Optimised Schwarz Methods for Field/Circuit Coupling

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GRADUATE SCHOOL
computational engineering

Outline of the Talk

1. Optimised Schwarz
2. Field/Circuit Convergence Optimisation
3. Future work

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Introduction

Domain Decomposition

Methods that take a Boundary Value Problem

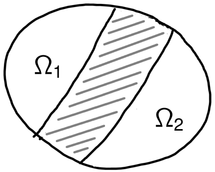
$$
\begin{aligned}
\mathrm{T}(x) &= \mathrm{f} && \text{in } \Omega \\
x &= x_0 && \text{on } \partial\Omega,
\end{aligned}
$$

and split the domain Ω into n subdomains to solve there smaller subproblems. In order to exchange information between them, some transmission conditions have to be defined at the boundaries.

- Subdomains can overlap.
- Typically solved iteratively.

Example

- For the case $n = 2$ and domain Ω, a subdivision into two overlapping domains $\Omega_1 \cup \Omega_2 = \Omega$ can be made.



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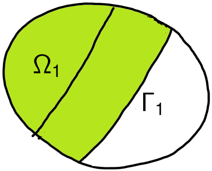

- With the systems

$$\begin{cases} T(x_1^k) = f_1 & \text{in } \Omega_1, \\ x_1^k = x_0 & \text{on } \partial\Omega_1 \backslash \Gamma_1, \\ x_1^k = x_2^{k-1} & \text{on } \Gamma_1 \end{cases}$$

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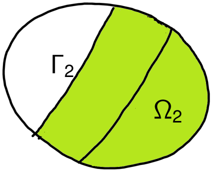

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Transmission conditions

- Study of different transmission conditions.
- Dirichlet T.C.

$$x_1^k = x_2^{k-1} \qquad x_2^k = x_1^{k-1}$$

- Dirichlet-Neuman T.C.

$$x_1^k = x_2^{k-1} \qquad \frac{\partial x_2^k}{\partial n} = \frac{\partial x_1^{k-1}}{\partial n}$$

- Robin-Robin T.C.

$$\frac{\partial x_i^k}{\partial n} + \alpha x_i^k = \frac{\partial x_j^{k-1}}{\partial n} + \alpha x_j^{k-1}, \text{ with } i, j = \{1, 2\} \text{ and } i \neq j.$$

Optimised Schwarz

Optimised Schwarz Methods study the optimal tranmission conditions for a certain problem.

Outline of the Talk

Waveform Relaxation Scheme

In frequency domain the general field/circuit coupling for a Jacobi-like scheme is

Circuit

$$\mathbf{A}j\omega\mathbf{x}^{k+1} + \mathbf{B}\mathbf{x}^{k+1} + \mathbf{P}\mathbf{i}_1^{k+1} = \hat{\mathbf{f}}(j\omega)$$
$$\mathbf{v}_1^{k+1} = \mathbf{P}^\top\mathbf{x}^{k+1}$$

With transmission condition

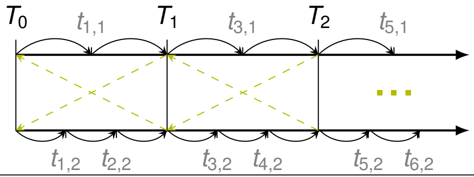
$$\mathbf{i}_1^{k+1} = \mathbf{i}_2^{k}$$

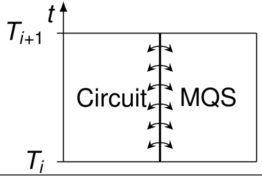
Magnetoquasistatic Field

$$\mathbf{M}j\omega\mathbf{u}^{k+1} + \mathbf{K}\mathbf{u}^{k+1} = \mathbf{X}\mathbf{i}_2^{k+1}$$
$$\mathbf{X}^\top j\omega\mathbf{u}^{k+1} = \mathbf{v}_2^{k+1}$$

With transmission condition

$$\mathbf{v}_2^{k+1} = \mathbf{v}_1^{k}.$$





Transmission conditions

- The scheme shows the source-coupling, which, from a Schwarz-type methods point of view, has Dirichlet transmission conditions.
- The first coupling equation $\mathbf{i}_1^{k+1} = \mathbf{i}_2^k$ can be changed to the more general expression

$$\alpha \mathbf{v}_1^{k+1} = \mathbf{i}_1^{k+1} - \mathbf{i}_2^k + \alpha \mathbf{v}_2^k.$$

- Compute contraction factor $\rho(\alpha)$ such that

$$||\mathbf{v}_1^{k+1} - \mathbf{v}_1^{k-1}|| = |\rho(\alpha)| \, ||\mathbf{v}_1^{k-1} - \mathbf{v}_1^{k-2}||.$$

This gives

$$\rho(\alpha) = (\mathbf{I} + \mathbf{x}_\mathrm{P}\alpha)^{-1} \left(-\mathbf{x}_\mathrm{P}(\mathbf{Z}(\omega)^{-1} - \alpha)\right),$$

with $\quad \mathbf{x}_\mathrm{P} = \mathbf{P}^\top (\mathbf{A}j\omega + \mathbf{B})^{-1}\mathbf{P}, \qquad \mathbf{Z}(\omega) = j\omega \mathbf{X}^\top (\mathbf{M}j\omega + \mathbf{K})^{-1}\mathbf{X}.$

Optimised Schwarz

- Contraction factor

$$\rho(\alpha) = (\mathbf{I} + \mathbf{x}_\mathrm{P}\alpha)^{-1}\left(-\mathbf{x}_\mathrm{P}(\mathbf{Z}(\omega)^{-1} - \alpha)\right).$$

- Iteration converges if $|\rho(\alpha)| < 1$. It can be sped up, if the contraction factor is decreased.
- Optimal convergence is obtained for $|\rho(\alpha)| = 0$, which is the case for

$$\alpha = \frac{1}{j\omega}\left(\mathbf{X}^\top(\mathbf{M}j\omega + \mathbf{K})^{-1}\mathbf{X}\right)^{-1},$$

  which is the admittance of the MQS system, as

$$\mathbf{v}_2^{k+1} = j\omega\mathbf{X}^\top(\mathbf{M}j\omega + \mathbf{K})^{-1}\mathbf{X}\mathbf{i}_2^{k+1}.$$

Series approximation

Inductance approximation

The aim is to approximate

$$\mathbf{L}(\omega) = \mathbf{X}^\top (\mathbf{M} j\omega + \mathbf{K})^{-1} \mathbf{X}.$$

the best possible way.

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Neumann Series

$$(\mathbf{I} - \mathbf{T})^{-1} = \sum_{i=0}^{\infty} \mathbf{T}^i,$$

with **T** being an operator on a normed space and $||\mathbf{T}|| < 1$.

Series approximation

Proposition

The inductance $\mathbf{L}(\omega) = \mathbf{X}^\top(\mathbf{M}j\omega + \mathbf{K})^{-1}\mathbf{X}$*, for a non-singular* $\mathbf{K}$*, can be approximated by the Neumann Series*

$$\mathbf{L}(\omega) = \mathbf{X}^\top\mathbf{K}^{-1}\sum_{i=0}^{\infty}(-j\omega\mathbf{M}\mathbf{K}^{-1})^i\mathbf{X},$$

if $||j\omega\mathbf{M}\mathbf{K}^{-1}||_2 < 1$*.*

For FIT (Finite Integration Technique), we have proved that the convergence condition is fulfilled for frequency

$$f < \frac{\pi}{8k_1^4 C_1 \sigma_{\max}\mu_{\max}},$$

with constants $h_{\max} = k_1 h_{\min}$ and $h_{\min} = \frac{C_1}{N+1}$.

Neumann Series approximation

The first term of the Neumann Series is

$$\mathbf{L} = \mathbf{X}^\top \mathbf{K}^{-1} \mathbf{X}.$$

The transmission condition with $\alpha = \frac{1}{j\omega}\mathbf{L}^{-1}$ is

Frequency Domain	Time Domain
$\frac{1}{j\omega}\mathbf{v}_1^{k+1} = \mathbf{X}^\top\mathbf{K}^{-1}\mathbf{X}\mathbf{i}_1^{k+1} - \mathbf{X}^\top\mathbf{K}^{-1}\mathbf{X}\mathbf{i}_2^{k} + \frac{1}{j\omega}\mathbf{v}_2^{k}.$	$\phi_1^{k+1} = \mathbf{X}^\top\mathbf{K}^{-1}\mathbf{X}\mathbf{i}_1^{k+1} - \mathbf{X}^\top\mathbf{K}^{-1}\mathbf{X}\mathbf{i}_2^{k} + \phi_2^{k},$ with ϕ_i being the mag. flux at system i.

The parameter coupling equation from (Schöps 2011[1]) is

$$\phi_\mathrm{L}^k = \mathbf{X}^\top\mathbf{K}^{-1}\mathbf{X}\mathbf{i}_\mathrm{L}^k - \mathbf{X}^\top\mathbf{K}^{-1}\mathbf{X}\mathbf{i}_\mathrm{M}^k + \phi_\mathrm{M}^k.$$

[1] Multiscale Modeling and Multirate Time-Integration of Field/Circuit Coupled Problems.

Outline of the Talk

Series approximation

Taylor Expansion

$$\mathbf{L}(\omega,\omega_0) = \sum_{k=0}^{\infty}(-j)^k \mathbf{X}^\top (\mathbf{M}j\omega_0 + \mathbf{K})^{-1} \left(\mathbf{M}(\mathbf{M}j\omega_0 + \mathbf{K})^{-1}\right)^k \mathbf{X}(\omega-\omega_0)^k$$

For $\omega_0 = 0$ the Neumann Series is obtained.

Idea

For convergence of the series with larger frequencies, expand around $\omega_0 \neq 0$.

Question

What is $j\omega_0$ in time domain?

Source coupling vs Parameter coupling

Ideas

- Show that the contraction factor

$$\rho(\alpha) = (\mathbf{I} + \mathbf{x}_\mathrm{P}\alpha)^{-1}\left(-\mathbf{x}_\mathrm{P}(\mathbf{Z}(\omega)^{-1} - \alpha)\right)$$

  is smaller for the parameter coupling than for the source coupling. That is

$$\begin{aligned}||\rho(0)|| =&|(-\mathbf{x}_\mathrm{P}\mathbf{Z}(\omega)^{-1})| \\ >&\left|\left(\mathbf{I} + \mathbf{x}_\mathrm{P}\frac{1}{j\omega}(\mathbf{X}^\top\mathbf{K}^{-1}\mathbf{X})^{-1}\right)^{-1}\left(-\mathbf{x}_\mathrm{P}\left(\mathbf{Z}(\omega)^{-1} - \frac{1}{j\omega}(\mathbf{X}^\top\mathbf{K}^{-1}\mathbf{X})^{-1}\right)\right)\right| \\ =&\rho\left(\frac{1}{j\omega}(\mathbf{X}^\top\mathbf{K}^{-1}\mathbf{X})^{-1}\right)\end{aligned}$$

- Show that $\mathrm{Re}(\mathbf{L}(\omega) = \mathbf{X}^\top(\mathbf{M}j\omega + \mathbf{K})^{-1}\mathbf{X}) \approx \mathbf{X}^\top\mathbf{K}^{-1}\mathbf{X} = \mathbf{L}$

Discussion...