# Dirac Fields in Hybrid LQC



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- Any fundamental theory includes **fermionic** fields.
- It is interesting to incorporate fermions in analyses of the **Early Universe** and of possible effects on the primordial perturbations.
- We want to extend the **hybrid LQC** formalism including Dirac fields. This will put to the **test** the very own consistency of the hybrid approach.
- We want to obtain a framework to discuss the effects of quantum geometry on realistic quantum matter fields.
- In this way we can explore issues such as the definition of a vacuum, the recovery of QFT, and backreaction.

#### Unperturbed model

 We start with a FLRW model with flat compact sections (threetori).

$$ds^{2} = \frac{4\pi}{3} \left[ -\left(\frac{4\pi e^{3\alpha(t)}}{3}\right)^{2} N_{0}^{2}(t) dt^{2} + e^{2\alpha(t)} h_{ij} d\theta_{i} d\theta_{j} \right].$$

We include a scalar field (the **inflaton**) subject to a potential  $W(\phi)$ .

- The phase space can be described with two canonical pairs:
  - 1) ( $\phi$ ,  $\pi_{\phi}$ ) for the inflaton.

#### Unperturbed model

2) (v, b) for the **FLRW geometry**, adopting the usual description in LQC, with  $\{b, v\}=2$  and



The sign of v determines the orientation of the triad.

The **volume** of the homogeneous sections is  $V = 2\pi \gamma \Delta_g^{1/2} |v|$ .

• The system is subject to a (rescaled) Hamiltonian constraint:

$$H_{0} = \frac{1}{2} \left( \pi_{\phi}^{2} - H_{0}^{(2)} \right), \qquad H_{0}^{(2)} = 3 \pi (vb)^{2} - 2 V^{2} W(\phi).$$

- We perturb the geometry and the inflaton, and truncate the action at second perturbative order.
- Using spatial, vector, and tensor harmonics, constructed from the Laplace-Beltrami operator on the spatial sections, we expand the perturbations in modes.
- **Zero-modes** are treated exactly at second perturbative order.
- In this perturbative scheme, the total system is a **constrained** system with a **canonical** structure.

 Linear perturbative constraints generate perturbative diffeomorphisms. Only perturbative quantities not affected by these transformations are physical:

GAUGE INVARIANTS.

- **Tensor perturbations** are gauge invariants.
- The **Mukhanov-Sasaki invariant** is related to the comoving curvature (*scalar*) perturbations. Its **momentum** can be chosen proportional to the time derivative.
- One can find **momenta** for the linear perturbative constraints, that commute with the gauge invariants.

 In all these considerations, the **background** variables (zeromodes) had been kept fixed.
 [Langlois]

- The variables for the perturbations can be completed into a canonical set for the whole system. [Pinto-Neto]
- Zero-modes are corrected with a fixed quadratic contribution of the perturbations.

The **corrected zero-modes** are the genuine free (*background*) variables.

 This correction of the zero-modes modifies the quadratic perturbative contribution to the global Hamiltonian constraint.

The resulting global Hamiltonian constraint is a gauge invariant.

- This **quadratic perturbative contribution**, additional to the Hamiltonian of the homogeneous sector, equals the Mukhanov-Sasaki Hamiltonian  ${}^{MS}H_2$  plus the tensor one  ${}^{T}H_2$ .
- The rest of the total Hamiltonian is a sum of linear perturbative constraints, with redefined Lagrange multipliers.

• We introduce a **massive Dirac field**  $\Psi$ :

$$S_{D} = \int d^{4}x \sqrt{|g|} \left[ i M \Psi^{\dagger} \gamma^{0} \Psi - \frac{1}{2} \left( i \Psi^{\dagger} \gamma^{0} e^{\mu}_{a} \gamma^{a} \nabla^{S}_{\mu} \Psi + Hermitian \ conj. \right) \right].$$
  
Mass Tetrad Dirac matrices

Connection

At our truncation order the Dirac field, regarded as a perturbation, couples directly with the (corrected) **FLRW geometry**.

- Adopting the Weyl representation for the Dirac matrices, we can describe the Dirac field by a pair of two-component spinors of definite chirality  $\varphi_A$ ,  $\overline{\chi}_{A'}(A, A'=1,2)$ , that are **Grassmann** variables.
- In the internal time gauge  $e_0^a = 0$   $(a \neq 0)$ , the gauge group reduces to SU(2).

 We expand the spinors in eigenmodes of the Dirac operator on the spatial sections, with <u>time-dependent</u> anticommuting coefficients:



 $\vec{\tau}$  may be any of the vertices of the cube with side  $\frac{1}{2}$ .

Eigenmode expansion:

$$\phi_{A}(x) = e^{-\frac{3\alpha}{2}} \left(\frac{3}{4\pi}\right)^{3/4} \sum_{\vec{k},(\pm)} \left[m_{\vec{k}} w_{A}^{\vec{k},(+)} + \overline{r}_{\vec{k}} w_{A}^{\vec{k},(-)}\right], \qquad \text{Same helicity} \\ \bar{\chi}_{A'}(x) = e^{-\frac{3\alpha}{2}} \left(\frac{3}{4\pi}\right)^{3/4} \sum_{\vec{k},(\pm)} \left[\overline{s}_{\vec{k}} \overline{w}_{A'}^{\vec{k},(+)} + t_{\vec{k}} \overline{w}_{A'}^{\vec{k},(-)}\right].$$

**Eigenvalues**:  $+\omega_k = 2\pi |\vec{k} + \vec{\tau}|$ , each with **degeneracy**  $g_k = O(\omega_k^2)$ .

Let us use the same annihilation and creation variables as **D'Eath** & Halliwell. For nonzero-modes, and  $(x, y)=(m, s) \circ (t, r)$ :

$$a_{\vec{k}}^{(x,y)} = \sqrt{\frac{\xi_k - \omega_k}{2\xi_k}} x_{\vec{k}} + \sqrt{\frac{\xi_k + \omega_k}{2\xi_k}} \overline{y}_{-\vec{k}}, \qquad \overline{b}_{\vec{k}}^{(x,y)} : \omega_k \to -\omega_k; \qquad \xi_k = \sqrt{\omega_k^2 + M^2 V^{2/3}}.$$
Particle annihilation
Antiparticle creation

• Variables: (x, y) = (m, s) or (t, r).

$$a_{\vec{k}}^{(x,y)} = \sqrt{\frac{\xi_k - \omega_k}{2\xi_k}} x_{\vec{k}} + \sqrt{\frac{\xi_k + \omega_k}{2\xi_k}} \overline{y}_{-\vec{k}}, \quad \overline{b}_{\vec{k}}^{(x,y)} : \omega_k \to -\omega_k; \quad \xi_k = \sqrt{\omega_k^2 + M^2 V^{2/3}}.$$

This choice provides an **instantaneous diagonalization** of the Dirac Hamiltonian.

- The choice is **unique** up to unitary transformations if:
- The FLRW background is treated classically.
- The dynamics of these variables must be unitarily implementable on Fock space.
- The Fock vacuum must be invariant under the Killing isometries of the spatial sections and the spin rotations generated by the helicity.
- The convention of particles and antiparticles must connect smoothly in the massless limit with the standard one.

#### Fermionic perturbations

 The D'Eath & Halliwell variables are volume dependent. Hence, the FLRW geometric momentum must be corrected to maintain the canonical structure:

$$b \to b + i \frac{M \omega_k V^{1/3}}{3 \xi_k^2 v} \sum_{(x, y), \vec{k}} \left( a_{\vec{k}}^{(x, y)} b_{\vec{k}}^{(x, y)} + \overline{a}_{\vec{k}}^{(x, y)} \overline{b}_{\vec{k}}^{(x, y)} \right)$$

Once this volume dependence is taken into account, the contribution of the fermionic nonzero-modes to the global **Hamiltonian constraint** becomes

$$\sum_{\vec{k}} H_{D}^{\vec{k}} = \sum_{(x,y),\vec{k}} \frac{\xi_{k} V^{2/3}}{2} (\bar{a}_{\vec{k}}^{(x,y)} a_{\vec{k}}^{(x,y)} - a_{\vec{k}}^{(x,y)} \bar{a}_{\vec{k}}^{(x,y)} + \bar{b}_{\vec{k}}^{(x,y)} b_{\vec{k}}^{(x,y)} - b_{\vec{k}}^{(x,y)} \bar{b}_{\vec{k}}^{(x,y)}) + 2\pi i \sum_{(x,y),\vec{k}} \frac{M \omega_{k} V^{1/3}}{2\xi_{k}^{2}} vb (a_{\vec{k}}^{(x,y)} b_{\vec{k}}^{(x,y)} + \bar{a}_{\vec{k}}^{(x,y)} \bar{b}_{\vec{k}}^{(x,y)}).$$
Instantaneous diagonalization
Particle production

### Hybrid quantization

- For the geometry zero-modes (v, b) we adopt the **polymeric** representation of LQC, with **superselection** of volume states.
- For the inflaton, a conventional **Schrödinger** representation.
- For the Mukhanov-Sasaki field, the tensor perturbations, and the Dirac field, a Fock representation (selected by unitarity criteria).
- The linear perturbative constraints imply that physical states depend only on zero-modes and gauge invariants.
- Only one relevant constraint remains: the global Hamiltonian one.

$$\boldsymbol{H}_{T} = \boldsymbol{H}_{0} + {}^{MS}\boldsymbol{H}_{2} + {}^{T}\boldsymbol{H}_{2} + \sum_{\vec{k}} \boldsymbol{H}_{D}^{\vec{k}}.$$

### Hybrid quantization: LQC

- In **Loop Quantum Cosmology**, the canonical variables can be chosen as the volume variable v, proportional to the cube of the scale factor, and the scaled **connection**  $b \propto \dot{\alpha}$ , with  $\{b, v\}=2$ .
- Only **holonomies** of the connection are meaningful. Their elements can be expressed in terms of  $e^{\pm i b/2}$ .
- These holonomies shift the volume in a constant, unit step.
- We adopt a volume representation with DISCRETE measure. It is not continuous.
- The unperturbed Hamiltonian constraint leaves invariant **superselection sectors** with volume eigenvalues that differ in multiples of 4 units.

### Hybrid quantization

• In particular:

$$\hat{H}_{0} = \frac{1}{2} \left( \hat{\pi}_{\phi}^{2} - \hat{H}_{0}^{(2)} \right), \qquad \hat{H}_{0}^{(2)} = \frac{3}{4 \pi \gamma^{2}} \hat{\Omega}_{0}^{2} - 2 \hat{V}^{2} W(\hat{\phi}),$$

- In the **fermionic part** of the global Hamiltonian:
- Products with the volume are symmetrized algebraically.
- We represent vb by an operator Â<sub>0</sub> like Ω̂<sub>0</sub>, but with double angle.
   ξ<sub>k</sub>=√ω<sub>k</sub><sup>2</sup>+M<sup>2</sup>V<sup>2/3</sup> is represented in terms of the volume, using the spectral theorem.
- We adopt normal ordering for creation and annihilation operators.

# Born-Oppenheimer

 We adopt a Born-oppenheimer ansatz, with the inflaton playing the role of internal time:

$$\Phi = \chi_0 \psi = \chi_0(V, \phi) \psi_s(N_s, \phi) \psi_T(N_T, \phi) \psi_D(N_D, \phi),$$
  
$$\chi_0(V, \phi) = \hat{U}_0(V, \phi) \chi(V).$$
  
Fock representation

 $\chi_0$ : Solution at the considered perturbative order.

 $\hat{U}_0$ : Evolution operator, with positive  $\hat{\tilde{H}}_0 = [\hat{\pi}_{\phi}, \hat{U}_0]\hat{U}_0^{-1}$ .

#### Approximation:

No change of FLRW geometry is mediated by the constraint.

The diagonal element in the FLRW geometry encodes all relevant information about the constraint.

#### Born-Oppenheimer

• With the ansatz  $\Phi = \chi_0(V, \phi)\psi$  and our approximation, we obtain a quadratic **master constraint** for the perturbations, in which the quantum effects on the FLRW geometry are incorporated, and the homogeneous inflaton appears as an **internal time**.

Neglecting some ignorable terms for the scalar perturbations:

Possible FLRW contribution

$$\begin{split} \|\chi_{0}\|^{2} \hat{\pi}_{\phi}^{2}\psi + \langle (\hat{\tilde{H}}_{0})^{2} - \hat{H}_{0}^{(2)} \rangle_{\chi_{0}} \psi \\ + 2 \langle \hat{\tilde{H}}_{0} \rangle_{\chi_{0}} \hat{\pi}_{\phi} \psi = -2 \langle {}^{MS} \hat{H}_{2} + {}^{T} \hat{H}_{2} \rangle_{\chi_{0}} \psi - 2 \langle \sum_{\vec{k}} \hat{H}_{D}^{\vec{k}} \rangle_{\chi_{0}} \psi. \\ \\ LQC \text{ inner product} \\ \end{split}$$

#### Born-Oppenheimer

 If the perturbations have a negligible contribution to the inflaton momentum compared to the average of the FLRW part, we arrive at **Schrödinger** equations for the different perturbations.

In particular:

$$\hat{\pi}_{\phi}\psi_{D} = -\frac{\langle \sum_{\vec{k}} \hat{H}_{D}^{k} \rangle_{\chi_{0}}}{\langle \hat{\tilde{H}}_{0} \rangle_{\chi_{0}}}\psi_{D} - \frac{C_{D}^{(\chi)}(\phi)}{2\langle \hat{\tilde{H}}_{0} \rangle_{\chi_{0}}}\psi_{D}.$$

Expectation values of the geometry

The constraint allows for a **backreaction**, which can add to zero:

$$C_{s}^{(\chi)}(\phi) + C_{T}^{(\chi)}(\phi) + C_{D}^{(\chi)}(\phi) = \langle (\hat{\tilde{H}}_{0})^{2} - \hat{H}_{0}^{(2)} \rangle_{\chi_{0}}.$$

 One can derive equations of motions for the perturbations directly from the master constraint, even without the above approximation.

#### Fermionic dynamics

 From the master constraint, the fermionic operators satisfy the Heisenberg equations:

$$d_{\eta} \hat{a}_{\vec{k}}^{(x,y)}(\eta) = -iF_{k}^{(\chi)} \hat{a}_{\vec{k}}^{(x,y)}(\eta) + G_{k}^{(\chi)} \hat{b}_{\vec{k}}^{(x,y)\dagger}(\eta), d_{\eta} \hat{b}_{\vec{k}}^{(x,y)\dagger}(\eta,\eta_{0}) = iF_{k}^{(\chi)} \hat{b}_{\vec{k}}^{(x,y)\dagger}(\eta) - G_{k}^{(\chi)} \hat{a}_{\vec{k}}^{(x,y)}(\eta).$$

where the evolution is described in terms of a <u>well-defined</u> **conformal time** that depends on the state of the FLRW geometry

$$d\eta = \frac{\langle \hat{V}^{2/3} \rangle_{\chi_0}}{\langle \hat{\tilde{H}}_0 \rangle_{\chi_0}} d\phi.$$

Here:

$$F_{k}^{(\chi)} = \frac{\langle \xi_{k}(\hat{V}) \hat{V}^{2/3} \rangle_{\chi_{0}}}{\langle \hat{V}^{2/3} \rangle_{\chi_{0}}}, \quad G_{k}^{(\chi)} = M \omega_{k} \frac{\langle \xi_{k}^{-1}(\hat{V}) \hat{V}^{1/6} \hat{\Lambda}_{0} \hat{V}^{1/6} \xi_{k}^{-1}(\hat{V}) \rangle_{\chi_{0}}}{2 \gamma \langle \hat{V}^{2/3} \rangle_{\chi_{0}}}$$

#### Fermionic dynamics

$$\begin{aligned} d_{\eta} \hat{a}_{\vec{k}}^{(x,y)}(\eta) &= -iF_{k}^{(\chi)} \hat{a}_{\vec{k}}^{(x,y)}(\eta) + G_{k}^{(\chi)} \hat{b}_{\vec{k}}^{(x,y)\dagger}(\eta), \\ d_{\eta} \hat{b}_{\vec{k}}^{(x,y)\dagger}(\eta,\eta_{0}) &= iF_{k}^{(\chi)} \hat{b}_{\vec{k}}^{(x,y)\dagger}(\eta) - G_{k}^{(\chi)} \hat{a}_{\vec{k}}^{(x,y)}(\eta). \end{aligned} F_{k}^{(\chi)} &= \frac{\langle \xi_{k}(\hat{V}) \hat{V}^{2/3} \rangle_{\chi_{0}}}{\langle \hat{V}^{2/3} \rangle_{\chi_{0}}}, \\ G_{k}^{(\chi)} &= M \omega_{k} \frac{\langle \xi_{k}^{-1}(\hat{V}) \hat{V}^{1/6} \hat{\Lambda}_{0} \hat{V}^{1/6} \xi_{k}^{-1}(\hat{V}) \rangle_{\chi_{0}}}{2 \chi \langle \hat{V}^{2/3} \rangle_{\chi_{0}}}. \end{aligned}$$

• Recall that  $\xi_k(\hat{V}) = \sqrt{\omega_k^2 + M^2 \hat{V}^{2/3}}$ . Therefore, fermions couple with an **infinite** sequence of expectation values on the geometry.

 The solution to the Heisenberg equations provides a Bogoliubov transformation from the initial operators.

There is no guarantee that it reproduces a transformation in an effective background.

# Quantum evolution

• Let  $(\alpha_k, \beta_k)$  be the coefficients of the Bogoliubov transformation. We must have  $|\alpha_k|^2 + |\beta_k|^2 = 1$ . We use the **parametrization**:

$$e^{i\omega_{k}(\eta-\eta_{0})}\alpha_{k} = \cos A_{k} + i\rho_{k}\frac{\sin A_{k}}{A_{k}}, \qquad \rho_{k} \in \mathbb{R}, \quad \Gamma_{k} \in \mathbb{C}$$
$$e^{-i\omega_{k}(\eta-\eta_{0})}\beta_{k} = -\Gamma_{k}\frac{\sin A_{k}}{A_{k}}, \qquad A_{k} = \sqrt{|\Gamma_{k}|^{2} + \rho_{k}^{2}}.$$

• Zero-modes aside,  $\hat{U}_D = \hat{U}_B \hat{U}_F$  solves the fermionic evolution.  $\hat{U}_F$  rotates the phase of the operators by  $\omega_k(\eta - \eta_0)$  and  $\hat{U}_B = e^{-\hat{T}_B}$ :

$$\hat{T}_{B} = \sum \left[ \Gamma_{k} \hat{a}_{\vec{k}}^{(x,y)\dagger} \hat{b}_{\vec{k}}^{(x,y)\dagger} - \bar{\Gamma}_{k} \hat{b}_{\vec{k}}^{(x,y)} \hat{a}_{\vec{k}}^{(x,y)} - i \rho_{k} (\hat{a}_{\vec{k}}^{(x,y)\dagger} \hat{a}_{\vec{k}}^{(x,y)\dagger} + \hat{b}_{\vec{k}}^{(x,y)\dagger} \hat{b}_{\vec{k}}^{(x,y)}) + i c_{k}^{(x,y)} \right].$$

$$\vec{k} \neq \vec{0} \text{ if } \vec{\tau} = \vec{0}; (x, y) \in \{(m, s), (r, t)\}.$$
Phase

# Unitarity

**The quantum evolution is unitary iff** the  $\beta$ -coefficients are square-summable. A careful asymptotic analysis proves that:

$$\beta_{k}(\eta) = i \frac{M}{4\omega_{k}^{2}} \Big[ \lambda_{0}^{(\chi)}(\eta_{0}) e^{-i\omega_{k}(\eta-\eta_{0})} - \lambda_{0}^{(\chi)}(\eta) e^{i\omega_{k}(\eta-\eta_{0})} \Big] + \mathcal{O}(\omega_{k}^{-3}).$$

Since the degeneracy goes like  $g_k = O(\omega_k^2)$ :

The quantum evolution is indeed well-defined and **unitary**.

For large frequency, the  $\beta$ -coefficients are proportional to the fermion mass: negligible production of particles.



 $\lambda_0^{(\chi)} = \frac{\langle \hat{V}^{1/6} \hat{\Lambda}_0 \hat{V}^{1/6} \rangle_{\chi_0}}{\nu \langle \hat{V}^{2/3} \rangle} \longrightarrow \qquad \text{It vanishes "at the bounce",} \\ \text{reducing the particle production}$ 

# Vacuum evolution



• It is an exact solution to the Schrödinger equation if the **backreaction** is  $C_D^{(\chi)}(\phi) = 2 \langle \hat{V}^{2/3} \rangle_{\chi_0} \sum_{k \in \mathcal{N}} \left[ G_k^{(\chi)} \Im(\Gamma_k) + \left( c_k^{(x,y)} \right)' \right].$ 

Recall that 
$$G_k^{(\chi)} = M \omega_k \frac{\langle \xi_k^{-1}(\hat{V}) \hat{V}^{1/6} \hat{\Lambda}_0 \hat{V}^{1/6} \xi_k^{-1}(\hat{V}) \rangle_{\chi_0}}{2 \gamma \langle \hat{V}^{2/3} \rangle_{\chi_0}}$$
. Phase  $\beta$ -coefficient

# Vacuum evolution

• **Backreaction:** 
$$C_D^{(\chi)}(\phi) = 2 \langle \hat{V}^{2/3} \rangle_{\chi_0} \sum_{(x,y),\vec{k}} \left[ G_k^{(\chi)} \Im(\Gamma_k) + (c_k^{(x,y)})' \right].$$

• Our asymptotic analysis gives:

$$G_k^{(\chi)}\Im(\Delta_k) = \frac{M^2}{8\omega_k^3}\lambda_0^{(\chi)}(\eta) \Big[\lambda_0^{(\chi)}(\eta) - \lambda_0^{(\chi)}(\eta_0)\cos[2\omega_k(\eta-\eta_0)]\Big] + \mathcal{O}(\omega_k^{-4}).$$

- Recalling that the degeneracy is  $g_k = O(\omega_k^2)$ , **regularization** of the backreaction, absorbing the divergent part in the phase, is *(barely)* needed.
- This considerably improves the situation found by D'Eath & Halliwell, who got, for each fermionic mode, a contibution  $O(\omega_k)$ .

# Conclusions

- We have completed the hybrid loop quantization of a perturbed FLRW cosmology with scalar and **Dirac fields**.
- We have deduced a master constraint for the perturbations using a Born-Oppenheimer approximation.
- In the resulting quantum dynamics, the fermions couple with the geometry through an **infinite** number of expectation values.
- We have solved this fermionic quantum dynamics and proven that it is **unitary**, even if the geometry is a quantum entity.

# Conclusions

- We have shown that the unitarily evolved vacuum for the Dirac field is a solution to the associated Schrödinger equation.
- Since the dynamics is unitary, the **production of particles** is finite. Furthermore, it is negligible for modes of large frequency.
- **Backreaction** effects in our vacuum require regularization, but the situation is much better than in traditional studies.
- Finally, there exists the possibility of choosing **another vacuum**, in the same unitary family, which improves the behaviour of the backreaction in such a way that regularization **may not be needed**.