HT Lecture on Nonlinear beam dynamics (I)

Motivations: nonlinear magnetic multipoles
Phenomenology of nonlinear motion
Simplified treatment of resonances (stopband concept)
Hamiltonian of the nonlinear betatron motion

HT Lecture on Nonlinear beam dynamics (II)

Hamiltonian of the nonlinear betatron motion
Resonance driving terms
Tracking
Dynamic Aperture and Frequency Map Analysis
Spectral Lines and resonances
Nonlinear beam dynamics experiments at Diamond

Hamiltonian of nonlinear betatron motion

The Hamiltonian for the nonlinear betatron motion is given by

$$H = \frac{p_x^2 + p_z^2}{2} - \frac{x^2}{2\rho^2} + \mathrm{Re}\sum_{n=1}^{M} \frac{k_n + ij_n}{(n+1)!}(x+iz)^{n+1}$$

We define H_0 the linear part (dependent only on dipoles and normal quadrupoles)

$$H_0(\overline{p},\overline{q};s) = \frac{p_x^2 + p_z^2}{2} - \frac{x^2}{2\rho^2} + \frac{k_1 x^2 - k_1 z^2}{2}$$

and V the nonlinear part dependent on the nonlinear magnetic multipoles

$$V(\overline{p},\overline{q};s) = \mathrm{Re}\sum_{n\geq 2}\left[k_n(s) + ij_n(s)\right]\frac{(x+iz)^{n+1}}{n+1!} = \sum_{n\geq 3} V_{mn} x^m z^n$$

$$k_n(s) = \frac{1}{B_0\rho}\frac{\partial^n B_z}{\partial x^n}$$ Normal multipoles

$$j_n(s) = \frac{1}{B_0\rho}\frac{\partial^n B_x}{\partial x^n}$$ skew multipoles

Normalisation of the linear part of the Hamiltonian

We define a canonical transformation that reduces the linear part of the Hamiltonian to a rotation

$$(\bar{x},\bar{p}) \rightarrow (\bar{J},\bar{\phi})$$ canonical transformation

with generating function

$$F(x,\phi_x,z,\phi_z;s) = -\frac{x^2}{2\beta_x(s)}\left[\tan\left(\phi_x + W_x(s)\right) + \alpha_x(s)\right] - \frac{z^2}{2\beta_z(s)}\left[\tan\left(\phi_z + W_z(s)\right) + \alpha_z(s)\right]$$

In detail

$$J_x = \gamma_x x^2 + 2\alpha_x x p_x + \beta_x p_x^2$$ linear Courant-Snyder invariant

$$\phi_x = -\arctan\left[\beta_x \frac{p_x}{x} + \alpha_x\right] - \int \frac{d\tau}{\beta_x}$$

This transformation reduces ellipses in phase space to circles and the motion to a rotation along these circles, for linear systems

Resonance driving terms (I)

The new Hamiltonian reads

$$H(\bar{J},\bar{\phi};s)=\frac{Q_x J_x+Q_z J_z}{R}+V(\bar{J},\bar{\phi};s)$$

$$V(\bar{J},\bar{\phi};s)=\frac{\varepsilon}{R}J_x^{\frac{m_x}{2}}J_z^{\frac{m_z}{2}}\sum_{j=0}^{m_x}\sum_{l=0}^{m_z}\sum_{p=-\infty}^{\infty}h_{jklmp}e^{i[(j-k)\phi_x+(l-m)\phi_z-p(s-s_0)/R]}$$

The h_{jklmp} are called resonance driving terms since they generate angle dependent terms in the Hamiltonian that are responsible for the resonant motion of the particles (i.e. motion on a chain of islands or on a separatrix).

On the islands the betatron tuned satisfy a resonant condition of the type

$NQ_x + MQ_y = p \quad \rightarrow$ resonance (N, M) $\quad N = j - k$ and $M = l - m$

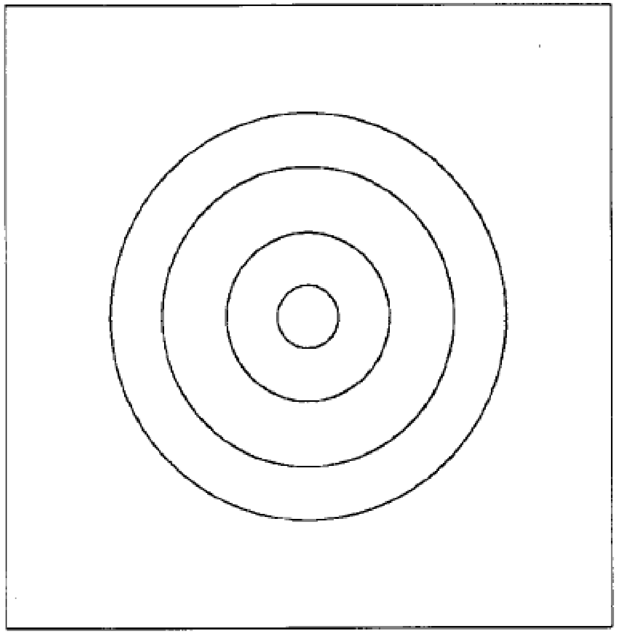
Terms of the type h_{jjkkp} produce detuning with amplitude to the lowest order in the multipolar gradient, but they can interfere with other terms in the Hamiltonian to create resonances (perturbative theory of betatron motion)

Without angle dependent term the motion will be just an amplitude dependent rotation

Non resonant and single resonance Hamiltonian

The dynamics with only detuning terms is an amplitude dependent rotation

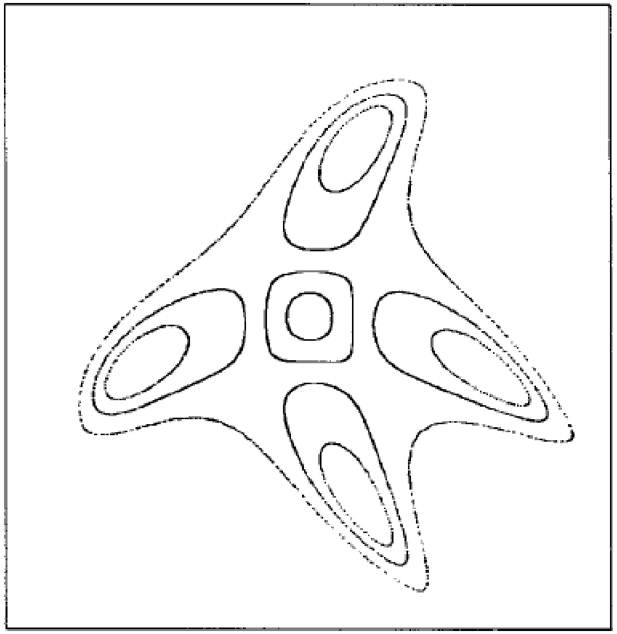
$$V(\bar{J},\bar{\phi};s)=\frac{\varepsilon}{R}J_x^{\frac{m_x}{2}}J_z^{\frac{m_z}{2}}\sum_{j=0}^{m_x}\sum_{l=0}^{m_z}\sum_{p=-\infty}^{\infty}h_{jjllp}e^{-ip(s-s_0)/R}$$



The dynamics with angle dependent terms exhibits fixed points, island

e.g. for the (4,0) resonance

$$V(\bar{J},\bar{\phi};s)=\frac{\varepsilon}{R}J_x^2\sum_{p=-\infty}^{\infty}h_{4000p}e^{i[4\phi_x-p(s-s_0)/R]}$$



Resonance driving terms (II)

The resonant driving terms are integrals over the whole length of the accelerator of functions which depend on the s-location of the multipolar magnetic elements

$$h_{jklmp} = \frac{1}{2\pi}\frac{1}{2^{\frac{j+k+l+m}{2}}}\binom{j+k}{j}\binom{l+m}{l}\cdot\int_{s_0}^{s_0+2\pi R} V_{j+k,l+q}(s)\cdot\beta_x^{\frac{j+k}{2}}(s)\beta_z^{\frac{l+m}{2}}(s)e^{i[(j-k)W_x(s)+(l-m)W_x(s)+p\frac{s-s_0}{R}]}ds$$

The solution for the stable betatron motion can be written as a quasi periodic signal

$$x(n)-ip_x(n)=\sqrt{2J_x}e^{i(2\pi Q_x n+\psi_0)}+$$
$$-2i\sum_{jklm} js_{jklm}(2J_x)^{\frac{j+k-1}{2}}(2J_y)^{\frac{l+m}{2}}e^{i\left[(1-j+k)(2\pi Q_x n+\psi_{x0})+(m-l)(2\pi Q_y n+\psi_{y0})\right]}$$

to first order in the multipoles strengths

with $$s_{jklm}=\frac{1}{1-\exp^{-2\pi i[(j-k)Q_x+(l-m)Q_z)]}}\sum_{p=-\infty}^{\infty}h_{jklmp}e^{-ip\frac{s-s_0}{R}}$$

Each resonance driving term h_{jklmp} contributes to the Fourier coefficient of a well precise spectral line

$$\nu(h_{jklm})=(1-j+k)Q_x+(m-l)Q_y$$

Resonance driving terms from sextupoles

Let us consider the driving terms generated by a normal sextupole. In the general definition of driving term

$$h_{jklmp} = \frac{1}{2\pi} \frac{1}{2^{\frac{j+k+l+m}{2}}} \binom{j+k}{j} \binom{l+m}{l} \cdot \int_{s_0}^{s_0+2\pi R} V_{j+k,l+q}(s) \cdot \beta_x^{\frac{j+k}{2}}(s) \beta_z^{\frac{l+m}{2}}(s) e^{i[(j-k)W_x(s)+(l-m)W_x(s)+p\frac{s-s_0}{R}]} ds$$

We substitute the function that give the azimuthal distribution of the normal sextupoles along the ring

$$V(\bar{x};s) = b_2(s)(x^3 - 3xz^2) = V_{30}(s)x^3 + V_{12}(s)xz^2$$

generate the following resonant driving terms (see Guignard, Bengtsson)

$$h_{3000p} = \frac{1}{2\pi} \frac{1}{2^{\frac{3}{2}}} \cdot \int_{s_0}^{s_0+2\pi R} V_{30}(s) \cdot \beta_x^{\frac{3}{2}}(s) e^{i[3W_x(s)+p\frac{s-s_0}{R}]} ds$$

from V_{30} j+k = 3; l+m =0

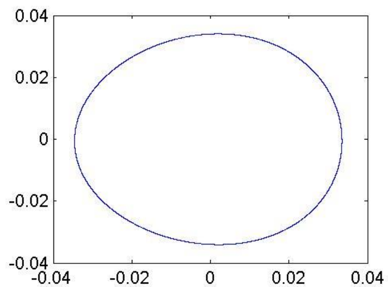
$$h_{3000p};\ h_{2100p};\ h_{1020p};\ h_{1011p};\ h_{1002p}$$

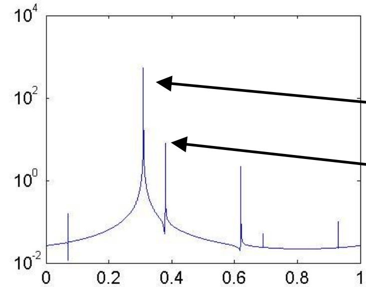
exciting the resonances (3, 0) (1, 0) (1, 2) (1, –2)

Example: third order resonance with a sextupole

Consider a linear lattice with a single sextupole kick. The resonance driving terms h_{3000} exciting the third order resonance (3,0) generates the frequency

$$\nu(h_{jklm}) = (1-j+k)Q_x + (m-l)Q_y \longrightarrow \nu(h_{3000}) = -2Q_x$$



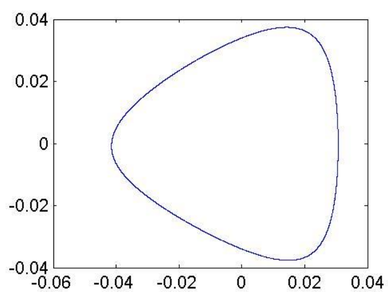


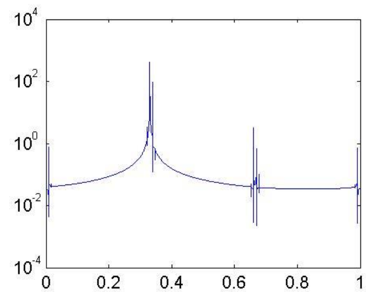
Far from the resonant values $Q_x = p/3$ e.g.

$Q_x = 0.31$

$-2Q_x = -0.62 \Rightarrow 0.38$

the lines Q_x and $-2Q_x$ are well separated





Approaching the resonant value $Q_x = 1/3$

$Q_x = 0.33 \quad -2Q_x = -0.66 \Rightarrow 0.34$

the tune spectral line (Q_x) and

the h_{3000} spectral line (-2Qx) coalesce

Resonance driving terms from octupoles

In an analogous way we can see that the normal octupoles in the circular ring

$$V(\bar{x};s) = b_4(s)(x^4 - 6x^2z^2 + z^4) = V_{40}(s)x^4 + V_{22}(s)x^2z^2 + V_{04}(s)z^4$$

generate the following resonant driving terms (see Guignard, Bengtsson)

$$h_{4000p} = \frac{1}{2\pi}\frac{1}{2^2}\cdot\int_{s_0}^{s_0+2\pi R} V_{40}(s)\cdot\beta_x^2(s)e^{i[4W_x(s)+p\frac{s-s_0}{R}]}ds$$

$h_{4000p};\ h_{3100p};\ h_{2200p};$ from V_{40} j+k = 4; l+m =0

$h_{2020p};\ h_{1120p};\ h_{2011p};$ from V_{22} j+k = 2; l+m =2

$h_{0040p};\ h_{0031p};\ h_{0022p};$ from V_{04} j+k = 0; l+m =4

exciting the resonances (4, 0) (2, 0) (2, 2) (2, –2) (0, 2) (0,4)

Resonance compensation

From the analysis of the Fourier expansion of the driving term we can infer simple rules to compensate the effect of strongly excited nonlinearities

The aim is to reduce the driving term

$$h_{jklmp} = \frac{1}{2\pi} \frac{1}{2^{\frac{j+k+l+m}{2}}} \binom{j+k}{j} \binom{l+m}{l} \cdot \int_{s_0}^{s_0+2\pi R} V_{j+k,l+q}(s) \cdot \beta_x^{\frac{j+k}{2}}(s) \beta_z^{\frac{l+m}{2}}(s) e^{i[(j-k)W_x(s)+(l-m)W_x(s)+p\frac{s-s_0}{R}]} ds$$

We have to find suitable distribution of nonlinear magnetic elements along the ring, i.e. suitable functions $V_{m,n}(s)$ that reduce or cancel those driving terms which are stronger in the uncorrected machine.

Typically two equal sextupoles at 60 degree phase advance apart compensate each other, in the (3,0) resonance driving term (and p=0 which is the strongest term)

In an analogous way two equal octupoles at 45 degree phase advance apart compensate each other in the (4,0) driving terms

However their effect on all the other resonances has to be assessed!

Can a sextupole excite a 4-th order resonance? (I)

Let us consider the nonlinear Hill's equation for the case of a linear lattice where a single sextupole kick is added

$$\frac{d^2x}{ds^2} + K(s)x = \frac{k_2}{2}x^2 \qquad\qquad K(s) = \frac{1}{\rho^2(s)} - k_1(s)$$

Let us use a perturbative procedure and try to solve this equation by successive approximations. The perturbation parameter ε is proportional to the sextupole strength k_2. We look for a solution of the type:

$$x(s) = x_0 + \varepsilon\, x_1(s) + \varepsilon^2\, x_2(s) + O(\varepsilon^3)$$

Substituting, ordering the contributions with the same perturbative order we have

$$\frac{d^2x_0}{ds^2} + K(s)x_0 = 0 \qquad \frac{d^2x_1}{ds^2} + K(s)x_1 = k_2(s)x_0^2(s) \qquad \frac{d^2x_2}{ds^2} + K(s)x_2 = 2k_2(s)x_0(s)x_1(s)$$

order zero: ε^0 first order: ε^1 second order: ε^2

Can a sextupole excite a 4-th order resonance? (II)

At each step we are using functions already calculated at the previous steps

$$x_0(s) = \sqrt{\varepsilon_y \beta_y(s)} \cos[\phi_x(s) + \phi_{x0}]$$

Linear solution

$$x_1(s) \propto A \cos[2\phi_x(s) + \phi_{x0}]$$

Term generated by the 3rd order resonance; linear with k_2 (first order)

$$x_2(s) \propto C \cos[3\phi_x(s) + \phi_{x0}] + D \cos[\phi_x(s) + \phi_{x0}]$$

Terms generated by the 4th order and 2nd order resonance; quadratic with k_2 (second order)

The series obtained from the successive approximation are in general divergent, however the first term of the series, judiciously chosen, offer a good approximation of the nonlinear betatron motion

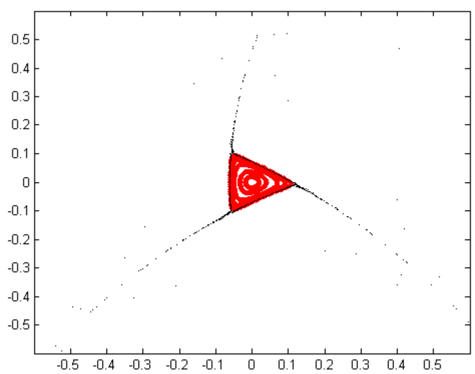
Can a sextupole excite a 4-th order resonance? (III)

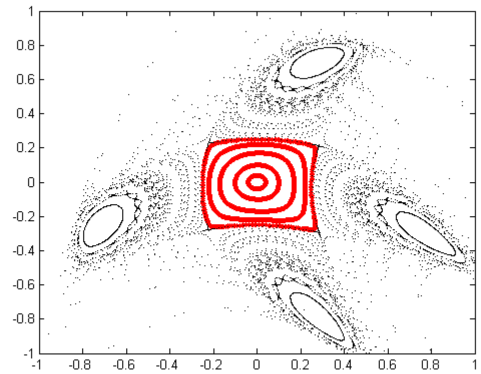
The equations can be solved numerically. The phase space plots of the motion of a charged particle in a lattice with a single thin sextupole are given by

Q = 1/3 +0.005

smaller scales in plot

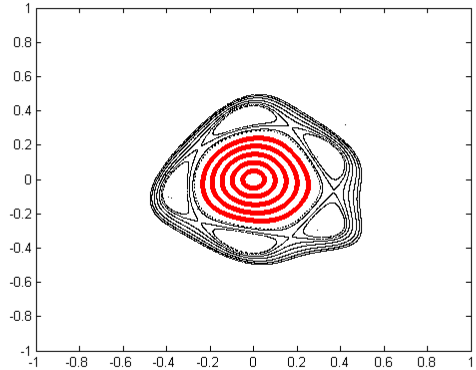
Tiny dynamic aperture

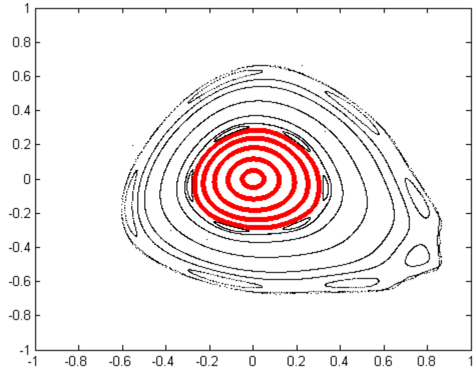




Q = 1/4 +0.005

Q = 1/5 +0.005





Q = 1/6 +0.005

Tracking particles close to the resonant tune value, starting at the same tune distance from the resonance, show that a sextupole can excite all higher order resonances. The islands width is smaller for higher orders, i.e. the corresponding resonances are weaker

Tracking (I)

Most accelerator codes have tracking capabilities: MAD, MADX, Tracy-II, elegant, AT, BETA, transport, …

Typically one defines a set of initial coordinates for a particle to be tracked for a given number of turns.

The tracking program "pushes" the particle through the magnetic elements. Each magnetic element transforms the initial coordinates according to a given integration rule which depends on the program used, e.g. transport (in MAD)

$$\vec{X} = (x, x', y, y', z, \delta) \ , \ \delta = \frac{\Delta P}{P_0}$$

$$\vec{X}_f = \mathbf{R}\vec{X}_i$$ Linear map

$$x_{j,f} = \sum_k \mathrm{R}_{jk} x_{j,i} + \sum_{kl} \mathrm{T}_{jkl} x_{j,i} x_{l,i} + \sum_{klm} \mathrm{U}_{jklm} x_{j,i} x_{l,i} x_{m,i} + \cdots$$

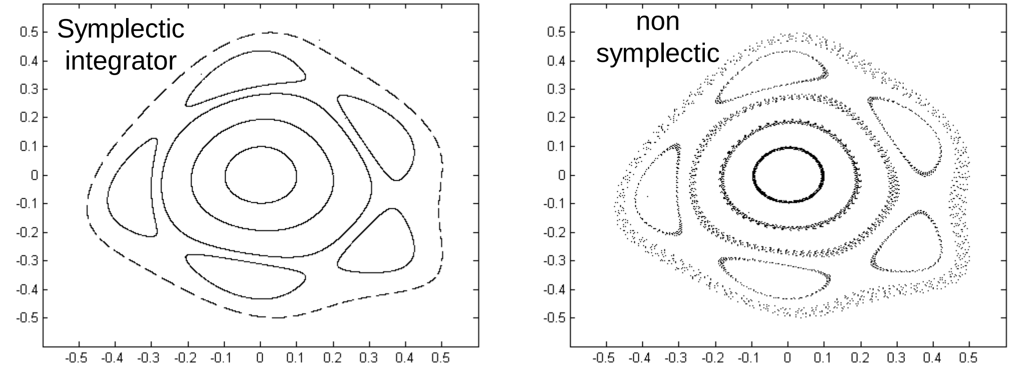
Nonlinear map up to third order as a truncated Taylor series

Tracking (II)

A Hamiltonian system is symplectic, i.e. the map which defines the evolution is symplectic (volumes of phase space are preserved by the symplectic map)

$$\bar{x}_f = M(\bar{x}_i) \qquad \text{M is symplectic transformation} \qquad J_{ab}(x_i) = \frac{\partial x_{a,f}}{\partial x_{b,i}} \qquad J^T S J = S$$

If the integrator is not symplectic one may found artificial damping or excitation effect



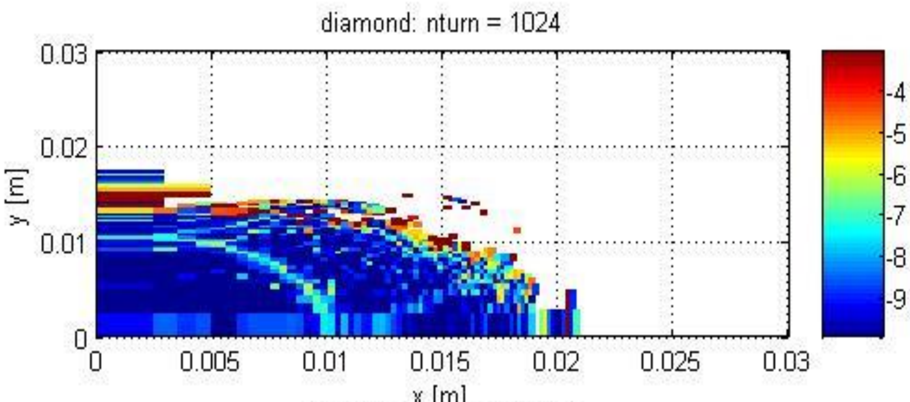
The well-known Runge-Kutta integrators are not symplectic. Likewise the truncated Taylor map is not symplectic. They are good for transfer line but they should not be used for circular machine in long term tracking analysis

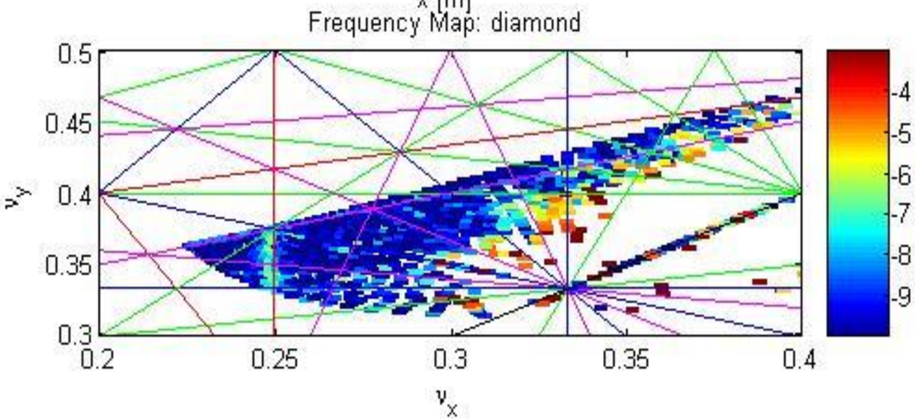
Elements described by thin lens kicks and drifts are always symplectic: long elements are usually sliced in many sections.

Frequency Map Analysis

The Frequency Map Analysis is a technique introduced in Accelerator Physics form Celestial Mechanics (Laskar).

It allows the identification of dangerous non linear resonances during design and operation. Strongly excited resonances can destroy the Dynamic Aperture.





To each point in the (x, y) aperture there corresponds a point in the (Q_x, Q_y) plane

The colour code gives a measure of the stability of the particle (blu = stable; red = unstable)

The indicator for the stability is given by the variation of the betatron tune during the evolution: i.e. tracking N turns we compute the tune from the first N/2 and the second N/2

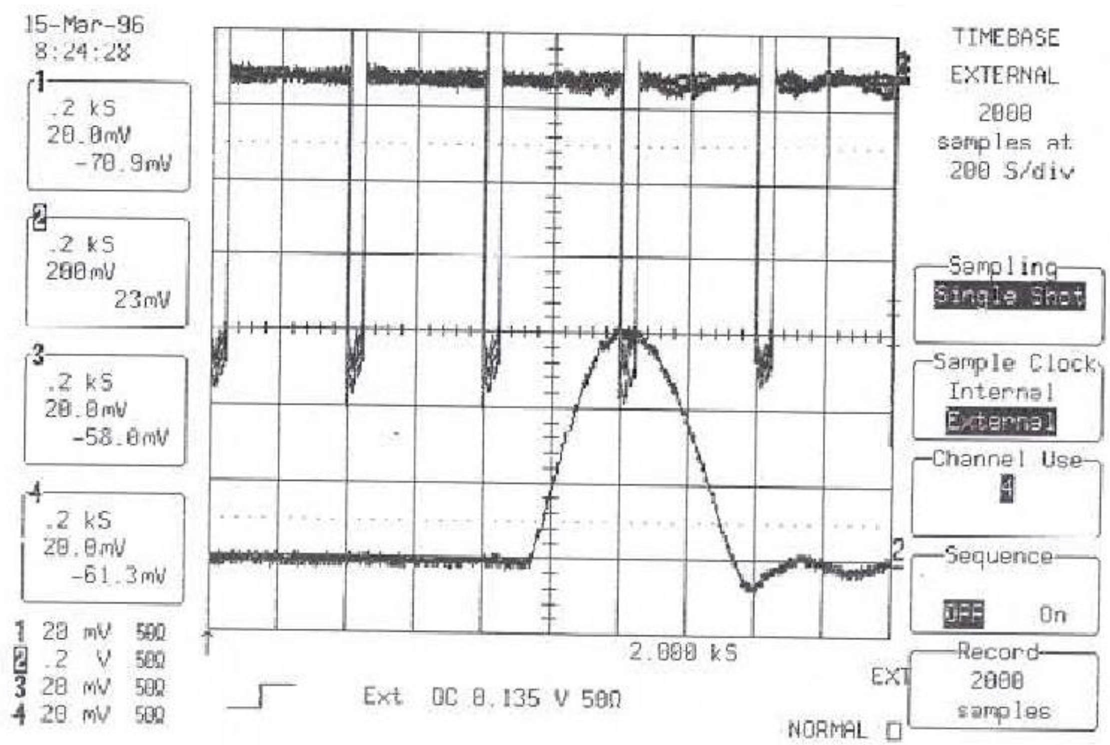
$$D = \log_{10} \sqrt{(Q_x^{(2)} - Q_x^{(1)})^2 + (Q_z^{(2)} - Q_z^{(1)})^2}$$

Frequency Map Measurement (I)

The measurement of the Frequency Map requires a set of two independent kickers to excite betatron oscillations in the horizontal and vertical planes of motion;

The Beam Position Monitors (BPMs) must have turn-by-turn capabilities (at least one !) in order to be able to measure the tunes from the induced betatron oscillations;

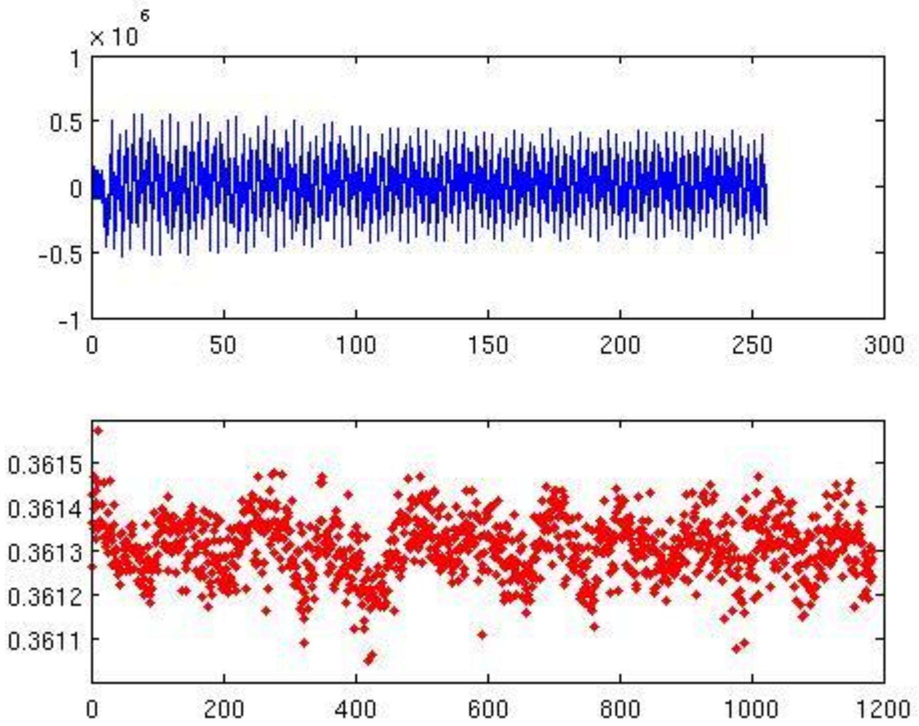
The betatron tune is generally the frequency corresponding to the maximum amplitude in the spectrum;



Frequency Map Measurement (II)

A example of betatron oscillations recorded after a kick in the vertical plane at diamond.

256 turns are recored: the time signals of many kicks is superimposed to check the reproducibility of the kick and of the oscillations, small variation in the betatron tunes are detected (2e-4)



FM measurement at the Advanced Light Source

Advanced Light Source

Energy = 1.5 – 1. 9 GeV

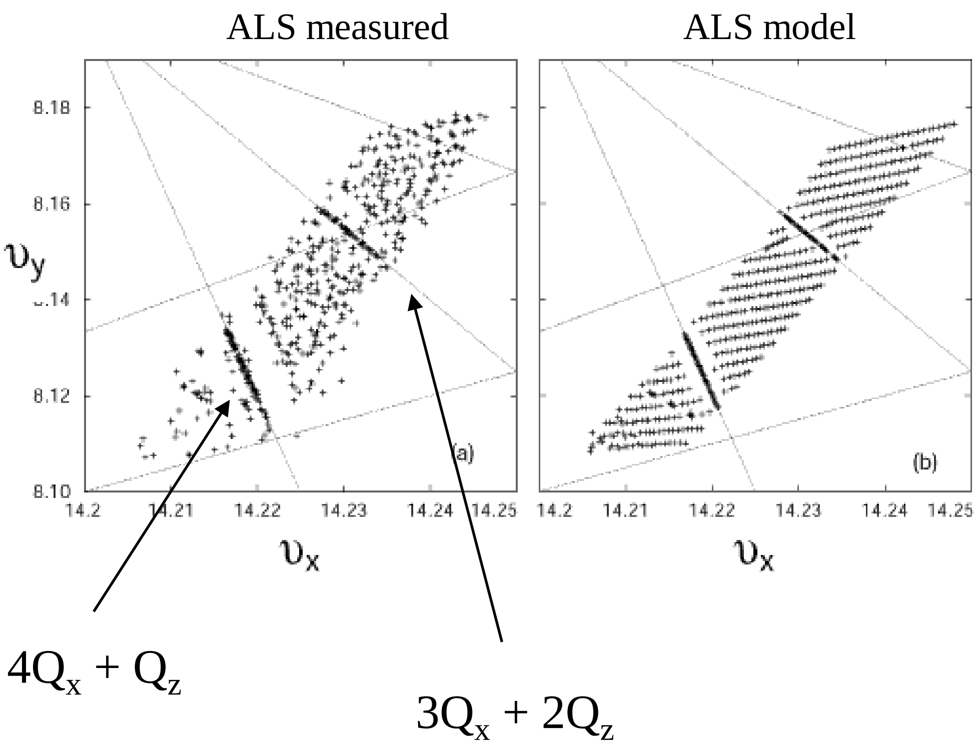
Circumference 198.6 m

Two single turn pinger H and V (600 ns)

Turn by Turn BPMs

40 electron bunches – 10 mA

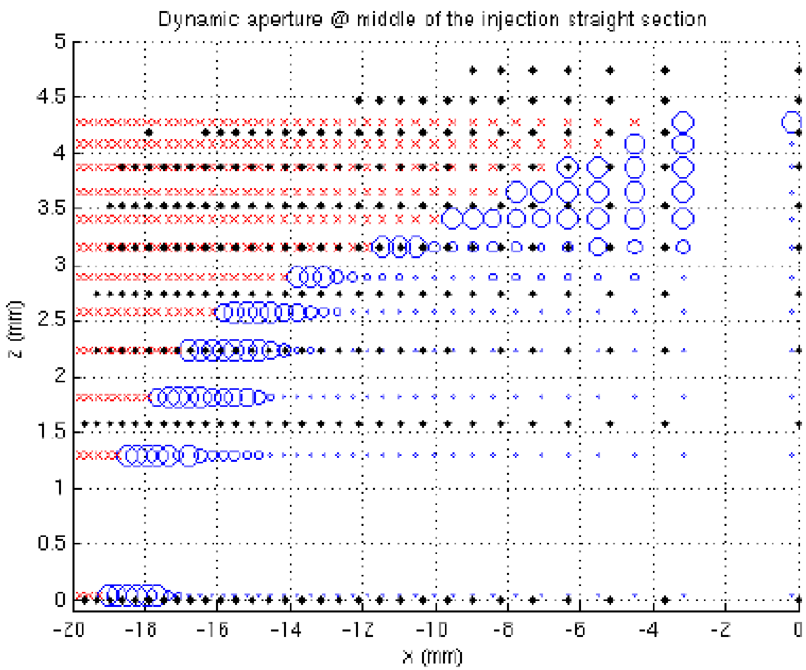
Used LOCO to set the linear lattice and restore super-periodicity (12–fold)



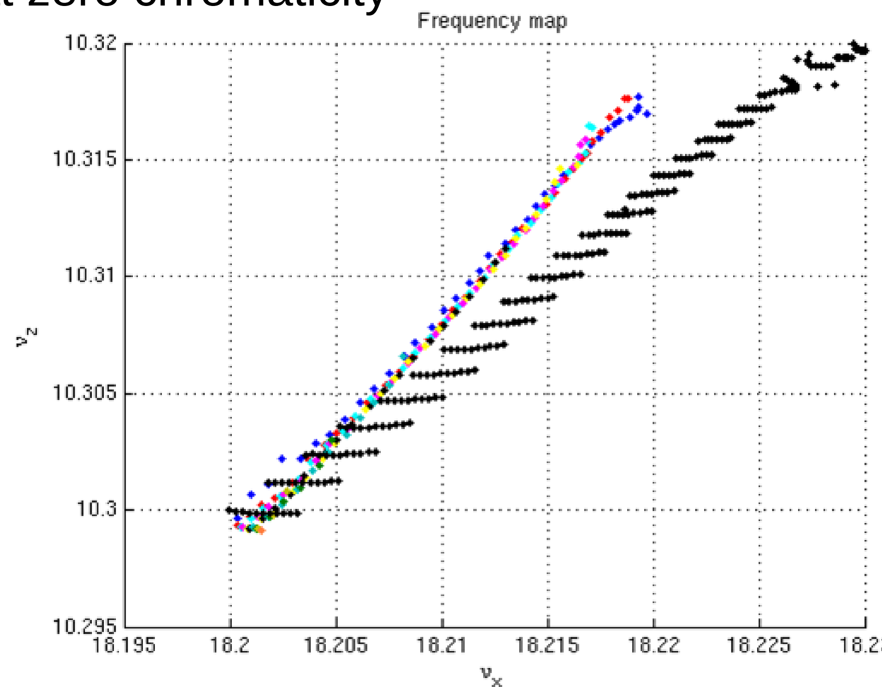
Very good comparison machine – model !

Dynamic Aperture: SOLEIL's example

SOLEIL bare lattice at zero chromaticity



Black–model; Blue–loss rate; Red unstable



Black–model; Colours measured

Tracking includes

Systematic multipole errors

- Dipole: up to 14-poles
- Quadrupoles: up to 28-poles
- Sextupoles: up to 54-poles
- Correctors (steerers): up to 22-poles
- Secondary coils in sext. → strong 10-pole term

From magnetic measurements:

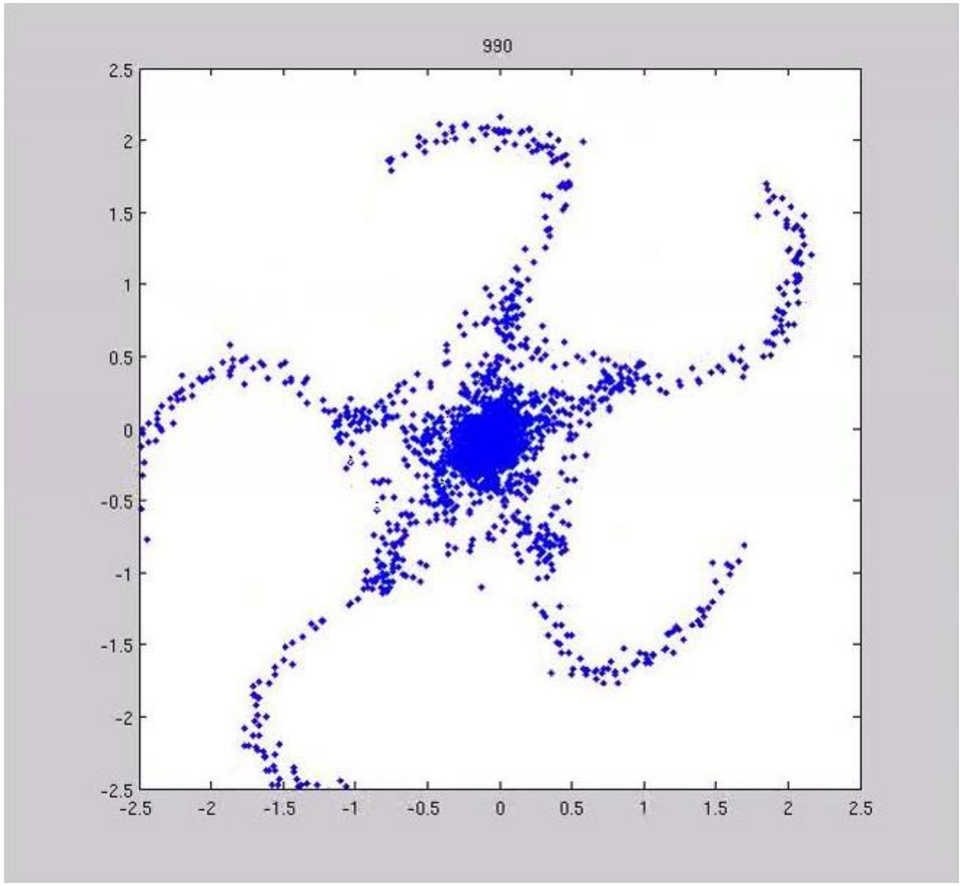
Dipole: fringe field, gradient error, edge tilt errors

Coupling errors (random rotation of quadrupoles)

No quadrupole fringe fields

Phase space orbit analysis

Using a kicker and two BPMs with a known phase advance we can reconstruct the orbit in phase space. Typically if the BPMs are at 90 degrees with the same β one can recover x and x’ and plot the phase space



Diamond horizontal phase space close to the 5th order resonance (2000 turns)

The damping in amplitude is not simply due to radiative damping but mainly to the fact that the centre of charge of the bunch is undergoing filamentation in phase space (decoherence)

Frequency spectrum measured at all BPMs at Diamond

All Diamond BPMs have turn-by-turn capabilities

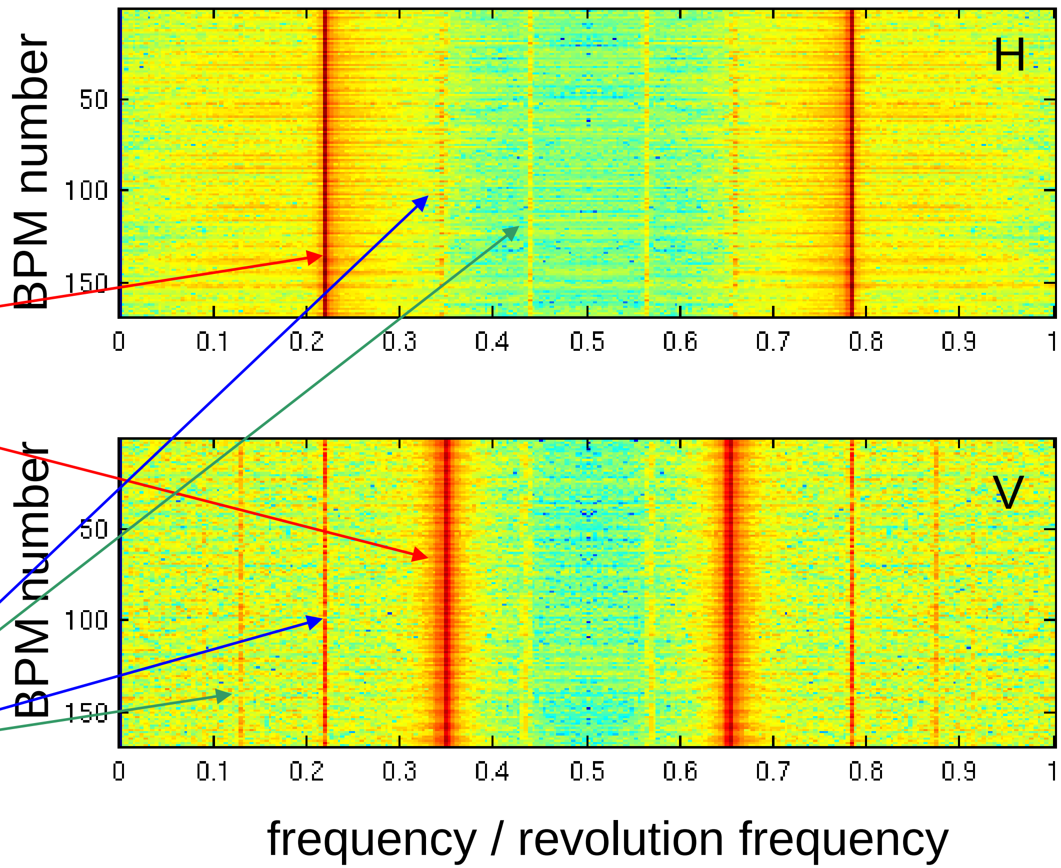
- excite the beam diagonally
- measure tbt data at all BPMs
- colour plots of the FFT

Q_X = 0.22 H tune in H

Q_y = 0.36 V tune in V

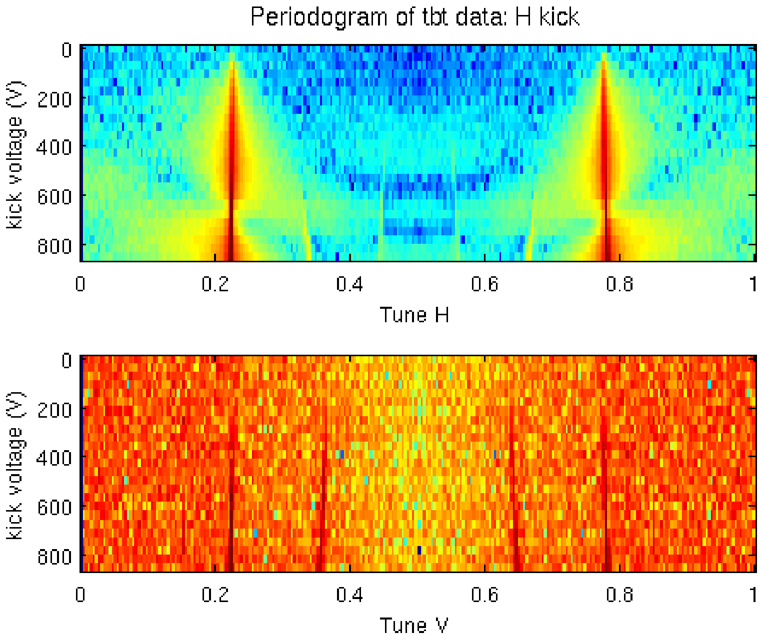
All the other important lines are linear combination of the tunes Q_x and Q_y

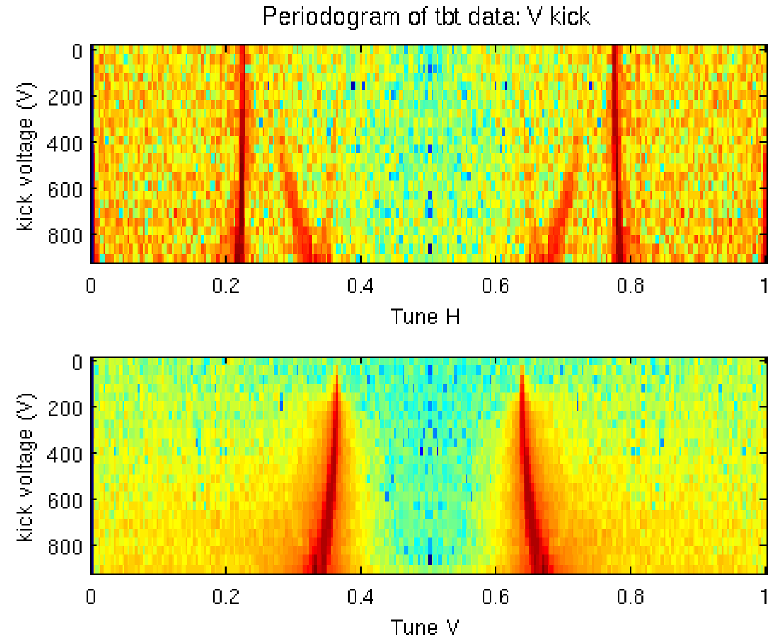
$m\,Q_x + n\,Q_y$



Detuning with amplitude and next to leading frequencies from turn-by-turn data

FFT as a function of the kicker strength





Q_x seen in the V plane: (1,1) resonance

$2Q_x$ seen in the H plane: (3,0) resonance

Q_y seen in the H plane: (1,1) resonance

$2Q_y$ seen in the H plane: (1,2) resonance

The information in the spectral lines can be used to compensate the resonant driving terms and improve the dynamic aperture of the ring

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