Infrared freezing of Euclidean QCD observables

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Outline of talk

- The Adler Function $D(Q^2)$

- Large-N_f results and the leading-b approximation

- Infrared freezing behaviour

- The skeleton expansion (Borel representations)

- non-perturbative effects/OPE

- IR behaviour of Bjorken sum rule- the GDH sum rule.

- Conclusions

The Adler Function

Consider the vacuum polarization function $\Pi^{\mu\nu}(Q^2)$,$Q^2 = -q^2 > 0$

$$\Pi^{\mu\nu}(Q^2) = 16\pi^2 i \int d^4x e^{iq.x} \langle 0|T(J_\mu(x)J_\nu(0))|0\rangle$$

Conservation of J_μ, $\partial_\mu J^\mu = 0$ then dictates the tensor structure

$$\Pi^{\mu\nu}(Q^2) = (q_\mu q_\nu - g_{\mu\nu} q^2)\Pi(Q^2)$$

Only $\Pi(Q^2) - \Pi(0)$ is observable, so it is useful to eliminate the constant and define the *Adler Function* $D(Q^2)$

$$D(Q^2) = -\frac{3}{4} Q^2 \frac{d}{dQ^2} \Pi(Q^2)$$

$$D(Q^2) = 3 \sum_f Q_f^2 [1 + \mathcal{D}(Q^2)]$$

The corrections to the parton model result, $\mathcal{D}(Q^2)$ are split into a perturbative and non-perturbative part

$$\mathcal{D}(Q^2) = \mathcal{D}_{PT}(Q^2) + \mathcal{D}_{NP}(Q^2)$$

$$\mathcal{D}_{PT}(Q^2) = a(Q^2) + d_1 a^2(Q^2) + d_2 a^3(Q^2) + \ldots + d_n a^{n+1}($$

Here $a(Q^2)$ is the running coupling. At one-loop level

$$a(Q^2) = \frac{2}{b \ln(Q^2/\Lambda^2)}$$

$b = (11N - 2N_f)/6$ is the first beta-function coefficient for SU(N) QCD with N_f quark flavours.

$$\mathcal{D}^{(L)}_{NP}(Q^2) = \sum_n \mathcal{C}_n \left(\frac{\Lambda^2}{Q^2}\right)^n$$

The leading OPE contribution for the Adler function is the dimension 4 gluon condensate contribution

$$G_0(a(Q^2)) = \frac{1}{Q^4} \langle 0|GG|0\rangle C_{GG}(a(Q^2)) \,,$$

Infrared Freezing ?

We are interested in the behaviour of $\mathcal{D}(Q^2) = \mathcal{D}_{PT}(Q^2) + \mathcal{D}_{NP}(Q^2)$ as $Q^2 \rightarrow 0$. Clearly at any fixed order perturbation theory breaks down at $Q^2 = \Lambda^2$, the Landau pole in the coupling, and $a(Q^2) \rightarrow \infty$. Clearly we need a resummation of perturbation theory to all-orders to address the freezing question, and we need to combine the resummation with the OPE condensates. The large-N_f limit provides a way of formulating this resummation.

Large-N_f approximation for vacuum polarization

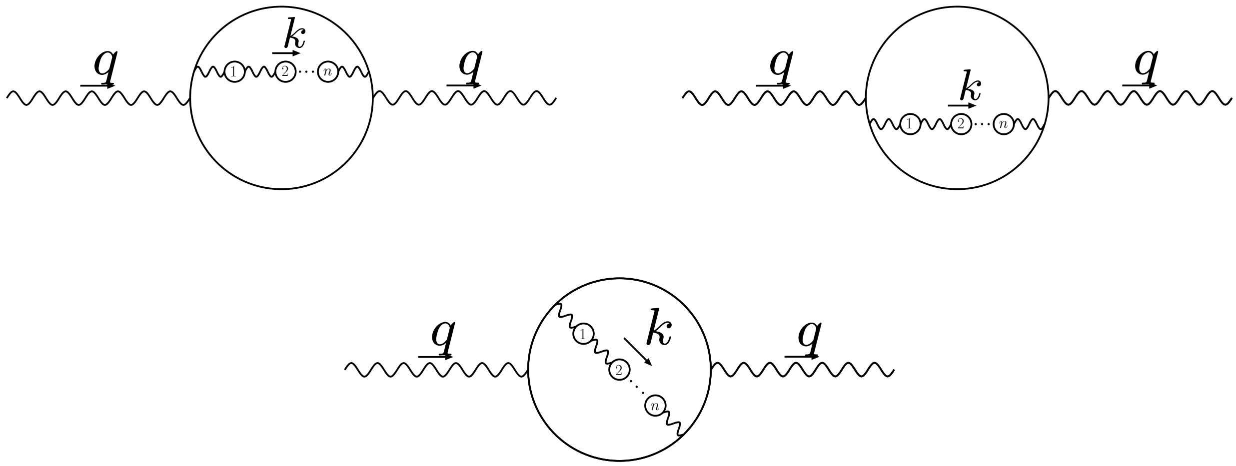
Consider the Adler D-function we discussed earlier with perturbative expansion

$$\mathcal{D}(Q^2) = a(Q^2) + d_1 a^2(Q^2) + d_2 a^3(Q^2) + \ldots + d_k a^{k+1} + \ldots$$

The coefficient d_n may be expanded in powers of N_f the number of quark flavours

$$d_n = d_n^{[n]} N_f^n + d_n^{[n-1]} N_f^{n-1} + \ldots + d_n^{[0]}$$

The leading large-N_f coefficient $d_n^{[n]}$ may be evaluated to all-orders since it derives from a restricted set of diagrams obtained by inserting a chain of fermion bubbles inside the quark loop



A crucial ingredient is the chain of n-bubbles, $B^{\mu\nu}_{(n)}(k^2)$.

$$B^{\mu\nu}_{(n)} = \frac{(k^2 g^{\mu\nu} - k_\mu k_\nu)}{(k^2)^2}\left[-\frac{N_f}{3}\left(\ln\frac{k^2}{\mu^2} + C\right)\right]^n$$

The constant C depends on the subtraction procedure used to renormalise the bubble. With $\overline{MS}$ subtraction $C = -\frac{5}{3}$. We shall choose to work in the "V-scheme" which corresponds to $\overline{MS}$ with the scale choice $\mu^2 = e^{-5/3}Q^2$, in which case $C = 0$.

Applying the Feynman rules to the three diagrams then gives $d_n^{[n]} a^{n+1}$

$$
\begin{aligned}
\sim\ \ & a \int \frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4} \\
\Bigg[\ \ & B^{\sigma\rho}_{(n)}(k^2)\,\mathrm{Tr}\left(\gamma_\nu \frac{1}{\not p + \not q + \not k}\gamma_\rho \frac{1}{\not p + \not q}\gamma_\mu \frac{1}{\not p}\gamma_\sigma \frac{1}{\not p + \not k}\right) \\
+\ \ & 2B^{\sigma\rho}_{(n)}(k^2)\,\mathrm{Tr}\left(\gamma_\nu \frac{1}{\not p + \not q}\gamma_\mu \frac{1}{\not p}\gamma_\sigma \frac{1}{\not p + \not k}\gamma_\rho \frac{1}{\not p}\right)\Bigg] \\
\sim\ \ & a \int \frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\Bigg[B^{\sigma\rho}_{(n)}(k^2) X_{\nu\rho\mu\sigma} + 2B^{\sigma\rho}_{(n)}(k^2)\bar{X}_{\nu\mu\sigma\rho}
\end{aligned}
$$

The loop integrals can be evaluated using the Gegenbauer polynomial x-space technique, with the result

$$
\begin{aligned}
d_n^{[n]}(V) &= \frac{-2}{3}(n+1)\left(\frac{-1}{6}\right)^n \Bigg[-2n - \frac{n+6}{2^{n+2}} \\
+ \quad & \frac{16}{n+1}\sum_{\frac{n}{2}+1>m>0} m(1-2^{-2m}) \\
\times \quad & (1-2^{2m-n-2})\zeta_{2m+1}\Bigg]\, n!\,.
\end{aligned}
$$

Leading-b approximation

This large-N_f result can describe QED vacuum polarization, but for QCD the corrections to the gluon propagator involve gluon and ghost loops, and are gauge (ξ)-dependent. The result for $\Pi_0(k^2)$ is proportional to $-N_f/3$ which is the first QED beta-function coefficient, b. In QCD one expects large-order behaviour of the form $d_n \sim K n^{\gamma}(b/2)^n n!$ involving the QCD beta-function coefficient $b = (33-2N_f)/6$, it is then natural to replace N_f by $(33/2-3b)$ to obtain an expansion in powers of b

$$d_n = d_n^{(n)} b^n + d_n^{(n-1)} b^{n-1} + \ldots + d_n^{(0)}$$

The leading-b term $d_n^{(L)} \equiv d_n^{(n)} b^n = (-3)^n d_n^{[n]} b^n$ can then be used to approximate d_n to all-orders, and an all-orders resummation of these terms performed to obtain $\mathcal{D}_{PT}^{(L)}(Q^2)$.

If we use the Borel method to define the all-orders perturbative result we obtain

$$\mathcal{D}^{(L)}_{PT}(Q^2) = \int_0^\infty dz\, e^{-z/a(Q^2)} B[\mathcal{D}^{(L)}_{PT}](z)\ .$$

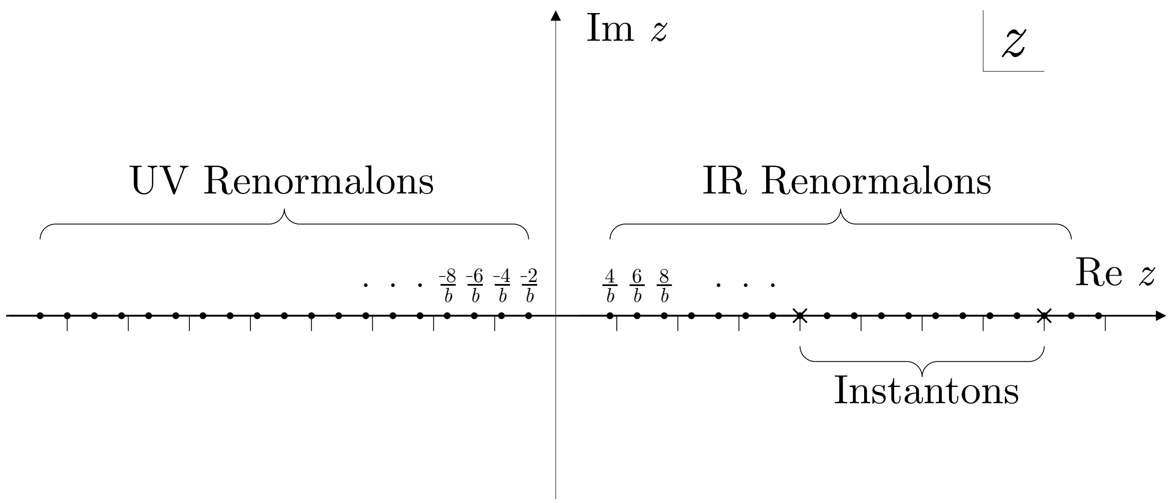
The Borel transform is given by

$$\begin{aligned} B[\mathcal{D}^{(L)}_{PT}](z) &= \sum_{n=1}^{\infty} \frac{A_0(n) - A_1(n) z_n}{\left(1 + \frac{z}{z_n}\right)^2} + \frac{A_1(n) z_n}{\left(1 + \frac{z}{z_n}\right)} \\ &+ \sum_{n=1}^{\infty} \frac{B_0(n) + B_1(n) z_n}{\left(1 - \frac{z}{z_n}\right)^2} - \frac{B_1(n) z_n}{\left(1 - \frac{z}{z_n}\right)} \end{aligned}$$

The residues are given by

$$\begin{aligned} A_0(n) &= \frac{8}{3}\frac{(-1)^{n+1}(3n^2+6n+2)}{n^2(n+1)^2(n+2)^2} \\ A_1(n) &= \frac{8}{3}\frac{b(-1)^{n+1}(n+\frac{3}{2})}{n^2(n+1)^2(n+2)^2} \end{aligned}$$

$$\begin{aligned} &B_0(1) = 0, \quad B_0(2) = 1, \quad B_0(n) = -A_0(-n) \quad n \geq \\ &B_1(1) = 0, \quad B_1(2) = -\frac{b}{4}, \quad B_1(n) = -A_1(-n) \quad n \end{aligned}$$



For the Adler function in leading-b approximation there are single and double poles in $B[\mathcal{D}_{PT}^{(L)}](z)$ at positions $z = z_n$ and $z = -z_n$ with $z_n \equiv 2n/b$, $n = 1, 2, 3$. The singularities on the positive real semi-axis are the *infrared* renormalons, IR_n and those on the negative real semi-axis are *ultraviolet* renormalons, UV_n. We shall see that they correspond to integration over the bubble-chain momentum k^2 in the regions $k^2 < Q^2$ and $k^2 > Q^2$, re spectively.

The (PV regulated) Borel integral may be evaluated in terms of Ei functions, but notice that the Borel integral diverges for $Q^2 < \Lambda^2$, and potentially for $Q^2 = \Lambda^2$!

$$
\begin{aligned}
\mathcal{D}_{PT}^{(L)}(Q^2) &= \sum_{n=1}^{\infty} [z_n e^{z_n/a(Q^2)} \mathrm{Ei}\left(\frac{z_n}{a(Q^2)}\right) \\
&\times \left[\frac{z_n}{a(Q^2)}(A_0(n) - z_n A_1(n)) - z_n A_1(n)\right] \\
&+ (A_0(n) - z_n A_1(n))] \\
&+ \sum_{n=1}^{\infty} z_n [e^{-z_n/a(Q^2)} \mathrm{Ei}\left(\frac{z_n}{a(Q^2)}\right) \\
&\times \left[\frac{z_n}{a(Q^2)}(B_0(n) + z_n B_1(n)) - z_n B_1(n)\right] \\
&- (B_0(n) + z_n B_1(n))]
\end{aligned}
$$

This expression has the property that it is finite and continuous at $Q^2 = \Lambda^2$ and freezes smoothly to a freezing limit of $\mathcal{D}_{PT}^{(L)}(0) = 0$. Similar behaviour is found for GLS/polarized Bjorken and unpolarized Bjorken DIS sum rules $\mathcal{K}_{PT}^{(L)}(Q^2)$ and $\mathcal{U}_{PT}^{(L)}(Q^2)$.

DIS sum rules

The polarised Bjorken (pBj) and GLS sum rules are defined as

$$\begin{aligned} K_{pBj} &\equiv \int_0^1 g_1^{ep-en}(x,Q^2)dx \\ &= \frac{1}{6}\left|\frac{g_A}{g_V}\right|\left(1-\frac{3}{4}C_F\mathcal{K}(Q^2)\right), \end{aligned}$$

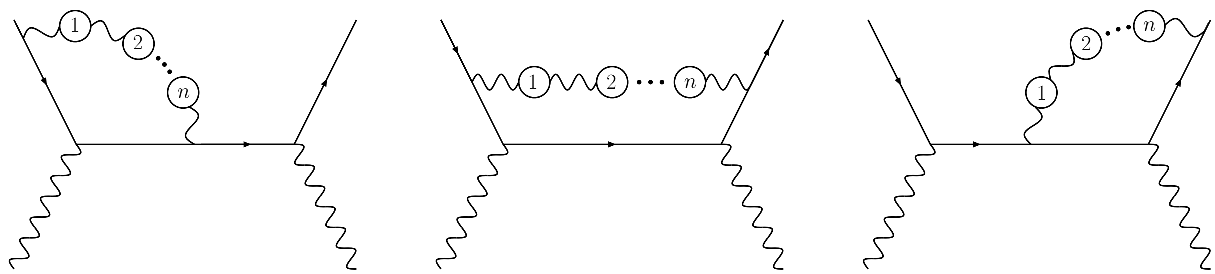
$$\begin{aligned} K_{GLS} &\equiv \frac{1}{2}\int_0^1 F_3^{\bar{\nu}p+\nu p}(x,Q^2)dx \\ &= 3\left(1-\frac{3}{4}C_F\mathcal{K}(Q^2)\right). \end{aligned}$$

$\mathcal{K}(Q^2)$ being the perturbative correction to the parton model result. We have neglected contributions due to "light-by-light" diagrams – which when omitted render the perturbative corrections to K_{GLS} and K_{pBj} identical.

Finally, the unpolarised Bjorken sum rule (uBj) is defined as

$$
\begin{aligned}
U_{uBj} &\equiv \int_0^1 F_1^{\bar{\nu}p-\nu p}(x, Q^2)dx \\
&= \left(1 - \frac{1}{2} C_F \mathcal{U}(Q^2)\right)
\end{aligned}
$$

Leading-b results for $\mathcal{K}_{PT}^{(L)}(Q^2)$ and $\mathcal{U}_{PT}^{(L)}(Q^2)$ can be computed from the diagrams



The expressions for $B[\mathcal{K}_{PT}^{(L)}](z)$ and $B[\mathcal{U}_{PT}^{(L)}](z)$ are

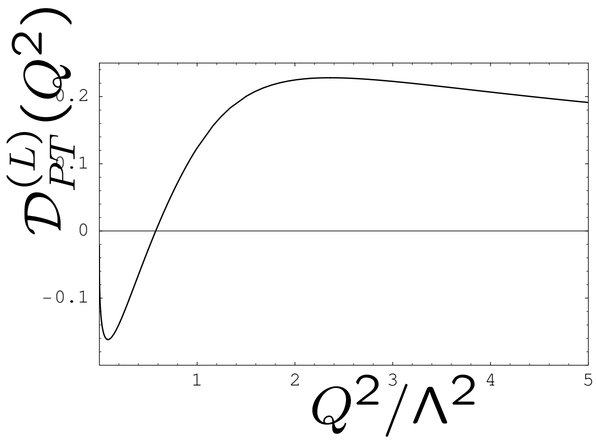
$$\begin{aligned} B[\mathcal{K}_{PT}^{(L)}](z) &= \frac{4/9}{\left(1+\frac{z}{z_1}\right)} - \frac{1/18}{\left(1+\frac{z}{z_2}\right)} + \frac{8/9}{\left(1-\frac{z}{z_1}\right)} \\ &- \frac{5/18}{\left(1-\frac{z}{z_2}\right)}. \end{aligned}$$

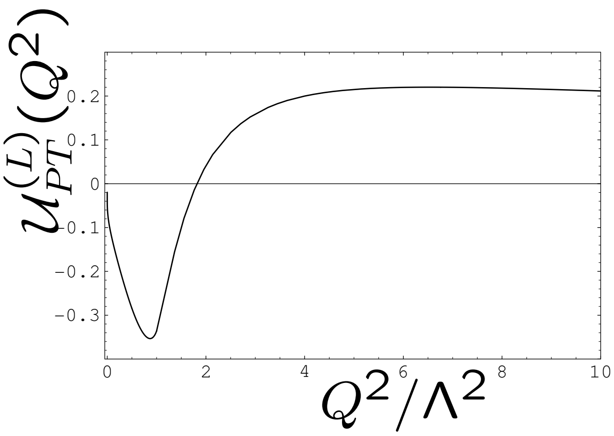
$$B[\mathcal{U}_{PT}^{(L)}](z) = \frac{1/6}{\left(1+\frac{z}{z_2}\right)} + \frac{4/3}{\left(1-\frac{z}{z_1}\right)} - \frac{1/2}{\left(1-\frac{z}{z_2}\right)}.$$

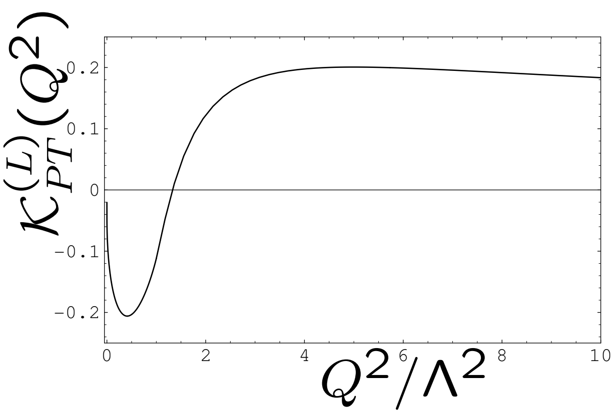
The resummed results are

$$\mathcal{K}_{PT}^{(L)}(Q^2) = \frac{1}{9b}\Bigg[-8e^{z_1/a(Q^2)}\mathrm{Ei}(-z_1/a(Q^2)) + 2e^{z_2/a(Q^2)}\mathrm{Ei}(-z_2/a(Q^2)) + 16e^{-z_1/a(Q^2)}\mathrm{Ei}(z_1/a(Q^2)) - 10e^{-z_2/a(Q^2)}\mathrm{Ei}(z_2/a(Q^2))\Bigg],$$

$$\mathcal{U}_{PT}^{(L)}(Q^2) = \frac{1}{3b}\Bigg[8e^{-z_1/a(Q^2)}\mathrm{Ei}(z_1/a(Q^2)) - 6e^{-z_2/a(Q^2)}\mathrm{Ei}(z_2/a(Q^2)) - 2e^{z_2/a(Q^2)}\mathrm{Ei}(-z_2/a(Q^2))\Bigg].$$







The observables vanish in the vicinity of $Q^2 = \Lambda^2$ and then freeze smoothly to zero through negative values.

There is potentially a divergence proportional to $\ln a(Q^2)$ at $Q^2 = \Lambda^2$. The coefficient of this divergent term is

$$-\sum_{n+1}^{\infty} z_n^2 [A_1(n) + B_1(n)]$$

For $\mathcal{K}_{PT}^{(L)}(Q^2)$ and $\mathcal{U}_{PT}^{(L)}(Q^2)$ the equivalent coefficients are $(-8+2 = 16-10 = 0)$ and $(8-6-2) = 0$, respectively. There is a relation between IR and UV renormalon residues which ensures the divergent term vanishes

$$z_{n+3}^2 B_1(n+3) = -z_n^2 A_1(n)$$

This ensures that

$$\sum_{n=1}^{\infty} z_n^2 [A_1(n) + B_1(n)] = 0$$

Another similar relation is

$$A_0(n) = -B_0(n+2)$$

We shall show that these relations are underwritten by continuity of the characteristic function in the skeleton expansion.

The skeleton expansion

In QED the insertion of chains of bubbles into a basic skeleton diagram produces a well defined skeleton expansion.

$$
\begin{aligned}
d_n^{(L)} a^{n+1} &= a \int_0^\infty dk^2 \omega_{\mathcal{D}} \left(\frac{k^2}{Q^2} \right) \Big(- \frac{ba}{2} \ln \left(\frac{-k^2}{Q^2} \right) \Big)^n , \\
\Rightarrow \mathcal{D}_{PT}^{(L)}(Q^2) &\simeq \sum_{n=0}^{\infty} d_n^{(L)} N_f^n a^{n+1}
\end{aligned}
$$

$$
= Q^2 \int_0^\infty \frac{d^2 k}{k^2} \omega_{\mathcal{D}} \left(\frac{k^2}{Q^2} \right) \frac{k^2}{Q^2} \left[\frac{a}{1 + \frac{ba}{2} \ln \left(\frac{-k^2}{Q^2} \right)} \right]
$$

which can be written as

$$
\mathcal{D}_{PT}^{(L)}(Q^2) = \int_0^\infty dt \, \omega_{\mathcal{D}}(t) a(tQ^2) \, .
$$

Here $\omega_{\mathcal{D}}(t)$ is the *characteristic function*. It satisfies the normalization condition

$$
\int_0^\infty dt \, \omega_{\mathcal{D}}(t) = 1 \, ,
$$

$\omega_{\mathcal{D}}(t)$ and its first three derivatives are continuous at $t=1$

$$\mathcal{D}_{PT}^{(L)} = \int_0^1 dt \omega_{\mathcal{D}}^{IR}(t) a(tQ^2) + \int_1^\infty dt \omega_{\mathcal{D}}^{UV} a(tQ^2)$$

$\omega_{\mathcal{D}}(t)$ can be derived by using classic QED work of Baker and Johnson on vacuum polarization

$$\Pi(Q^2) = \int_0^\infty dt\, \omega_\Pi(t) a(tQ^2)$$

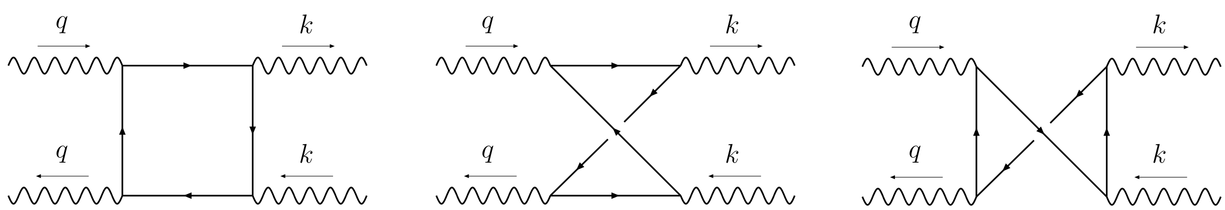
where the characterisitic function $\omega_\Pi(t)$ is given by

$$\omega_\Pi(t) = -\frac{4}{3}\begin{cases} t\,\Xi(t) & t \le 1 \quad \leftrightarrow \quad \text{IR} \\ \frac{1}{t}\,\Xi\left(\frac{1}{t}\right) & t \ge 1 \quad \leftrightarrow \quad \text{UV} \end{cases}$$

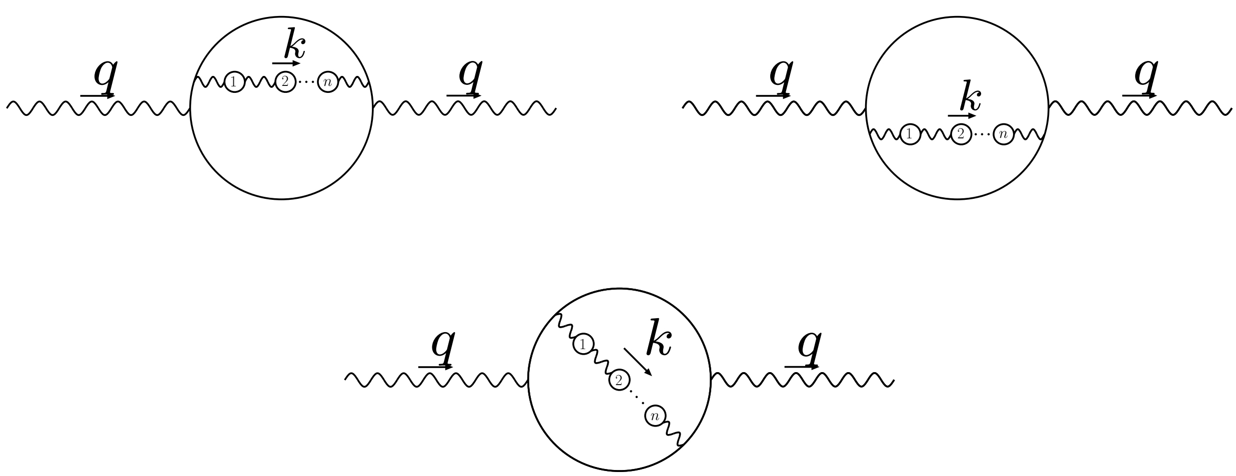
$\Xi(t)$ is given by

$$\begin{aligned} \Xi(t) &\equiv \frac{4}{3t}[1 - \ln t + \left(\frac{5}{2} - \frac{3}{2}\ln t\right)t \\ &+ \frac{(1+t)^2}{t}[L_2(-t) + \ln t \ln(1+t)]] \end{aligned}$$

$\Xi(t)$ corresponds to the Bethe-Salpeter kernel for the scattering of light-by-light involving the diagrams



Notice that by attaching the ends of the fermion bubble chain to the momentum k external propagators of these *one-loop* diagrams one reproduces the topology of the three *two-loop* bubble chain diagrams.



The skeleton expansion can be transformed into the Borel representation for $Q^2 > \Lambda^2$ by a change of variable

$$\begin{aligned}
\omega_{\mathcal{D}}^{IR}(t) &= \frac{b}{2}\sum_{n=1}^{\infty} -z_{n+1}^2 B_1(n+1)t^n \\
&- \ln t \sum_{n=2}^{\infty} (n+1)^2 [B_0(n+1) \\
&+ z_{n+1} B_1(n+1)] t^n \\
\omega_{\mathcal{D}}^{UV}(t) &= \frac{b}{2}\sum_{n=1}^{\infty} z_{n-1}^2 A_1(n-1)t^{-n} \\
&+ \ln t \sum_{n=2}^{\infty} (n-1)^2 [A_0(n-1) \\
&- z_{n-1} A_1(n-1)] t^{-n}.
\end{aligned}$$

For $Q^2 < \Lambda^2$ the skeleton expansion is equivalent to a modified Borel representation

$$\mathcal{D}_{PT}^{(L)}(Q^2) = \int_0^{-\infty} dz\, e^{-z/a(Q^2)} B[\mathcal{D}_{PT}^{(L)}](z)\,.$$

One can then show that continuity of $\omega_{\mathcal{D}}(t)$ and its first three derivatives at $t = 1$, and equivalently finiteness of $\mathcal{D}^{(L)}_{PT}(Q^2)$ and its first three derivatives $d/d\ln Q$ at $Q^2 = \Lambda^2$ is underwritten by the following relations between the $A_{0,1}$ and $B_{0,1}$ residues

$$\sum_{n=1}^{\infty} z_n^2[A_1(n) + B_1(n)] = 0$$

$$\sum_{n=1}^{\infty} [2z_n^3(A_1(n) - B_1(n)) - z_n^2(A_0(n) + B_0(n)] = 0 \, .$$

$$\sum_{n=1}^{\infty} [3z_n^4(A_1(n) + B_1(n)) - 2z_n^3(A_0(n) - B_0(n))] = 0 \, .$$

$$\sum_{n=1}^{\infty} [4z_n^5(A_1(n) - B_1(n)) - 3z_n^4(A_0(n) + B_0(n))] = 0 \, .$$

OPE and IR renormalon ambiguities

The regular OPE is a sum over the contributions of condensates with different mass dimensions. In the case of the Adler function the dimension four gluon condensate is the leading contribution

$$G_0(a(Q^2)) = \frac{1}{Q^4}\langle 0|GG|0\rangle C_{GG}(a(Q^2)) \,,$$

where $C_{GG}(a(Q^2))$ is the Wilson coefficient. The OPE is of the form

$$\mathcal{D}^{(L)}_{NP}(Q^2) = \sum_n \mathcal{C}_n \left(\frac{\Lambda^2}{Q^2}\right)^n$$

The n^{th} term in this expansion will be of the form

$$\mathcal{C}_n(a(Q^2)) = C_n[a(Q^2)]^{\delta_n}(1 + O(a)) \,.$$

The exponent δ_n corresponding to the anomalous dimension of the condensate operator concerned.

Non-logarithmic UV divergences lead to an ambiguous imaginary part in the coefficient so that $C_n = C_n^{(R)} \pm iC_n^{(I)}$. If one considers an IR_n renormalon singularity in the Borel plane to be of the form $K_n/(1 - z/z_n)^{\gamma_n}$ then one finds an ambiguous imaginary part arising of the form

$$Im[\mathcal{D}_{PT}] = \pm K_n \frac{\pi z_n^{\gamma_n}}{\Gamma(\gamma_n)} e^{-z_n/a(Q^2)} a^{1-\gamma_n}[1 + O(a)] \, .$$

Here the $\pm$ ambiguity comes from routing the contour above or below the real z-axis in the Borel plane. This is structurally the same as the ambiguous OPE term above, and if $C_n^{(I)} = K_n \pi z_n^{\gamma_n}/\Gamma(\gamma_n)$ and $\delta_n = 1 - \gamma_n$, then the PT Borel and NP OPE ambiguities can cancel against each other. Taking a PV of the Borel integral corresponds to averaging over the $\pm$ possibilities.

The NP component

It is easy to show that the ambiguous imaginary part in $\mathcal{D}_{PT}^{(L)}$ arising from IR renormalons for $Q^2 > \Lambda^2$ and UV renormalons for $Q^2 < \Lambda^2$ can be written directly in terms of $\omega_{\mathcal{D}}^{IR}$ and $\omega_{\mathcal{D}}^{UV}$

$$Im[\mathcal{D}_{PT}^{(L)}(Q^2)] = \pm\frac{2\pi}{b}\frac{\Lambda^2}{Q^2}\omega_{\mathcal{D}}^{IR}\left(\frac{\Lambda^2}{Q^2}\right) \quad (Q^2 > \Lambda^2)$$
$$Im[\mathcal{D}_{PT}^{(L)}(Q^2)] = \pm\frac{2\pi}{b}\frac{\Lambda^2}{Q^2}\omega_{\mathcal{D}}^{UV}\left(\frac{\Lambda^2}{Q^2}\right) \quad (Q^2 < \Lambda^2)$$

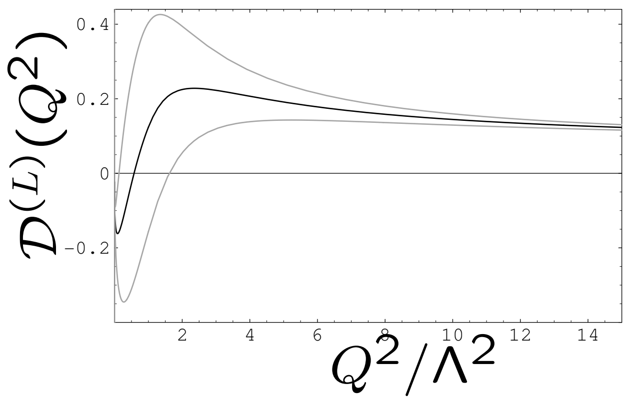
Continuity at $Q^2 = \Lambda^2$ then follows from continuity of $\omega_{\mathcal{D}}(t)$ at $t = 1$. In principle the real part of the OPE condensates are independent of the imaginary, but continuity and finiteness involve the set of relations between $A_{0,1}$ and $B_{0,1}$ that we have just noted. Continuity naturally follows if we write

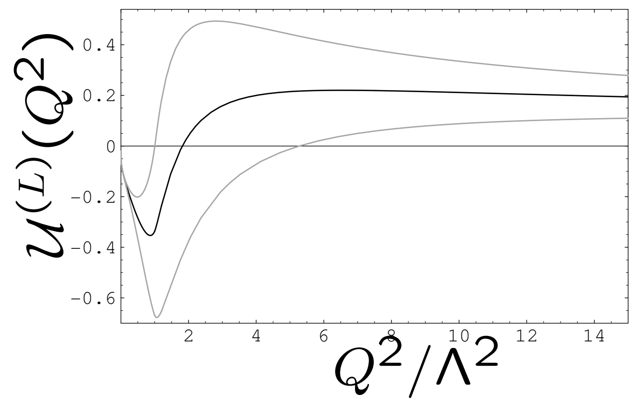
$$\mathcal{D}_{NP}^{(L)}(Q^2) = \left(\kappa \pm \frac{2\pi i}{b}\right)\int_0^{\Lambda^2/Q^2} dt \, \left(\omega_{\mathcal{D}}(t) + t\frac{d\omega_{\mathcal{D}}(t)}{dt}\right) \, .$$

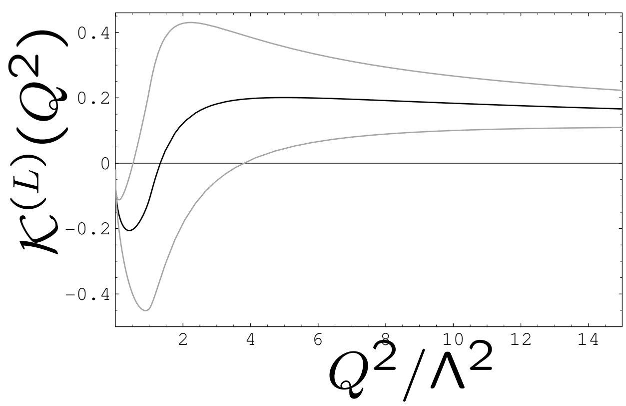
Here κ is an overall real non-perturbative constant. If the PT component is PV regulated then one averages over the $\pm$ possibilities for contour routing, combining with $\mathcal{D}_{PT}^{(L)}$ one can then write down the overall result for $\mathcal{D}^{(L)}(Q^2)$

$$\begin{aligned}\mathcal{D}^{(L)}(Q^2) &= \int_0^\infty dt \left[\omega_{\mathcal{D}}(t)a(tQ^2)\right.\\ &+ \left.\kappa\left(\omega_{\mathcal{D}}(t)+t\frac{d\omega_{\mathcal{D}}(t)}{dt}\right)\theta(\Lambda^2-tQ^2)\right].\end{aligned}$$

The Q^2 evolution is fixed by the non-perturbative constant κ and by Λ.







The bold curves show the choice $\kappa = 0$, i.e. just the PT component as in the earlier plots. The upper and lower curves correspond to the choices $\kappa = 1$ and $\kappa = -1$, respectively.

The GDH sum rule

Consider the Q^2-dependent integral

$$I_1(Q^2) = \frac{2M^2}{Q^2}\int_0^1 g_1(x,Q^2)\,dx$$

There is an exact low-energy sum rule due to Gerasimov-Drell-Hearn (GDH)

$$I_1(0) = \frac{-\mu_A^2}{4}$$

where μ_A is the nucleon anomalous magnetic moment in nuclear magnetons. For the polarized Bjorken sum rule

$$\begin{aligned} K_{pBj}(Q^2) &\equiv \int_0^1 g_1^{ep-en}(x,Q^2)dx \\ &= \frac{1}{6}\left|\frac{g_A}{g_V}\right|\left(1-\frac{3}{4}C_F\mathcal{K}(Q^2)\right), \end{aligned}$$

we would then expect that as $Q^2 \to 0$

$$\frac{2M^2}{Q^2}K_{pBj}(Q^2) \to \frac{(\mu_{A,n}^2 - \mu_{A,p}^2)}{4}$$

Conclusions

The one-chain term of a QCD skeleton expansion naturally results in IR freezing behaviour as $Q^2 \to 0$. The freezing limit is just the parton model result.

Continuity and finiteness of the Euclidean observable at $Q^2 = \Lambda^2$ are underwritten by the continuity of the characteristic function $\omega(t)$ at $t = 1$.

Continuity and finiteness translate into previously unexplained relations between UV and IR renormalon residues

The ambiguous imaginary part induced by IR or UV renormalon ambiguities may be written directly in terms of the characteristic function. Continuity then suggests that non-perturbative effects may similarly be written in terms of the characteristic function, and are characterized by a single overall non-perturbative constant κ.