Wall Crossing Invariants from Spectral Networks

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A construction of the BPS monodromy for theories of class S, directly from the Coulomb branch geometry

- Does not involve knowledge of the BPS spectrum
- Manifest wall-crossing invariance
- Topological nature and symmetries of the superconformal index

The $d = 4,\ \mathcal{N} = 2$ super-Poincaré algebra $\mathfrak{s} = \mathfrak{s}_0 \oplus \mathfrak{s}_1$

$$\mathfrak{s}_0 = iso(1,3) \oplus su(2)_R \oplus u(1)_R \oplus \mathbb{C}$$
$$\mathfrak{s}_1 = (2,1;2)_{+1} \oplus (1,2;2)_{-1}$$

encodes the BPS bound

$$M \geq |Z|$$

BPS states are massive representations saturating this bound

$$M\,|\psi\rangle = |Z|\,|\psi\rangle\,, \qquad \mathcal{Q}_\vartheta\,|\psi\rangle = 0 \qquad (\vartheta = \mathrm{Arg}Z)$$

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when BPS states interact, they can form **BPS boundstates**

$$E_{bound} = |Z_1 + Z_2| - |Z_1| - |Z_2| \leq 0$$

these become **marginally stable** when

$$Z_1 // Z_2$$

4d $\mathcal{N}$=2 quantum field theories

- Coulomb branch $\mathcal{B}$ of vacua: $G \to U(1)^r$
- Quantized e.m. charges $\gamma \in \Gamma \simeq \mathbb{Z}^{2r}$, with $\mathbb{Z}$-valued DSZ pairing $\langle \cdot , \cdot \rangle$
- Z_γ is topological and linear in γ [Olive-Witten]

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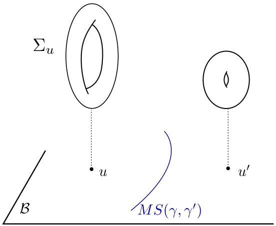
The IR dynamics is encoded by a family (Σ_u, λ_u) of Riemann surfaces and differentials over $\mathcal{B}$ **[Seiberg-Witten]**

$$\Gamma_u \simeq H_1(\Sigma_u, \mathbb{Z}) \qquad Z_\gamma(u) = \frac{1}{\pi} \oint_\gamma \lambda_u$$



$Z_\gamma(u)$ is meromorphic in $u \in \mathcal{B}$, **walls of marginal stability** divide $\mathcal{B}$ into chambers

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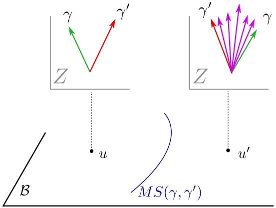

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Wall crossing: BPS boundstates can form/decay, the BPS spectrum must be determined chamber-wise.

The Kontsevich-Soibelman wall crossing formula

- BPS multiplets: $[(1/2,0)\oplus(0,1/2)]\otimes\mathfrak{h}$, with $\mathfrak{h}=(j,j_R)$ of $\mathfrak{so}(3)\oplus\mathfrak{su}(2)_R$
- Counted by a protected spin character:

$$\Omega(\gamma,u;y)=\mathrm{Tr}_{\mathfrak{h}_\gamma}\, y^{2J_3}(-y)^{2R_3}=\sum_{m\in\mathbb{Z}} a_m(\gamma,u)\cdot(-y)^m$$

  where $|a_m(\gamma,u)|$ counts $|\gamma,m\rangle$

- Quantum torus algebra: $X_\gamma X_{\gamma'}=y^{\langle\gamma,\gamma'\rangle}X_{\gamma+\gamma'}$

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Jumps of the BPS spectrum are controlled by an $\mathrm{Arg}\,Z_\gamma$-ordered product of quantum dilogarithms

$$\prod_{\gamma,m}^{\mathrm{Arg}Z(u)\nearrow}\Phi((-y)^m X_\gamma)^{a_m(\gamma,u)}=\prod_{\gamma',m'}^{\mathrm{Arg}Z(u')\nearrow}\Phi((-y)^{m'}X_{\gamma'})^{a_{m'}(\gamma',u')}$$

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Motivation

- The BPS monodromy $\mathbb{U}$ is of central importance in wall crossing. It is also a spectrum generating function, BPS state counting follows from knowledge of $\mathbb{U}$ [Kontsevich-Soibelman, Gaiotto-Moore-Neitzke, Dimofte-Gukov].
- Relation to various specializations of the superconformal index [Cecotti-Neitzke-Vafa, Iqbal-Vafa, Cordova-Shao, Cecotti-Song-Vafa-Yan].
- Graphs encoding $\mathbb{U}$ are an important link in the Network/Quiver correspondence

Surface defects are good probes of BPS spectra and phases of wall-crossing in 4d $\mathcal{N}=2$ theories [Gaiotto-Moore-Neitzke]

2d-4d system

- 2d $\mathcal{N}=(2,2)$ theory on $\mathbb{R}^{1,1} \subset \mathbb{R}^{1,3}$
- chiral matter transforming under a global symmetry G
- 4d vectormultiplets couple to 2d chirals, gauging G
- Adjoint 4d scalars give masses for 2d chirals, 2d theory is massive with effective superpotential $\widetilde{W}(u)$ controlled by 4d Coulomb moduli
- Finite number of massive vacua, with solitons interpolating between them

Example: U(1) GLSM with a charged chiral doublet of SU(2) global symmetry, coupled to 4d SU(2) SYM

- On the 4d Coulomb branch $\langle\Phi\rangle$ breaks $SU(2) \to U(1)_{4d}$, and generates masses for 2d chiral multiplets
- Effective theory of the 2d field-strength σ

$$\widetilde{W} = t\,\sigma - \mathrm{Tr}\,(\sigma + \Phi)\log(\sigma + \Phi)/e$$

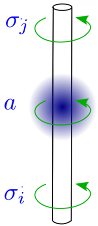
- 4d quantum dynamics **[Gaiotto-Gukov-Seiberg]**

$$\partial^2_\sigma \widetilde{W} = \left\langle \mathrm{Tr}\,\frac{1}{\sigma + \Phi} \right\rangle \qquad \Rightarrow \qquad \partial_\sigma \widetilde{W} = t - arcosh\left(\frac{\sigma^2 - u}{2\Lambda^2}\right)$$

- 2d chiral ring equations coincide with SU(2) SYM **Seiberg-Witten curve**, presented as a 2-fold ramified covering π over the t-plane (FI-θ coupling)
- Discrete set of **massive vacua** $\pi^{-1}(t) \in \Sigma_u$: one per sheet $\sigma_i(t, u)$
- A defect vacuum is a source of $U(1)_{4d}$ monodromy for 4d IR gauge field, similar to flux in a solenoid **[Gukov-Witten]**

New soliton field configurations of 2d and 4d d.o.f. introduced by the defect

- a 2d **topological charge (ij)** for $\sigma_i(t,u) \to \sigma_j(t,u)$
- a 2d **flavor charge** $\gamma \in \Gamma$, corresponding to 4d gauge charge
- space-dependent monodromy for 4d $U(1)^r$ gauge fields: boundary values classified by (ij), profile classified by γ



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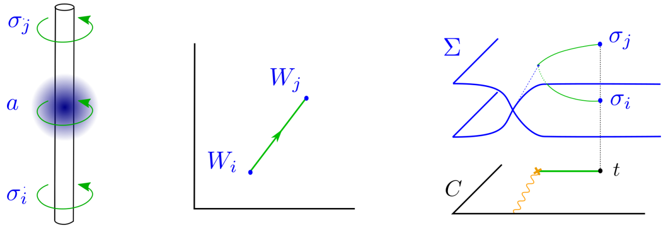
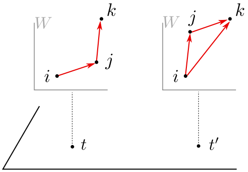
BPS equations $\partial_{x_1}\sigma = \alpha \cdot \partial_\sigma \widetilde{W}$ with slope $\alpha = \Delta\widetilde{W}/|\Delta\widetilde{W}|$,

$$Z_{ij,\gamma}(t,u) = \widetilde{W}_j(t,u) - \widetilde{W}_i(t,u) + Z_\gamma(u)$$



2d wall crossing: vacua σ_i depend on t, so does $Z_{ij} = \widetilde{W}_j - \widetilde{W}_i$.
Marginal stability when $Z_{ij} // Z_{jk}$, 2d spectrum jumps **[Cecotti-Vafa]**



$Z_{ij} // Z_{jk}$

$(ij) + (jk) \to (ik)$

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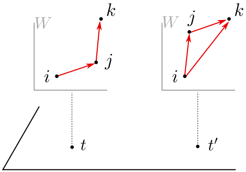


$$Z_{ij}//Z_{jk}$$

$$(ij) + (jk) \to (ik)$$

In 2d-4d systems a new kind of boundstate appears: pure flavor in 2d, gauge charge in 4d **[Hanany-Hori]**

$$(ij, \gamma') + (ji, \gamma'') \to (ii, \gamma) \ \sim \ \gamma$$

2d-4d mixing: Boundstates of solitons of opposite type mix with 4d BPS states, in this way the surface defect **probes the 4d BPS spectrum** **[Gaiotto-Moore-Neitzke]**

$$Z_{ij} \ // \ Z_{ji} \ // \ Z_\gamma$$

Class $\mathcal{S}$ theories $\mathcal{S}[\mathfrak{g}_{ADE}, C, D]$: twisted compactifications of the 6d (2,0) theory on the Riemann surface C "UV curve" **[Gaiotto, Gaiotto-Moore-Neitzke]**

- Coulomb branch geometry is encoded by Hitchin systems **[Martinec-Warner, Gorski-Krichever-Marshakov-Mironov-Morozov, Donagi-Witten]** due to their 6d origin **[Gaiotto-Moore-Neitzke]**
- Seiberg-Witten curve identified with **spectral curve** $\Sigma_u \subset T^*C$, naturally presented as ramified covering of C
- **Canonical surface defect**: UV curve C generalizes the FI parameter space, Σ_u is the 2d vacuum manifold.
- A_n theories: M theory engineering by wrapping M5 branes on $C \times \mathbb{R}^{1,3}$, with M2 ending on $\{z\} \times \mathbb{R}^{1,1}$ **[Hanany-Hori, Witten, Klemm-Lerche-Mayr-Vafa-Warner, Tong ...]**

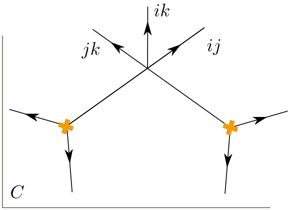
Webs of trajectories associated to the covering $\Sigma \to C$

Geometric data

- Trajectories from branch points: $(\partial_\tau, \lambda_j - \lambda_i) = e^{i\vartheta}$ (BPS equation)
- New trajectories from joints: (ij) + (jk) = (ik)

Combinatorial data

- Soliton data on each trajectory $\{(a, \mu(a)) \,|\, a \in H_1^{rel}(\Sigma_u, \mathbb{Z}), \mu \in \mathbb{Z}\}$



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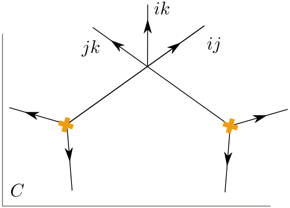
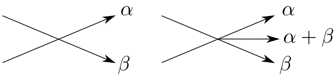


$\mathcal{W}(\vartheta, u)$ **counts 2d-4d BPS states** on surface defect at $z \in C$ with $\mathrm{Arg} Z_a = \vartheta$
...without any field theory computation: **2d-4d spectrum is determined by $\mathcal{W}$**

Spectral networks carry a natural **Lie algebraic structure**, underlying the generalization to ADE theories of class $\mathcal{S}$, and beyond canonical defects [PL-Park]



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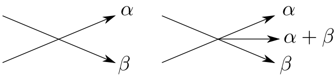

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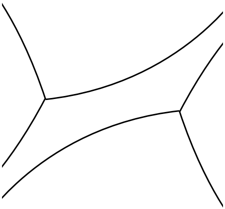
4d BPS spectra from 2d-4d mixing

- Varying ϑ the **topology of a network jumps**, inducing wall crossing of 2d-4d BPS spectrum
- Jumps occur when $Z_{ij}//Z_{ji}//Z_\gamma$: marginal stability for **2d-4d mixing**
- **Finite edges** appear at $\vartheta = \mathrm{Arg} Z_\gamma$ corresponding to 4d BPS states



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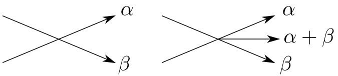

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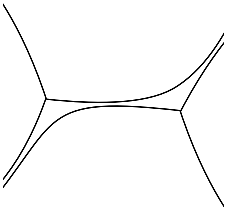

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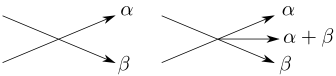

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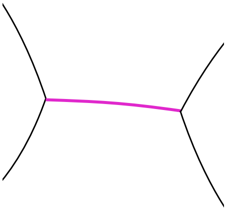

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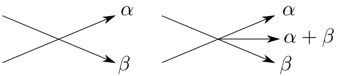

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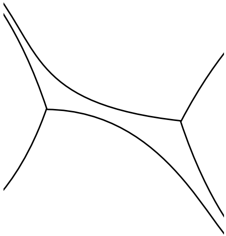

1. Formal generating series of 2d-4d BPS states preserving $\mathcal{Q}_\vartheta$ **[Galakhov-PL-Moore]**

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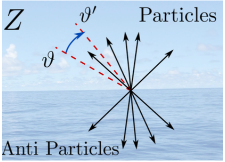
2. Piecewise-constant in ϑ, jumps across 4d BPS rays, at phases $\mathrm{Arg}\, Z_\gamma$ **[Gaiotto-Moore-Neitzke]**

$$F' = \left[\prod \Phi((-y)^m X_\gamma)^{a_m}\right] F \left[\prod \Phi((-y)^m X_\gamma)^{a_m}\right]^{-1}$$



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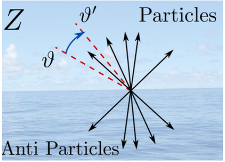

3. 4d BPS degeneracies $a_m(\gamma, u)$ control jumps in ϑ (at fixed u), Comparing $F(\vartheta, u)$ to $F(\vartheta + \pi, u)$ gives the **whole 4d spectrum**:

$$F(\vartheta + \pi, u) = \mathbb{U}\, F(\vartheta, u) \mathbb{U}^{-1}$$

Can use spectral networks to compute $F(\vartheta, u)$, $F(\vartheta + \pi, u)$ and obtain $\mathbb{U}$

- still choosing a chamber of $\mathcal{B}$ and some 4d BPS states
- still impractical: complexity of 2d-4d wall crossing

Marginal Stability

Let $\mathcal{B}_c \subset \mathcal{B}$ be a locus where central charges of **all 4d BPS particles** have **the same phase**

$$\mathcal{B}_c := \{u \in \mathcal{B}\,,\ \mathrm{Arg}\, Z_\gamma(u) = \mathrm{Arg}\, Z_{\gamma'}(u) \equiv \vartheta_c(u)\}$$

Because of marginal stability, the **4d BPS spectrum is ill-defined** at $u_c \in \mathcal{B}_c$.

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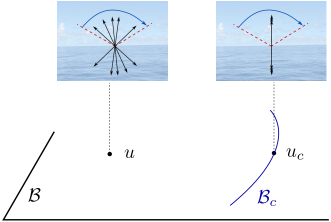
At $u_c \in \mathcal{B}_c$ the generating function of 2d-4d $\mathcal{Q}_\vartheta$-BPS states is well defined

$$F(\vartheta, u_c) = \sum_{ij,\gamma} \Omega(\vartheta, u_c, ij, \gamma; y)\, X_{ij,\gamma}$$



$$\text{at } u: \quad F' = \left[\prod \Phi((-y)^m X_\gamma)^{a_m(\gamma,u)}\right] \cdot F \cdot \left[\prod \Phi((-y)^m X_\gamma)^{a_m(\gamma,u)}\right]^{-1}$$

$$\text{at } u_c: \quad F' = \mathbb{U} \cdot F \cdot \mathbb{U}^{-1}$$

- $F(\vartheta, u_c)$ exhibits a **single jump** at ϑ_c which captures the **full BPS monodromy**
- From the viewpoint of 2d-4d states nothing special happens at the critical locus: can "parallel transport" both F and F' to $\mathcal{B}_c$
- Redefining $\mathbb{U}$ as the jump $F \to F'$, **extends its definition to $\mathcal{B}_c$**

$\mathbb{U}$ is determined by considering **several surface defects** at once. Each contributes $F' = \mathbb{U}\, F\, \mathbb{U}^{-1}$. Both F, F' are computed by **spectral networks**.

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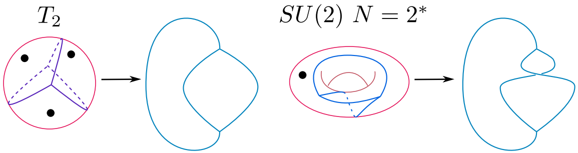
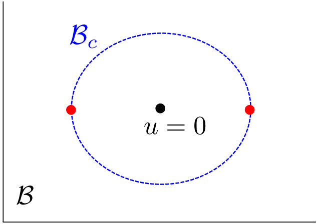
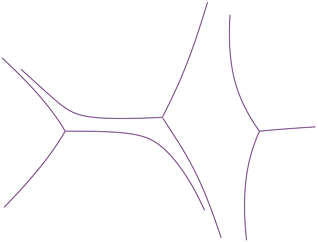
The **graph** topology, together with a notion of framing, **determine** $\mathbb{U}$.



First Example: Argyres-Douglas





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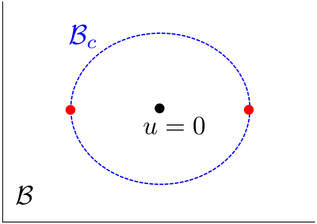

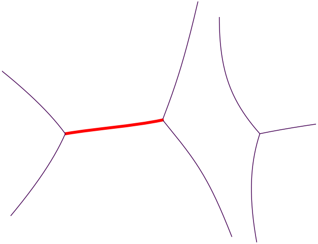

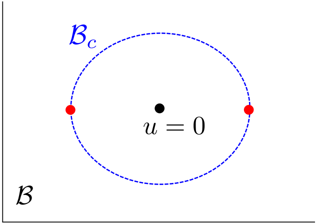
$\mathcal{B}_c$
$u = 0$
$\mathcal{B}$

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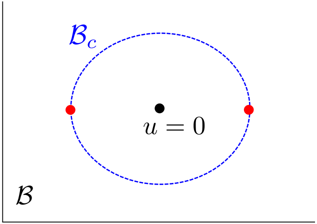

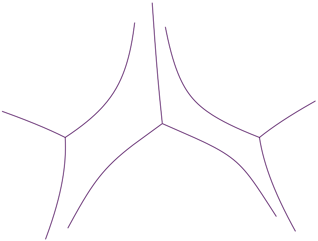

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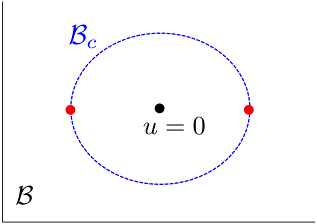

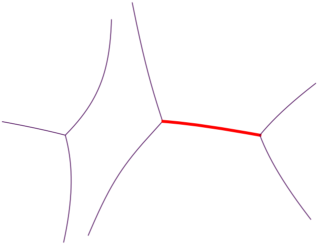

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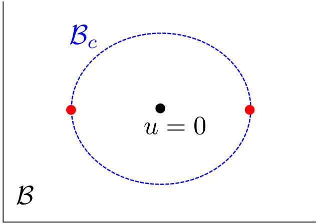

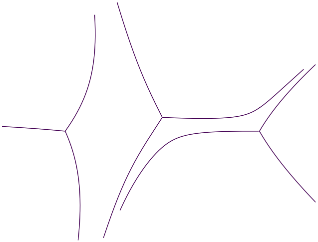

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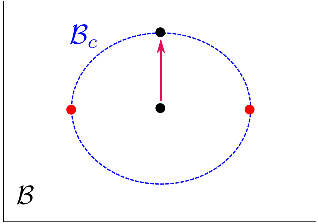

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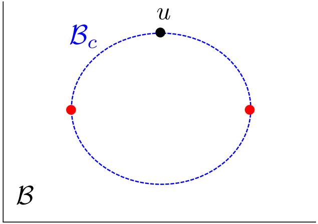

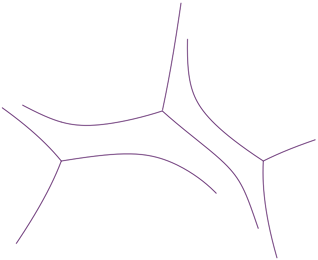

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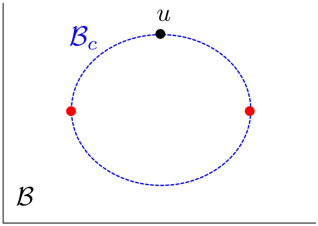
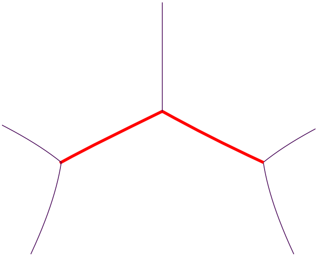
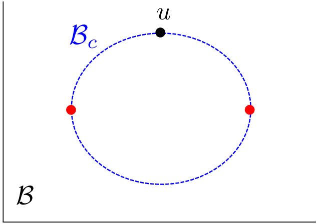
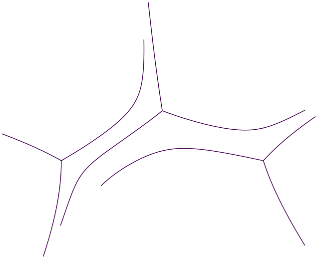






u
$\mathcal{B}_c$
$\mathcal{B}$



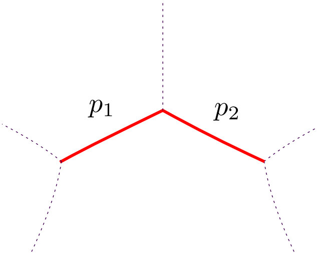
First Example: Argyres-Douglas

The graph has 2 edges, each contributes an equation

$$F'_p = \mathbb{U}\, F_p\, \mathbb{U}^{-1}$$

with

$$\begin{aligned} F_{p_1} &= 1 + y^{-1} X_{\gamma_1} + y^{-1} X_{\gamma_1+\gamma_2} \\ F_{p_2} &= 1 + y^{-1} X_{\gamma_2} \\ F'_{p_1} &= 1 + y^{-1} X_{\gamma_1} \\ F'_{p_2} &= 1 + y^{-1} X_{\gamma_2} + y^{-1} X_{\gamma_1+\gamma_2} \end{aligned}$$



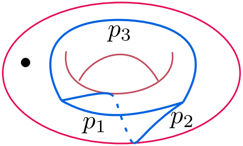
Together, they determine the monodromy

$$\begin{aligned} \mathbb{U} &= 1 - \frac{y}{(y)_1}\left(X_{\gamma_1} + X_{\gamma_2}\right) + \frac{y^2}{(y)_1^2} X_{\gamma_1+\gamma_2} + \frac{y^2}{(y)_2}\left(X_{2\gamma_1} + X_{2\gamma_2}\right) + \ldots \\ &= \Phi(X_{\gamma_1})\Phi(X_{\gamma_2}) \end{aligned}$$

The graph has three edges p_1, p_2, p_3;
each contributes one equation

$$F'_p = \mathbb{U}\, F_p\, \mathbb{U}^{-1}$$

with



$$F_{p_1} = \frac{1+X_{\gamma_1}+(y+y^{-1})X_{\gamma_1+\gamma_3}+X_{\gamma_1+2\gamma_3}+(y+y^{-1})X_{\gamma_1+\gamma_2+2\gamma_3}+X_{\gamma_1+2\gamma_2+2\gamma_3}+X_{2\gamma_1+2\gamma_2+2\gamma_3}}{(1-X_{2\gamma_1+2\gamma_2+2\gamma_3})^2}$$

$$F'_{p_1} = \frac{1+X_{\gamma_1}+(y+y^{-1})X_{\gamma_1+\gamma_2}+X_{\gamma_1+2\gamma_2}+(y+y^{-1})X_{\gamma_1+2\gamma_2+\gamma_3}+X_{\gamma_1+2\gamma_2+2\gamma_3}+X_{2\gamma_1+2\gamma_2+2\gamma_3}}{(1-X_{2\gamma_1+2\gamma_2+2\gamma_3})^2}$$

$F_{p_{2,3}}$ & $F'_{p_{2,3}}$ are obtained by cyclic shifts of $\gamma_1, \gamma_2, \gamma_3$.

The solution:

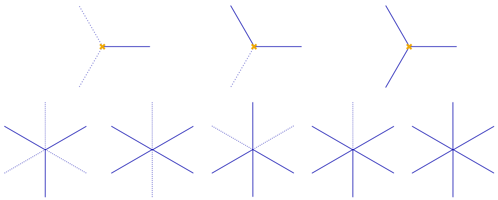
$$\mathbb{U} = \left(\prod_{n\geq 0}^{\nearrow} \Phi\left(X_{\gamma_1+n(\gamma_1+\gamma_2)}\right)\right)$$

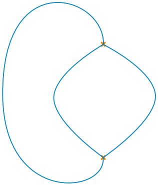
$$\times \Phi\left(X_{\gamma_3}\right) \Phi\left((-y)X_{\gamma_1+\gamma_2}\right)^{-1} \Phi\left((-y)^{-1}X_{\gamma_1+\gamma_2}\right)^{-1} \Phi\left(X_{2\gamma_1+2\gamma_2+\gamma_3}\right)$$

$$\times \left(\prod_{n\geq 0}^{\searrow} \Phi(X_{\gamma_2+n(\gamma_1+\gamma_2)})\right)$$

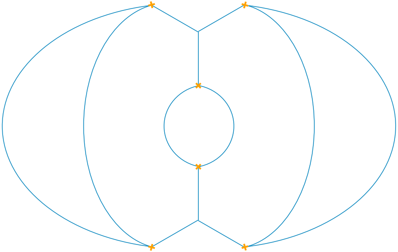
Spectral networks rules constrain the types of graphs that can occur.

Graphs can have two types of **nodes**: **branch points** or **joints**.
Combinatorics of 2d-4d soliton propagation depends on the node type.

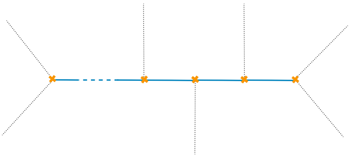




T_2



T_3



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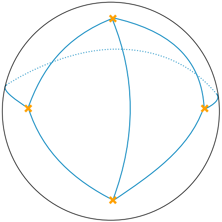

$SU(2)\ N_f = 4$

Framing

F_p, F'_p can be determined from the graph by simple rules, based on

- the **topology** of a graph
- a **framing**: a cyclic ordering of edges at each node

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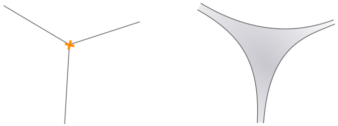
Graphs of A_1 theories have no joints, only branch points. Topology and framing define a **ribbon graph**.



To each ($\overline{\mathbb{Q}}$-algebraic) Riemann surface C is associated a holomorphic map $\mathfrak{B} : C \to \mathbb{P}^1$, with ramification at $0, 1, \infty$ [Belyi].
The preimage $\mathfrak{B}^{-1}([0,1])$ is a ribbon graph on C, a dessin d'enfants [Grothendieck].
The ribbon graph is the union of critical leaves of a foliation on C by a Strebel differential [Harer, Mumford, Penner, Thurston, Mulase-Penkava].

Symmetries of a graph: automorphisms preserving both its topology and framing, they are **inherited by** $\mathbb{U}$.

These symmetries are often **hidden** by the Kontsevich-Soibelman factorization $\mathbb{U} = \prod \Phi(X)$. Instead they become **manifest on the graph** (Ex. $\mathbb{Z}_3$ symmetry in $\mathcal{N} = 2^*$).

Graph symmetries

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Graph symmetries show that $\mathbb{U}$ shares important properties of the superconformal index.

- Punctures on C encode **global symmetries** of a Class $\mathcal{S}$ theory [Gaiotto, Chacaltana-Distler-Tachikawa].
- The index is computed by correlators of a TQFT on C [Gadde-Pomoni-Rastelli-Razamat], it is a symmetric function of the flavor fugacities.
- Symmetries of the graph **permute punctures**, implying that $\mathbb{U}$ is a **symmetric** function of the corresponding **flavor fugacities**, like the index.

Conclusion

1. To a class S theory associate a **canonical "critical graph"** on the UV curve, emerging from a degenerate spectral network at $\mathcal{B}_c$.

2. A new definition of the BPS monodromy, encoded by the **topology and framing** of the graph.

3. Does not use BPS spectrum. **Manifest invariance** under wall-crossing. At the critical locus $\mathcal{B}_c$ the BPS spectrum is ill-defined.

4. Simpler than computing $\mathbb{U}$ by using BPS spectra. **Symmetries** of $\mathbb{U}$ are manifest from the graph.

Open questions

- Existence conditions for the critical locus $\mathcal{B}_c$ where the critical graph emerges
- Equivalence relations among graphs: different topology, same $\mathbb{U}$ on different components of $\mathcal{B}_c$
- Constructive approach by gluing graphs **[Gabella-PL in progress]**
- Relation to BPS quivers **[Gabella-PL-Park-Yamazaki in progress]**

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Thank You.