Constraining single energy partial wave analysis

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Kinematics of $\eta$ photoproduction

Problems in the unconstrained single energy partial wave analysis

Constrained single energy PWA by imposing the fixed-t analyticity

Preliminary results with
- Pseudo data MAID
- Real data from MAMI and GRAAL

Conclusions
\[ p_i \] - four momentum of incoming nucleon

\[ p_f \] - four momentum of outgoing nucleon

\[ k \] - four momentum of incident photon

\[ q \] - four momentum of \( \eta \) meson

Madelstam variables:

\[ s = w^2 = (p_i + k)^2 \]

\[ t = (q - k)^2 \]

\[ u = (p_i - q)^2 \]

\[ \nu = \frac{s - u - q^2}{4m} \]

\[ s + t + u = 2m^2 + m_\eta^2 \]

\( m \) - mass of nucleon,

\( m_\eta \) - mass of eta meson
Observables, amplitudes and multipoles in $\eta$ photoproduction

16 observables

<table>
<thead>
<tr>
<th>Spin Observable</th>
<th>Type</th>
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<tbody>
<tr>
<td>$\sigma_0$</td>
<td>$S$</td>
</tr>
<tr>
<td>$\Sigma$</td>
<td>(single spin)</td>
</tr>
<tr>
<td>$\hat{T}$</td>
<td>$BT$</td>
</tr>
<tr>
<td>$\hat{P}$</td>
<td>(beam–target)</td>
</tr>
<tr>
<td>$\hat{G}$</td>
<td>$BR$</td>
</tr>
<tr>
<td>$\hat{H}$</td>
<td>(beam–recoil)</td>
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<tr>
<td>$\hat{E}$</td>
<td></td>
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<tr>
<td>$\hat{F}$</td>
<td></td>
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<tr>
<td>$\hat{O}_x'$</td>
<td>$BR$</td>
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<tr>
<td>$\hat{O}_z'$</td>
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<tr>
<td>$\hat{C}_x'$</td>
<td>(beam–recoil)</td>
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<tr>
<td>$\hat{C}_z'$</td>
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<tr>
<td>$\hat{T}_x'$</td>
<td>$TR$</td>
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<tr>
<td>$\hat{T}_z'$</td>
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<tr>
<td>$\hat{L}_x'$</td>
<td>(target–recoil)</td>
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<tr>
<td>$\hat{L}_z'$</td>
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Observables are represented by one set of four complex amplitudes:

- CGLN amplitudes $(F_k(W, \cos \theta), k = 1, 2, 3, 4)$
- helicity amplitudes $(H_k(W, \cos \theta), k = 1, 2, 3, 4)$
- invariant amplitudes $(B_k(s, t), k = 1, 2, 6, 8)$

Amplitudes are given by expansion in terms of electric $(E_{\ell\pm})$ and magnetic $(M_{\ell\pm})$ multipoles (Details in Tiator’s talks).
Example, differential cross section in terms of helicity amplitudes

\[ \frac{d\sigma}{d\Omega} = \frac{q}{2k} (|H_1|^2 + |H_2|^2 + |H_3|^2 + |H_4|^2) \]

Expansion of CGLN in terms of multipoles (truncated - up to \( \ell = L_{\text{max}} \))

\[ F_1 = \sum_{\ell \geq 0}^{L_{\text{max}}} \left\{ (\ell M_\ell^+ + E_\ell^+)P_{\ell+1}' + [(\ell + 1)M_\ell^- + E_\ell^-]P_{\ell-1}' \right\}, \]

\[ F_2 = \sum_{\ell \geq 1}^{L_{\text{max}}} [(\ell + 1)M_\ell^+ + \ell M_\ell^-]P_\ell', \]

\[ F_3 = \sum_{\ell \geq 1}^{L_{\text{max}}} [(E_\ell^+ - M_\ell^+)P_{\ell+1}'' + (E_\ell^- - M_\ell^-)P_{\ell-1}'' ] \]

\[ F_4 = \sum_{\ell \geq 2}^{L_{\text{max}}} (M_\ell^+ - E_\ell^+ - M_\ell^- - E_\ell^-)P_\ell'' \]
Some concepts in analysis of experimental data

- **Energy dependent (ED)** partial waves (multipoles) are parametrized as a function of energy (model dependent). In this talk we will use three ED MAID solutions (Details in Kashevarov’s talk).

- **Single energy (SE)** multipoles are determined at a single energy.

- **Amplitude analysis (AA)** amplitudes are parametrized as a function of energy in energy range where data are available.
Unconstrained SE PWA

Input: pseudo data (relative error 0.1%) created from MAID solution - Solution I.

We fitted 8 observables
\[ \frac{d\sigma}{d\Omega}, F \frac{d\sigma}{d\Omega}, T \frac{d\sigma}{d\Omega}, E \frac{d\sigma}{d\Omega}, P \frac{d\sigma}{d\Omega}, G \frac{d\sigma}{d\Omega}, C_x \frac{d\sigma}{d\Omega}, O_x \frac{d\sigma}{d\Omega}, \] (complete set of observables).

\[ L_{\text{max}} = 5, \] 40 real parameters in the fit. Multipoles with \( L > 5 \) set to zero.
The starting parameters of the fit were randomly selected in a 30% range around the "true" solution.

Problem is more serious - uniqueness problem. How to resolve it?
One must impose more stringent constraints taking into account analyticity of scattering amplitudes.

**Dispersion relations? Not easy to apply!**

**Important step forward:**

In a series of papers E. Pietarinen proposed a substitute for dispersion relations. In his method invariant amplitudes are expanded in terms of analytic functions having the same analytic structure.

- E. Pietarinen: Amplitude analysis using fixed-t analyticity of invariant amplitudes
  - E. Pietarinen, Nuovo Cim. 12 (1972) 522
In our PWA of $\eta$ photoproduction data we use the same approach as it was done in KH80 analysis of $\pi N$ scattering data.

- The method consists of two analyses:
  - Fixed-t amplitude analysis (Fixed-t AA) - determination of the invariant scattering amplitudes from exp. data at a given fixed-t value
  - Constrained single energy partial wave analysis - SE PWA
  - Fixed-t amplitude analysis and single energy PWA are coupled. Results from one analysis are used as constraint in another in an iterative procedure.
Imposing the fixed-t analyticity in PWA of scattering data

Single energy PWA is performed along red lines. Fixed-t amplitude analysis is performed along green lines.
Coupled fixed-t amplitude analysis and single energy PWA

Initial solution

Fixed-t AA

SE PWA

Connection between SE PWA and fixed-t AA

- Multipoles obtained from SE PWA at a given set of energies are used to calculate helicity amplitudes which are used as constraint in the fixed-t amplitude analysis.

- The whole procedure has to be iterated until reaching reasonable agreement in two subsequent iterations.
The simplest case - $\pi N$ elastic scattering at fixed-$t$.
Apart from nucleon poles, crossing symmetric invariant amplitudes are analytic function in a complex $\nu^2$ plane $\nu_{th}^2 \leq \nu^2 < \infty$, ($\nu_{th} = m_\pi + \frac{t}{4m}$).

Conformal mapping:

\[
z = \frac{\alpha - \sqrt{\nu_{th}^2 - \nu^2}}{\alpha + \sqrt{\nu_{th}^2 - \nu^2}}
\]

Points on the cut in complex $\nu^2$ plane is mapped on the circle.
Fixed-t amplitude analysis of $\eta$ photoproduction data

For a given $t$ invariant amplitudes are represented by two Pietarinen series:

$$B = B_N + \sum_{i=0}^{N_1} b_i^{(1)} z_1^i + \sum_{i=0}^{N_2} b_i^{(2)} z_2^i.$$ 

$B$ stands for crossing symmetric invariant amplitudes $B_1, B_2, B_6$ and $\frac{B_8}{\nu}$. $B_N$ are known nucleon pole contributions. Conformal variables $z_1$ and $z_2$ are defined as:

$$z_1 = \frac{\alpha - \sqrt{\nu_{th1}^2 - \nu^2}}{\alpha + \sqrt{\nu_{th1}^2 - \nu^2}}, \quad z_2 = \frac{\beta - \sqrt{\nu_{th2}^2 - \nu^2}}{\beta + \sqrt{\nu_{th2}^2 - \nu^2}}.$$
Fixed-t amplitude analysis

Coefficients \( \{b_i^{(1)}\} \), \( \{b_i^{(2)}\} \) are obtained by minimizing a quadratic form

\[
\chi^2 = \chi^2_{\text{data}} + \chi^2_{\text{PW}} + \Phi
\]

\( \chi^2_{\text{data}} \) contains data at a given t-value.

\( \chi^2_{\text{PW}} \) contains as the “data” the helicity amplitudes from the SE PWA analysis. \( \Phi \) is Pietarinen’s convergence test function.

\[
\Phi = \lambda_1 \Phi_1 + \lambda_2 \Phi_2 + \lambda_3 \Phi_3 + \lambda_4 \Phi_4.
\]

\[
\Phi_1 = \sum_{n=0}^{N} (n+1)^3(c^+_n)^2, \ldots, \Phi_4 = \sum_{n=0}^{N} (n+1)^3(b^-_n)^2.
\]

\( \lambda_1, \lambda_2, \lambda_3 \) and \( \lambda_4 \) are determined in an iterative procedure.

In a first iteration helicity amplitudes are calculated from initial, already existing PW solution.

In subsequent iterations helicity amplitudes are calculated from multipoles obtained in SE PWA of the same set of experimental data.
Fixed-t amplitude analysis

At each $t$-value perform FT AA
Minimize:
\[ \chi^2 = \chi^2_{data} + \chi^2_{PW} + \Phi \]
Parameters: Coefficients in Pietarinen’s expansions

Use results from SE PWA to calculate helicity amplitudes which are used as a constraint in FT amplitude analysis

At each of $N$ energies perform SE PWA
Minimize:
\[ \chi^2 = \chi^2_{data} + \chi^2_{FT} \]
Parameters: multipoles
How does it work? - testing with pseudo data

Input data:
We used pseudo data created from Solution 1 with relative error 0.1%.
In present calculations we fitted 8 observables (Complete set of observables) $\frac{d\sigma}{d\Omega}, F \frac{d\sigma}{d\Omega}, T \frac{d\sigma}{d\Omega}, P \frac{d\sigma}{d\Omega}, \Sigma \frac{d\sigma}{d\Omega}, G \frac{d\sigma}{d\Omega}, C_x \frac{d\sigma}{d\Omega}$ and $O_x \frac{d\sigma}{d\Omega}$. Observables at $t = -0.2\text{GeV}^2$. 

![Graphs of observables at different energies](image-url)
Fixed-t amplitude analysis-pseudo data

IA from start solution

At each $t$-value perform FT AA
Minimize:
$$\chi^2 = \chi^2_{data} + \chi^2_{PW} + \Phi$$
Parameters: Coefficients in Pietarinen’s expansions

At each of N energies perform SE PWA
Minimize:
$$\chi^2 = \chi^2_{data} + \chi^2_{FT}$$
Parameters: multipoles

Use results from SE PWA to calculate helicity amplitudes which are used as a constraint in FT amplitude analysis

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Fixed-t amplitude analysis-pseudo data

Fit of pseudo data at $t = -0.2\text{GeV}^2$ and $t = -0.5\text{GeV}^2$
Fixed - t amplitude analysis - helicity amplitudes

Obtained helicity amplitudes at $t = -0.2\text{GeV}^2$ and $t = -0.5\text{GeV}^2$.

Full lines (Re FT and Im FT) are results from 3rd iteration. Dots (Re SE and Im SE) are results from 2nd iteration.
From fixed-$t$ to single energy PWA

IA from start solution

At each $t$-value perform FT AA
Minimize:
$$
\chi^2 = \chi^2_{data} + \chi^2_{PW} + \Phi
$$
Parameters: Coefficients in Pietarinen’s expansions

At each of $N$ energies perform SE PWA
Minimize:
$$
\chi^2 = \chi^2_{data} + \chi^2_{FT}
$$
Parameters: multipoles

Use results from SE PWA to calculate helicity amplitudes which are used as a constraint in FT amplitude analysis
Fit to the data at $W = 1543.24\,\text{MeV}$ and $W = 1769.8\,\text{MeV}$. 

Elab= 800.00 MeV, $W= 1543.24 \,\text{MeV}$; 

Data

$\frac{d\sigma}{d\Omega}$

$\cos \theta$

Fit

Elab= 800.00 MeV, $W= 1543.24 \,\text{MeV}$;

Data

$\frac{d\sigma}{d\Omega}$

$\cos \theta$

Fit

Elab= 1200.00 MeV, $W= 1769.80 \,\text{MeV}$;

Data

$\frac{d\sigma}{d\Omega}$

$\cos \theta$

Fit

Elab= 1200.00 MeV, $W= 1769.80 \,\text{MeV}$;

Data

$\frac{d\sigma}{d\Omega}$

$\cos \theta$
Helicity amplitudes $H_k(W, \cos \theta)$ at $W = 1543.24\,\text{MeV}$ and $W = 1660.3\,\text{MeV}$.

To be pointed out: Real and imaginary parts of helicity amplitudes (blue and red dots- Re FT and Im FT) are obtained from independent fixed-t AA at different $t$-values.
Red and blues solid lines are multipoles from Solution 1. (3rd iteration)
Real case - $\eta$ photoproduction data base

Data base consists of following experimental data

- **A2 Collaboration at MAMI**
  - Differential cross section $\frac{d\sigma}{d\Omega}$
    - CBall/MAMI: E. McNicoll et al., PRC 82(2010) 035208
  - Target asymmetry $T$
  - Double-polarisation asymmetry $F$

- **GRAAL collaboration**
  - Beam asymmetry $\Sigma$
    - O. Bartalini et al., EPJ A 33 (2007) 169
Preparing input data

Kinematical grid of MAMI $d\sigma/d\Omega$

$t [\text{GeV}^2]$ versus $W [\text{GeV}]$
Observables at $t = -0.2\text{GeV}^2$ and $t = -0.5\text{GeV}^2$. 
At each $t$-value perform FT AA
Minimize:
$$\chi^2 = \chi^2_{\text{data}} + \chi^2_{\text{PW}} + \Phi$$
Parameters: Coefficients in Pietarinen's expansions

Use results from SE PWA to calculate helicity amplitudes which are used as a constraint in FT amplitude analysis

At each of N energies perform SE PWA
Minimize:
$$\chi^2 = \chi^2_{\text{data}} + \chi^2_{\text{FT}}$$
Parameters: multipoles
In order to explore model dependence of our solution we started with two MAID solutions and performed complete analysis (Solution II and Solution III).

Invariant amplitudes from initial Solution II and III $t = -0.2 \text{GeV}^2$. 

![Graphs showing real and imaginary parts of invariant amplitudes](image)
Fit to the data at $t = -0.2\text{GeV}^2$ and $t = -0.5\text{GeV}^2$. 
Constrained SE PWA-real data

IA from start solution

At each $t$-value perform FT AA
Minimize:
$$\chi^2 = \chi_{data}^2 + \chi_{PW}^2 + \Phi$$
Parameters: Coefficients in Pietarinen’s expansions

At each of $N$ energies perform SE PWA
Minimize:
$$\chi^2 = \chi_{data}^2 + \chi_{FT}^2$$
Parameters: multipoles

Use results from SE PWA to calculate helicity amplitudes which are used as a constraint in FT amplitude analysis.
Constrained SE PWA-real data

Fit to the data at $W = 1554.27\text{MeV}$ and $W = 1657.62\text{MeV}$.

Fit to the data at $W = 1765.4\text{MeV}$ and $W = 1840.09\text{MeV}$.
Helicity amplitudes at $W = 1602.01\, \text{MeV}$.

(a) Helicity amplitudes obtained using Solution II as constraint.

(b) Helicity amplitudes obtained using Solution III as constraint.

To be pointed out: Real and imaginary parts of helicity amplitudes (blue and red dots- Re FT and Im FT) are obtained from independent fixed-t AA at different $t$-values.
Helicity amplitudes at $W = 1602.01\text{MeV}$.

(c) Initial helicity amplitudes—solutions II and III.

(d) Helicity amplitudes after three iterations using solutions II and III as a constraint.
Constrained SE PWA-multipoles

$s, p, d$ waves

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Multipoles obtained in our coupled fixed-t-single energy PWA are:

- model independent - almost
- consistent with fixed-s and fixed-t analyticity
- consistent with crossing symmetry
Unconstrained single energy PWA

SE PWA with fixed-t constraint

Constraining single energy partial wave analysis