3c) becomes extremely complicated ODE of the fifth order

$$\begin{aligned}
&(4(\chi_0-2)(\chi_0-1)+\mu Q^2 Z^3)\left\{-2\mu^2 Z^3 Z'''' + 2\mu Z^2 Z'''(Q' Z''' + \mu' Z + \mu Z')\right.\\
&\qquad\left. +2Z' Z'''(\mu Z + Q Z''')^2 - Z'''^2(\mu Z + Q Z''')(Q Z' + 2\chi_0)\right\}\\
&-(Q Z' - 2\chi_0 + 4)(\mu Z + Q Z''')^3\left(\mu Z^3 + \gamma_0\right) + 4Q\mu Z^3 Z'''(\mu Z + Q Z''')^2 = 0
\end{aligned}$$

where

$$\mu(v) := \frac{3Q''^2}{Q'} - 2Q''', \ Z(v) = \frac{1}{Q'}$$

Concluding Remarks

Disadvantages of our approach

- No new solutions have been found so far (most promising case is $\chi_0 = 2$)
- Houser solution has not been reconstructed so far
- No transformation which reduce the order of the final differential equation has been found so far

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- Final equation is ODE and it can be written in the form

$$Q''''' = G(Q, Q', Q'', Q''', Q'''')$$

  with G being the rational function. It always has solution for arbitrary initial values. It works in complex case, real Lorentzian case and real neutral case.
- We formulated the theorem which is complex counterpart of the theorem formulated by W.D. Halford (1979) and C.D. Collinson (1969, 1980)

Theorem

For any vacuum $\mathcal{HH}$-spaces of the type $[N] \otimes [II,D,III,N]$ with twisting congruence of null geodesics arising as intersection of SD null strings with ASD null strings there exist at most two homothetic Killing vectors. They must be noncommuting.

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Concluding Remarks

- The form of the key function is valid for any spaces for which ASD Weyl spinor is of the type $[\mathrm{N}]$. Such key function can be used in neutral geometry (for example, the problem of the Einstein, para-Hermite spaces of the type $[\mathrm{D}]^{ee} \otimes [\mathrm{N}]^{e}$)