A DIGRESSION ON DIVERGENCES IN ELECTROMAGNETISM

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FOREWORD

- The equations ruling the construction of an electromagnet have some aspects that can be used as a paradigm in teaching some aspects of physics and mathematics

CONTENTS

MAXWELL EQUATIONS

- Maxwell equations

$$\vec{\nabla}\cdot\vec{B}=0$$

$$\vec{\nabla}\times\vec{E}+\frac{1}{c}\frac{\partial B}{\partial t}=0$$

$$\vec{\nabla}\cdot\vec{E}=4\pi\rho$$

$$\vec{\nabla}\times\vec{B}-\frac{1}{c}\frac{\partial E}{\partial t}=\frac{4\pi}{c}\vec{J}$$

$$\vec{B}=\vec{\nabla}\times\vec{A}$$

$$\vec{E}=-\vec{\nabla}\phi-\frac{1}{c}\frac{\partial A}{\partial t}$$

James Clerk Maxwell,
Scottish
(13 June 1831 – 5 November 1879)

- Hide few monsters:
  - The presence of *c* means that they cannot be invariant with Galileo
  - So Maxwell has inside the special relativity

Ridley Scott, Alien (1979)

MAXWELL EQUATIONS

- Maxwell equations

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} + \frac{1}{c}\frac{\partial B}{\partial t} = 0$$

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho$$

$$\vec{\nabla} \times \vec{B} - \frac{1}{c}\frac{\partial E}{\partial t} = \frac{4\pi}{c}\vec{J}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial A}{\partial t}$$

James Clerk Maxwell, Scottish
(13 June 1831 – 5 November 1879)

- Hide few monsters:
  - Equations have divergences for infinitely small charge density and infinitely large current density
  - So the charge cannot be a point, it has some dimension

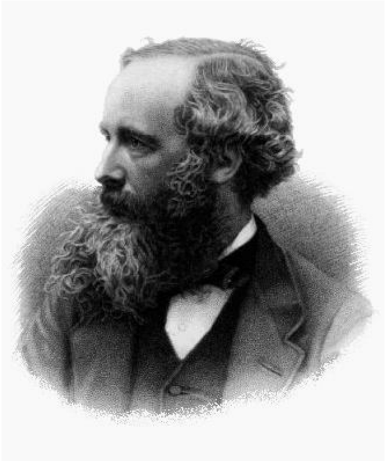
CONTENTS

- Maxwell equations
- The field of a current line
  - Analytic functions and Taylor series
  - Convergence domain and singularities
- The two divergences of electromagnetism
- An outloook on scales

MAXWELL EQUATIONS

- Maxwell equations for magnetic field

$$\nabla \cdot B = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \qquad \nabla \times B = \mu_0 J + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$$



James Clerk Maxwell,
Scottish
(13 June 1831 – 5 November 1879)

- In absence of charge and magnetized material

$$\nabla \times B = \left(\frac{\partial B_y}{\partial z} - \frac{\partial B_z}{\partial y}, \frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z}, \frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x} \right) = 0$$

- If $\frac{\partial B_z}{\partial z} = 0$ (constant longitudinal field), then

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0 \qquad \frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x} = 0$$

ANALYTIC FUNCTIONS

- A complex function of complex variables is analytic if it coincides with its power series

$$f(z)=\sum_{n=1}^{\infty} C_n z^{n-1} \qquad f_x(x,y)+if_y(x,y)=\sum_{n=1}^{\infty} C_n (x+iy)^{n-1} \qquad (x,y)\in D$$

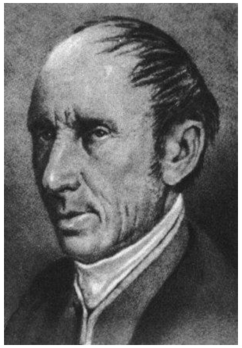
on a domain D !

  - Note: domains are usually a painful part, we talk about it later

- A necessary and sufficient condition to be analytic is that

$$\begin{cases} \dfrac{\partial f_x}{\partial x}-\dfrac{\partial f_y}{\partial y}=0 \\ \dfrac{\partial f_x}{\partial y}+\dfrac{\partial f_y}{\partial x}=0 \end{cases}$$

called the Cauchy-Riemann conditions



Augustin Louis Cauchy
French
(August 21, 1789 – May 23, 1857)

- If $\frac{\partial B_z}{\partial z} = 0$

  Maxwell gives
$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = 0$$
$$\frac{\partial B_y}{\partial y} + \frac{\partial B_x}{\partial x} = 0$$

$$\begin{cases} \frac{\partial f_x}{\partial x} - \frac{\partial f_y}{\partial y} = 0 \\ \frac{\partial f_x}{\partial y} + \frac{\partial f_y}{\partial x} = 0 \end{cases}$$

Georg Friedrich Bernhard Riemann, German (November 17, 1826 - July 20, 1866)

and therefore the function $B_y + iB_x$ is analytic

$$B_y(x,y) + iB_x(x,y) = \sum_{n=1}^{\infty} C_n (x+iy)^{n-1} \qquad (x,y) \in D$$

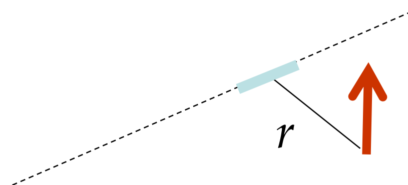
where C_n are complex coefficients

- Advantage: we reduce the description of a function from R^2 to R^2 to a (simple) series of complex coefficients
  - Attention !! We lose something (the function outside *D*)

FIELD OF A CURRENT LINE

- Field given by a current line (Biot-Savart law)
  - Differential form (international system)

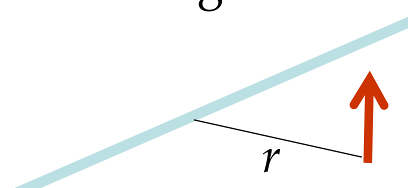
$$d\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{I}\times d\vec{r}}{|r|^3}$$



Félix Savart,
French
(June 30, 1791-March 16, 1841)

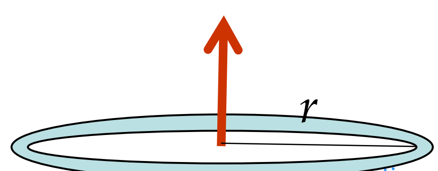
  - Infinite current line
    - A factor two is given by the atan integration

$$\vec{B}(\vec{r}) = \frac{\mu_0}{2\pi}\frac{\vec{I}\times\vec{r}}{|r|^2}$$



  - Field in a centre of a circular loop, radius *r*

$$B = \frac{\mu_0 I}{2r}$$



Jean-Baptiste Biot,
French
(April 21, 1774 – February 3, 1862)

FIELD OF A CURRENT LINE: COMPLEX NOTATION

- Field given by a current line (Biot-Savart law)

$$\vec{B}(\vec{r}) = \frac{\mu_0}{2\pi} \frac{\vec{I} \times \vec{r}}{|r|^2}$$

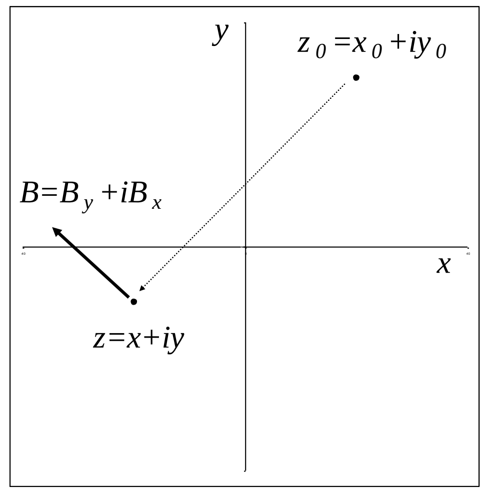
  - Infinite current line

$$B_x(x,y) = -\frac{\mu_0 I}{2\pi} \frac{y - y_0}{(x - x_0)^2 + (y - y_0)^2},$$

$$B_y(x,y) = \frac{\mu_0 I}{2\pi} \frac{x - x_0}{(x - x_0)^2 + (y - y_0)^2},$$

  - Complex notation

$$B_y(x,y) + \mathrm{i}B_x(x,y) = \frac{\mu_0 I}{2\pi} \frac{(x - x_0) - \mathrm{i}(y - y_0)}{(x - x_0)^2 + (y - y_0)^2}$$



  - Using the relation

$$\frac{a - \mathrm{i}b}{a^2 + b^2} = \frac{a - \mathrm{i}b}{(a + \mathrm{i}b)(a - \mathrm{i}b)} = \frac{1}{(a + \mathrm{i}b)}$$

  - We obtain the compact very useful notation

$$B(z) = \frac{\mu_0 I}{2\pi(z - z_0)}$$

Félix Savart,
French
(June 30, 1791-March 16, 1841)

Jean-Baptiste Biot,
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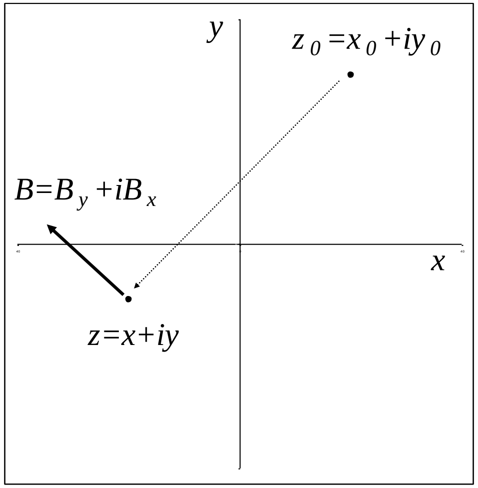
FIELD HARMONICS OF A CURRENT LINE

- Field given by a current line (Biot-Savart law)

$$B(z) = B_y(z) + iB_x(z)$$



$$B(z) = \frac{I\mu_0}{2\pi(z - z_0)} = -\frac{I\mu_0}{2\pi z_0}\frac{1}{1 - \frac{z}{z_0}}$$

Félix Savart,
French
(June 30, 1791-March 16, 1841)

using

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \ldots = \sum_{n=1}^{\infty} t^{n-1} \qquad |t| < 1 \text{ !!!}$$

and defining a R_{ref} as a scale length we get

$$B(z) = -\frac{I\mu_0}{2\pi z_0}\sum_{n=1}^{\infty}\left(\frac{z}{z_0}\right)^{n-1} = -\frac{I\mu_0}{2\pi z_0}\sum_{n=1}^{\infty}\left(\frac{R_{ref}}{z_0}\right)^{n-1}\left(\frac{x+iy}{R_{ref}}\right)^{n-1}$$

Jean-Baptiste Biot,
French
(April 21, 1774 – February 3, 1862)

FIELD HARMONICS OF A CURRENT LINE

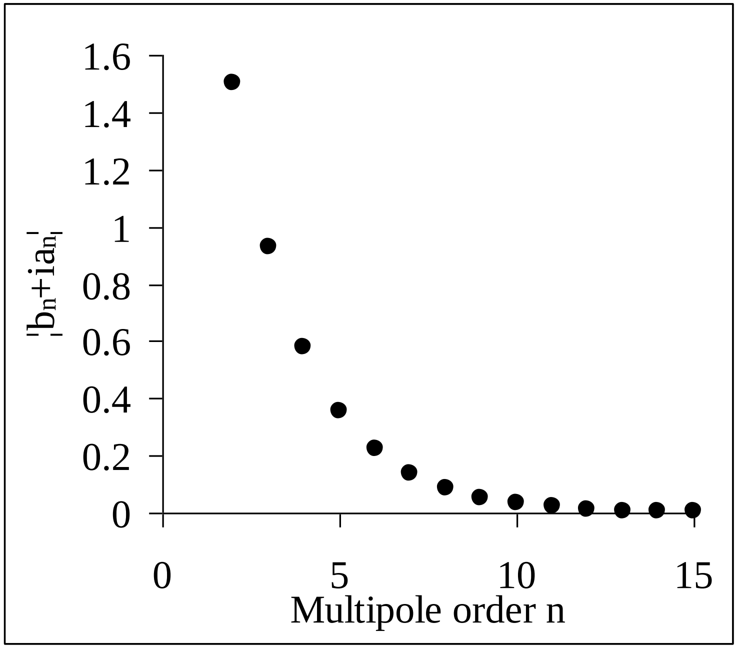
- Now we can compute the multipoles of a current line at z_0

$$B(z) = -\frac{I\mu_0}{2\pi z_0}\sum_{n=1}^{\infty}\left(\frac{z}{z_0}\right)^{n-1} = -\frac{I\mu_0}{2\pi z_0}\sum_{n=1}^{\infty}\left(\frac{R_{ref}}{z_0}\right)^{n-1}\left(\frac{x+iy}{R_{ref}}\right)^{n-1} \qquad |x+iy|<|z_0|$$

$$B_y + iB_x = 10^{-4} B_1 \sum_{n=1}^{\infty}(b_n + ia_n)\left(\frac{x+iy}{R_{ref}}\right)^{n-1}$$

$$B_1 = -\frac{I\mu_0}{2\pi}\,\mathrm{Re}\left(\frac{1}{z_0}\right)$$

$$b_n + ia_n = -\frac{I\mu_0 10^4}{2\pi z_0 B_1}\left(\frac{R_{ref}}{z_0}\right)^{n-1}$$

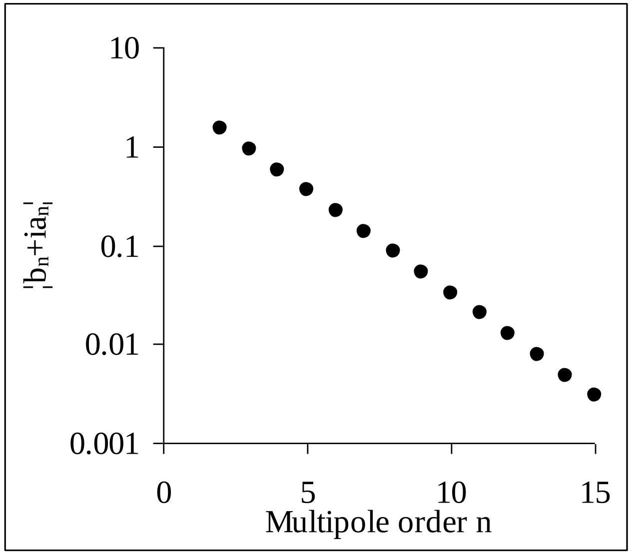


2. FIELD HARMONICS OF A CURRENT LINE

- Multipoles given by a current line decay with the order

$$b_n + i a_n = -\frac{I \mu_0 10^4}{2\pi z_0 B_1} \left(\frac{R_{ref}}{z_0} \right)^{n-1}$$

$$\ln\left(\left|b_n + i a_n\right|\right) = \ln\left(\frac{|I| \mu_0 10^4}{2\pi R_{ref} B_1} \right) + n \ln\left(\frac{R_{ref}}{|z_0|} \right) = p + nq$$



- The slope of the decay is the logarithm of ($R_{ref}/|z_0|$)
  - At each order, the multipole decreases by a factor $R_{ref}/|z_0|$
  - The decay of the multipoles tells you the ratio $R_{ref}/|z_0|$, i.e. where is the coil w.r.t. the reference radius –
  - like a radar … one can detect assembly errors in the magnet through magnetic field shape

2. FIELD HARMONICS OF A CURRENT LINE

- Multipoles given by a current line decay with the order

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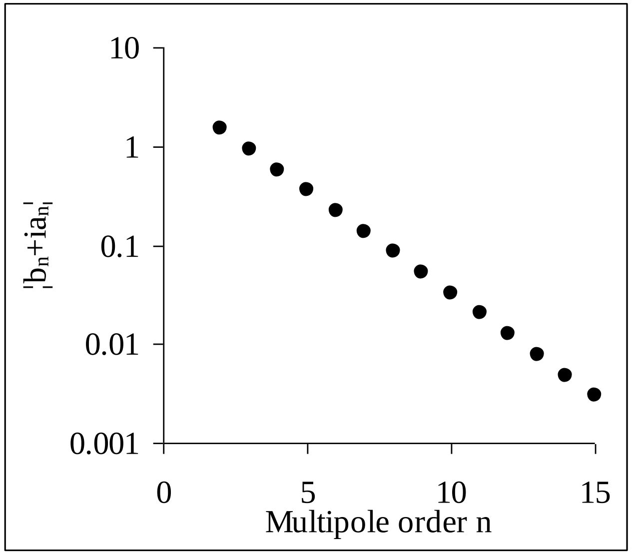
$$\ln\left(\left|b_n + i a_n\right|\right) = \ln\left(\frac{|I| \mu_0 10^4}{2\pi R_{ref} B_1} \right) + n \ln\left(\frac{R_{ref}}{|z_0|} \right) = p + nq$$



- The semilog scale is the natural way to plot multipoles
  - This is the point of view of Biot-Savart
- But usually specifications are on a linear scale
  - In general, multipoles must stay below one or a fraction of units – see later
  - This explains why only low order multipoles, in general, are relevant

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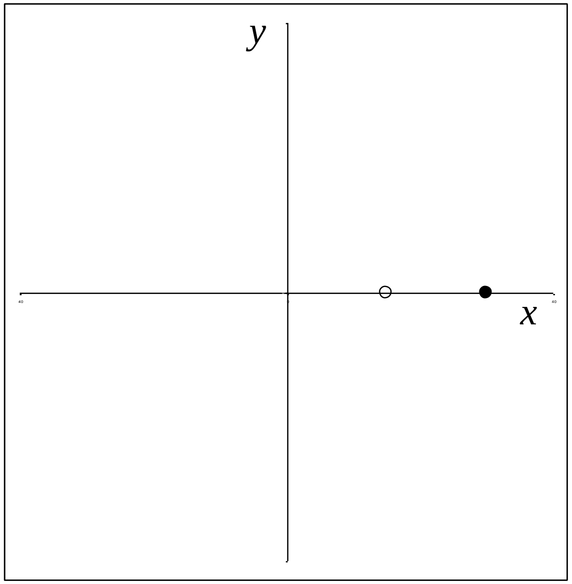
3. VALIDITY LIMITS OF FIELD HARMONICS

- When we expand a function in a power series we lose something

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \ldots = \sum_{n=1}^{\infty} t^{n-1} \qquad |t| < 1$$



- Example 1. In t=1/2, the function is

$$\frac{1}{1-1/2} = \frac{1}{1/2} = 2$$

and using the series one has

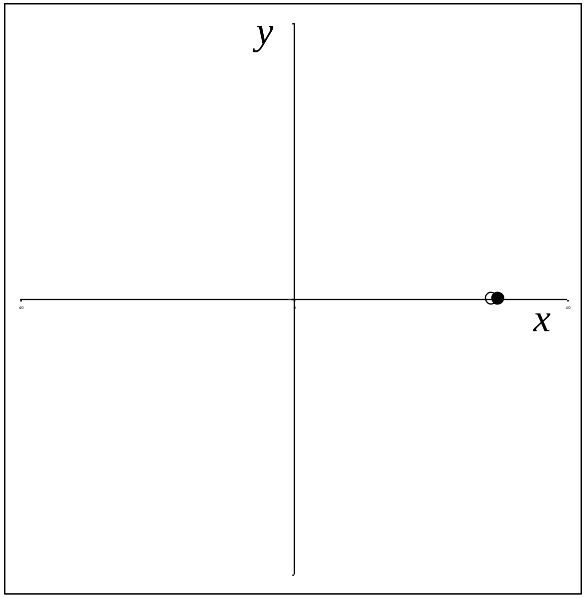
$$1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \ldots = 1 + 0.5 + 0.25 + 0.125 = 1.875 + \ldots$$

not bad … with 4 terms we compute the function within 7%

3. VALIDITY LIMITS OF FIELD HARMONICS

- When we expand a function in a power series we lose something

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \ldots = \sum_{n=1}^{\infty} t^{n-1} \qquad |t| < 1$$



- Ex. 2. In t=1, the function is infinite

$$\frac{1}{1-1} = \frac{1}{0} = \infty$$

and using the series one has

$$1 + 1 + (1)^2 + (1)^3 + \ldots = 1 + 1 + 1 + 1 + \ldots$$

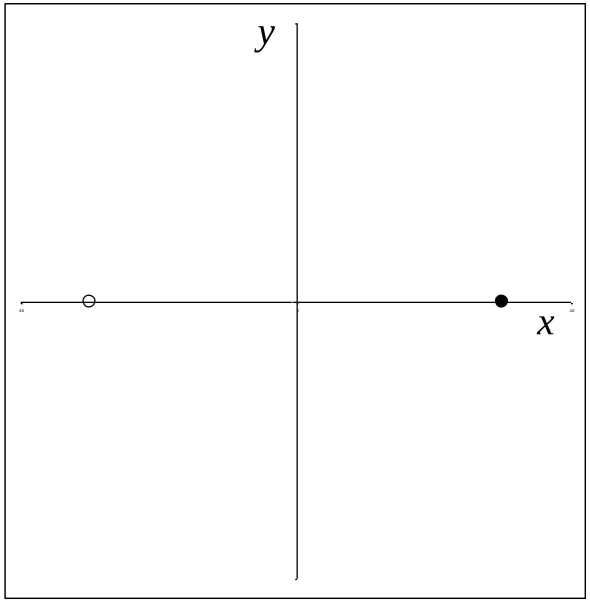
which diverges … this makes sense

- When we expand a function in a power series we lose something

$$\frac{1}{1-t}=1+t+t^2+t^3+\ldots=\sum_{n=1}^{\infty}t^{n-1} \qquad |t|<1$$

  - Ex. 3. In t=-1, the function is well defined

$$\frac{1}{1+1}=\frac{1}{2}$$



BUT using the series one has

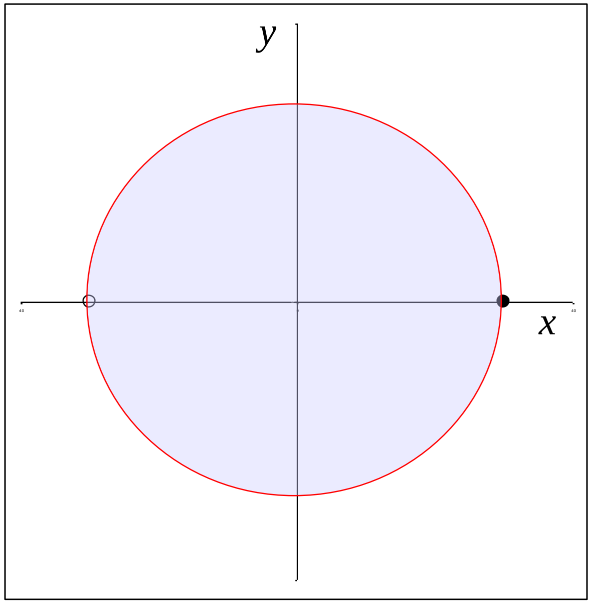
$$1+(-1)+(-1)^2+(-1)^3+\ldots=1-1+1-1+\ldots$$

even if the function is well defined, the series does not work: we are outside the convergence radius

3. VALIDITY LIMITS OF FIELD HARMONICS

- When we expand a function in a power series we lose something

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \ldots = \sum_{n=1}^{\infty} t^{n-1} \qquad |t| < 1$$



- Ex. 3. In t=-1, the function is well defined

$$\frac{1}{1+1} = \frac{1}{2}$$

BUT using the series one has

$$1 + (-1) + (-1)^2 + (-1)^3 + \ldots = 1 - 1 + 1 - 1 + \ldots$$

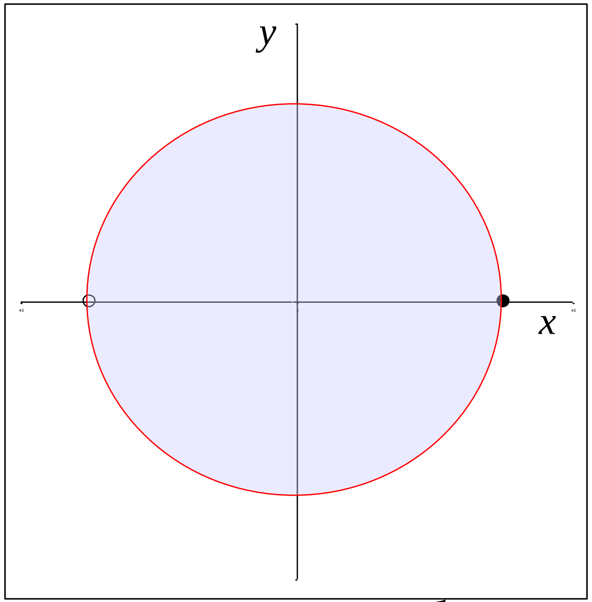
even if the function is well defined, the series does not work: we are outside the convergence radius

3. VALIDITY LIMITS OF FIELD HARMONICS

- Now let us consider t=-2

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \ldots = \sum_{n=1}^{\infty} t^{n-1}$$

$$|t| < 1$$



$$\frac{1}{1+2} = \frac{1}{3}$$

using the series one has

$$1 + (-2) + (-2)^2 + (-2)^3 + \ldots = 1 - 2 + 4 - 8 + \ldots$$

therefore when we can resum the series (if we manage to work out the original function) to

$$1 - 2 + 4 - 8 + 16 - 32 + \ldots = \frac{1}{3}$$

see more on G. H. Hardy, Divergent series https://archive.org/details/DivergentSeries

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- The field of a current line
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CLASSICAL RADIUS OF ELECTRON

- Maxwell equations present a divergence for a point charge
  - Stored energy is proportional to the square of electric field

$$U = \int \frac{\varepsilon_0 E^2}{2} dV$$

  - Integrating over a sphere larger than R

$$U = 4\pi \frac{\varepsilon_0}{2} \int_R^{\infty} \frac{e^2}{16\pi^2 \varepsilon_0^2 r^4} r^2 \, dr = \frac{e^2}{8\pi\varepsilon_0 R}$$

  - We obtain a divergence proportional to the size of the sphere
  - The classical electron radius is defined by equating this energy to half of the rest mass energy

$$U(r_e) = \frac{mc^2}{2} \qquad r_e = \frac{e^2}{4\pi\varepsilon_0 mc^2}$$

CLASSICAL RADIUS OF ELECTRON

- One can prove that the energy associated to a charge *e* on the surface of a sphere whose radius is r_e is $mc^2/2$

$$U = \frac{1}{4\pi\varepsilon_0} \iint_{r=r_e} \frac{dq_1 dq_2}{r_1 - r_2} = \frac{mc^2}{2}$$

  - So the total energy associated to electric field in space and to the charges on the sphere surface is equal to the electron mass
  - The electron classical radius is 2.8×10^{-15} m

$$r_e = \frac{e^2}{4\pi\varepsilon_0 mc^2}$$

  - The scale length of Thompson scattering is the classical electron radius:

$$\sigma_t = \frac{8\pi}{3}\left(\frac{e^2}{4\pi\varepsilon_0 mc^2}\right)^2$$

DIVERGENCE OF A CURRENT LOOP

- A less know divergence is related to the energy stored in the magnetic field
- We consider a loop of current of radius r_L and current I
- The loop creates a dipole-like magnetic field whose energy is

$$U = \int \frac{B^2}{2\mu_0} dV$$

- Close to the current line the field is proportional to the inverse of the distance to the current line

$$B = \frac{\mu_0 I}{2\pi r}$$

DIVERGENCE OF A CURRENT LOOP

- And one can approximate the integral with the contribution close to the current line

$$U \approx \frac{2\pi r_L}{2\mu_0} 2\pi \int_{r_w}^{R} \left(\frac{\mu_0 I}{2\pi r} \right)^2 r dr = \frac{r_L \mu_0 I^2}{2} \int_{r_w}^{R} \frac{dr}{r} = \frac{r_L \mu_0 I^2}{2} \ln\left(\frac{R}{r_w} \right)$$

where r_w is the wire radius

- The energy diverges with the logarithm of the radius of the wire, so a point charge also has a divergence in the energy associated to the magnetic field, not only to the electric field
- The inductance of a loop of current depends on the wire size

$$U = \frac{1}{2} L I^2 \qquad L \approx r_L \mu_0 \ln\left(\frac{R}{r_w} \right)$$

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A REMINDER ON BOHR RADIUS

- The classical electron radius 10^{-15} m is about four order of magnitudes smaller than the size of atoms

- Bohr radius can be computed from the electrostatic potential of an electron in the field created by a proton, whose energy is

$$E = \frac{p^2}{2m} + \frac{e^2}{4\pi\varepsilon_0 r}$$

- Plus the quantization of the angular momentum

$$\oint p\, dr = nh$$

- For a circular orbit

$$pr = n\frac{h}{2\pi} = n\hbar$$

A REMINDER ON BOHR RADIUS

- Replacing the momentum using quantization, one has

$$E(r) = \frac{h^2}{8\pi^2 r^2 m} + \frac{e^2}{4\pi\varepsilon_0 r} = \frac{e^2}{4\pi\varepsilon_0} \frac{1}{r^2} \left(\frac{\varepsilon_0 h^2}{2\pi m e^2} + r \right)$$

- And minimizing the energy one finds the Bohr radius

$$\frac{dE}{dr} = \frac{e^2}{4\pi\varepsilon_0} \frac{1}{r^3} \left(-\frac{\varepsilon_0 h^2}{\pi m e^2} + r \right)$$

$$r_B = \frac{\varepsilon_0 h^2}{\pi m e^2}$$

- The Bohr radius is 5.3×10^{-11} m

ALPHA: THE FINE STRUCTURE CONSTANT

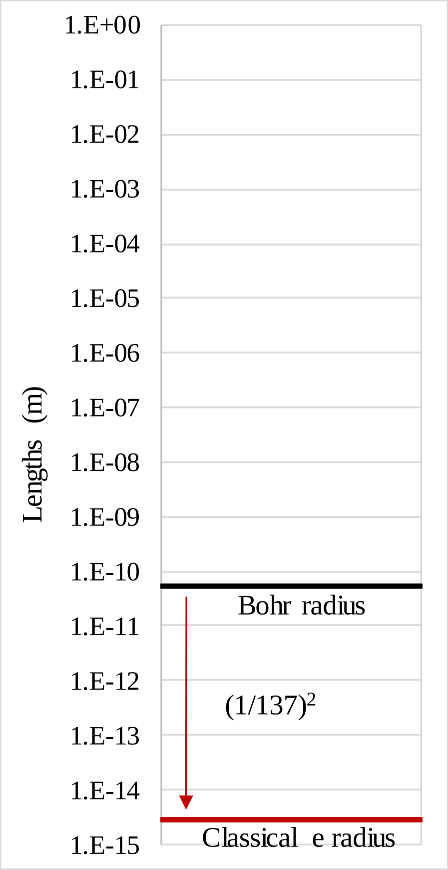
- What is rarely found in textbooks is that the ratio between the Bohr radius and the classical electron radius is ≈$(137)^2$, the square of an important physical constant

$$r_B = \frac{\varepsilon_0 h^2}{\pi m e^2} = \frac{4\varepsilon_0^2 h^2 c^2}{e^4} \frac{e^2}{4\pi\varepsilon_0 m c^2} = \left(\frac{2\varepsilon_0 hc}{e^2}\right)^2 r_e$$

- The quantity between brackets is the inverse of the fine structure constant

$$\alpha = \frac{e^2}{2\varepsilon_0 hc} \approx \frac{1}{137}$$

- It has a very misleading name, since in reality is the coupling constant of electromagnetism e^2 normalized through h and c – *and is the perturbative expansion parameter of QED*



ALPHA AND COMPTON LENGTH

- In between the Bohr radius and the classical electron radius one finds another very important quantity

$$\frac{r_e}{\alpha} = \frac{2\varepsilon_0 hc}{e^2}\frac{e^2}{4\pi\varepsilon_0 mc^2} = \frac{\hbar}{mc}$$

- This is called reduced Compton wavelength

$$r_C = \frac{\hbar}{mc}$$

- Whose size is 3.8×10^{-13} m