

# Constraining single energy partial wave analysis

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- Kinematics of  $\eta$  photoproduction
- Problems in the unconstrained single energy partial wave analysis
- Constrained single energy PWA by imposing the fixed-t analyticity
- Preliminary results with
  - Pseudo data MAID
  - Real data from MAMI and GRAAL
- Conclusions



# $\eta$ photoproduction

$p_i$  - four momentum of incoming nucleon

$p_f$  - four momentum of outgoing nucleon

$k$  - four momentum of incident photon

$q$  - four momentum of  $\eta$  meson

Madelstam variables:

$$s = w^2 = (p_i + k)^2$$

$$t = (q - k)^2$$

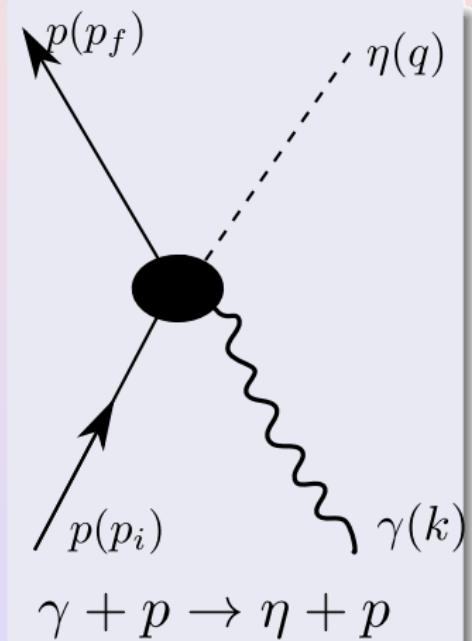
$$u = (p_i - q)^2$$

$$\nu = \frac{s-u}{4m}$$

$$s + t + u = 2m^2 + m_\eta^2;$$

$m$  - mass of nucleon,

$m_\eta$  - mass of eta meson



# Observables, amplitudes and multipoles in $\eta$ photoproduction

## 16 observables

Spin Observable	Type
$\sigma_0$	
$\hat{S}$	$\mathcal{S}$
$\hat{T}$	(single spin)
$\hat{P}$	
$\hat{G}$	
$\hat{H}$	$\mathcal{BT}$
$\hat{E}$	(beam-target)
$\hat{F}$	
$\hat{O}_{x'}$	
$\hat{O}_{z'}$	$\mathcal{BR}$
$\hat{C}_{x'}$	(beam-recoil)
$\hat{C}_{z'}$	
$\hat{T}_{x'}$	
$\hat{T}_{z'}$	$\mathcal{TR}$
$\hat{L}_{x'}$	(target-recoil)
$\hat{L}_{z'}$	

Observables are represented by one set of four complex amplitudes:

- CGLN amplitudes ( $F_k(W, \cos \theta)$ ,  $k = 1, 2, 3, 4$ )
- helicity amplitudes ( $H_k(W, \cos \theta)$ ,  $k = 1, 2, 3, 4$ )
- invariant amplitudes ( $B_k(s, t)$ ,  $k = 1, 2, 6, 8$ )

Amplitudes are given by expansion in terms of electric ( $E_{\ell\pm}$ ) and magnetic ( $M_{\ell\pm}$ ) multipoles (Details in Tiator's talks).



# Observables, amplitudes and multipoles in $\eta$ photoproduction

Example, differential cross section in terms of helicity amplitudes

$$\frac{d\sigma}{d\Omega} = \frac{q}{2k}(|H_1|^2 + |H_2|^2 + |H_3|^2 + |H_4|^2)$$

Expansion of CGLN in terms of multipoles (truncated - up to  $\ell = L_{max}$ )

$$F_1 = \sum_{\ell \geq 0}^{L_{max}} \{(\ell M_{\ell+} + E_{\ell+}) P'_{\ell+1} + [(\ell+1)M_{\ell-} + E_{\ell-}] P'_{\ell-1}\},$$

$$F_2 = \sum_{\ell \geq 1}^{L_{max}} [(\ell+1)M_{\ell+} + \ell M_{\ell-}] P'_{\ell},$$

$$F_3 = \sum_{\ell \geq 1}^{L_{max}} [(E_{\ell+} - M_{\ell+}) P''_{\ell+1} + (E_{\ell-} - M_{\ell-}) P''_{\ell-1}]$$

$$F_4 = \sum_{\ell \geq 2}^{L_{max}} (M_{\ell+} - E_{\ell+} - M_{\ell-} - E_{\ell-}) P''_{\ell}$$



# Some concepts in analysis of experimental data

- Energy dependent (ED)  
partial waves (multipoles) are parametrized as a function of energy (model dependent). In this talk we will use three ED MAID solutions (Details in Kashevarov's talk).
- Single energy (SE)  
multipoles are determined at a single energy.
- Amplitude analysis (AA)  
amplitudes are parametrized as a function of energy in energy range where data are available.



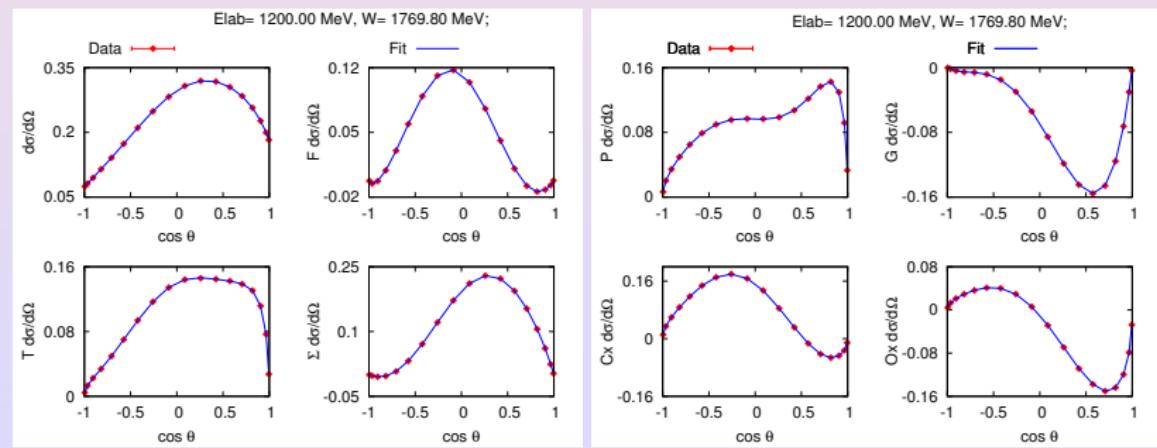
# Unconstrained SE PWA

Input: pseudo data (relative error 0.1%) created from MAID solution - Solution I.

We fitted 8 observables

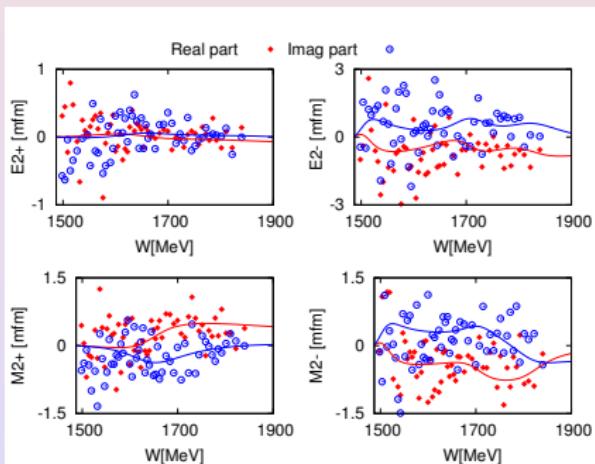
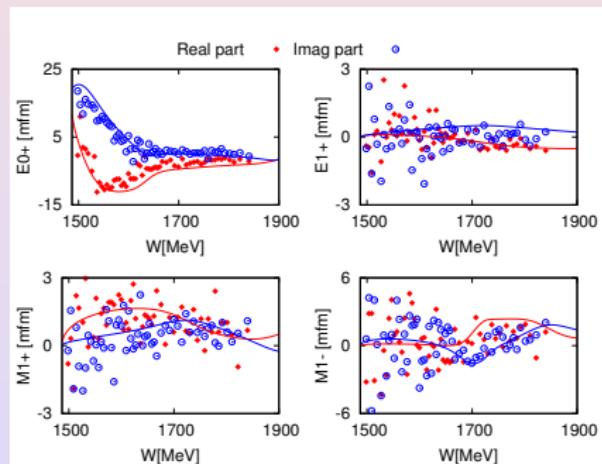
$\frac{d\sigma}{d\Omega}$ ,  $F \frac{d\sigma}{d\Omega}$ ,  $T \frac{d\sigma}{d\Omega}$ ,  $E \frac{d\sigma}{d\Omega}$ ,  $P \frac{d\sigma}{d\Omega}$ ,  $G \frac{d\sigma}{d\Omega}$ ,  $C_x \frac{d\sigma}{d\Omega}$ ,  $O_x \frac{d\sigma}{d\Omega}$ , (complete set of observables).

$L_{max} = 5$ , 40 real parameters in the fit. Multipoles with  $L > 5$  set to zero.



# Unconstrained SE PWA

The starting parameters of the fit were randomly selected in a 30% range around the "true" solution.



Problem is more serious - uniqueness problem. How to resolve it?



One must impose more stringent constraints taking into account analyticity of scattering amplitudes.

Dispersion relations? Not easy to apply!

### Important step forward:

In a series of papers E. Pietarinen proposed a substitute for dispersion relations. In his method invariant amplitudes are expanded in terms of analytic functions having the same analytic structure.

- S. Bowcock, H. Burkhardt, Rep. Prog Phys 38 (1975) 1099
- E. Pietarinen: Amplitude analysis using fixed-t analyticity of invariant amplitudes
  - E. Pietarinen, Nuovo Cim. 12 (1972) 522
  - E. Pietarinen, Nucl. Phys. B49 (1972) 315 Discussion of uniqueness problem
  - E. Pietarinen, Nucl. Phys. 8107 (1976) 21 Discussion of uniqueness problem
  - J. Hamilton, J. L. Peterson, New developments in dispersion theory, Vol.1, Nordita, 1975.



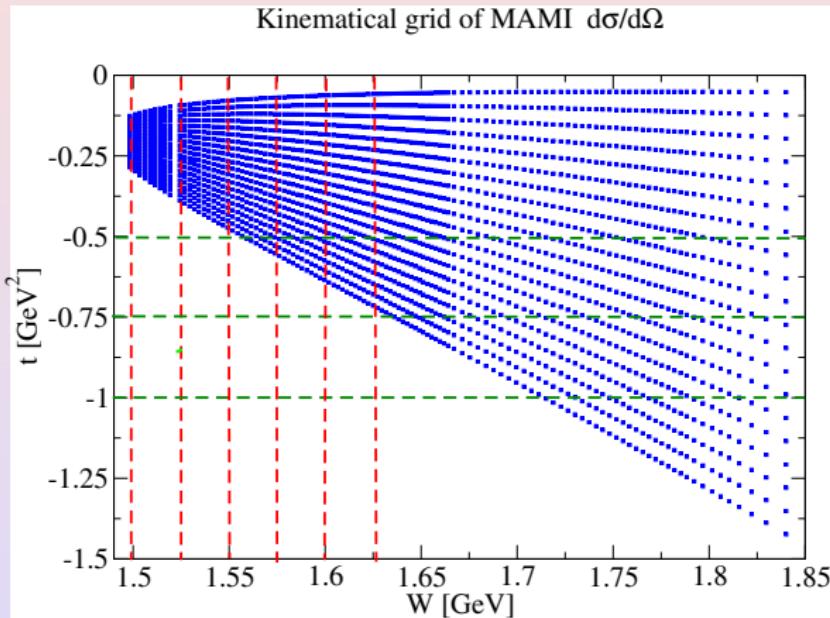
# How to impose fixed-t analyticity in PWA of scattering data?

In our PWA of  $\eta$  photoproduction data we use the same approach as it was done in KH80 analysis of  $\pi N$  scattering data.

- The method consists of two analyses:
  - Fixed-t amplitude analysis (Fixed-t AA) - determination of the invariant scattering amplitudes from exp. data at a given fixed-t value
  - Constrained single energy partial wave analysis - SE PWA
  - Fixed-t amplitude analysis and single energy PWA are coupled. Results from one analysis are used as constraint in another in an iterative procedure.

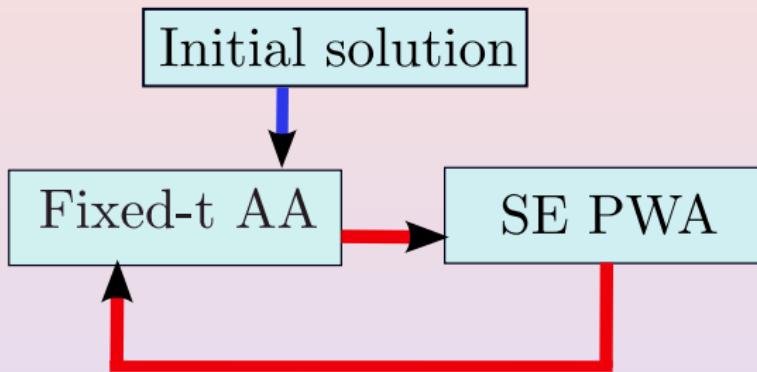


# Imposing the fixed-t analyticity in PWA of scattering data



Single energy PWA is performed along red lines. Fixed-t amplitude analysis is performed along green lines.





## Connection between SE PWA and fixed-t AA

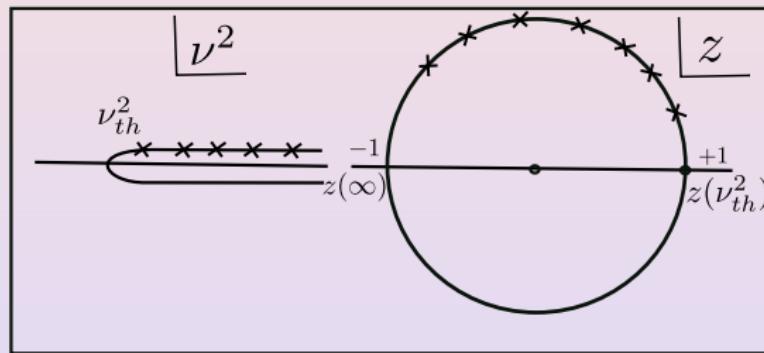
- Multipoles obtained from SE PWA at a given set of energies are used to calculate helicity amplitudes which are used as constraint in the fixed-t amplitude analysis.
- The whole procedure has to be iterated until reaching reasonable agreement in two subsequent iterations



# Fixed-t amplitude analysis - Pietarinen's expansion method

The simplest case- $\pi N$  elastic scattering at fixed-t.

Apart from nucleon poles, crossing symmetric invariant amplitudes are analytic function in a complex  $\nu^2$  plane  $\nu_{th}^2 \leq \nu^2 < \infty$ , ( $\nu_{th} = m_\pi + \frac{t}{4m}$ ).



Conformal mapping:

$$z = \frac{\alpha - \sqrt{\nu_{th}^2 - \nu^2}}{\alpha + \sqrt{\nu_{th}^2 - \nu^2}}$$

Points on the cut in complex  $\nu^2$  plane is mapped on the circle.



# Fixed-t amplitude analysis of $\eta$ photoproduction data

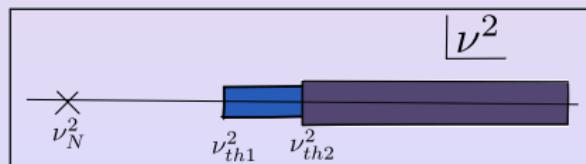
For a given  $t$  invariant amplitudes are represented by two Pietarinen series :

$$B = B_N + \sum_{i=0}^{N_1} b_i^{(1)} z_1^i + \sum_{i=0}^{N_2} b_i^{(2)} z_2^i.$$

$B$  stands for crossing symmetric invariant amplitudes  $B_1, B_2, B_6$  and  $\frac{B_8}{\nu}$ .

$B_N$  are known nucleon pole contributions. Conformal variables  $z_1$  and  $z_2$  are defined as:

$$z_1 = \frac{\alpha - \sqrt{\nu_{th1}^2 - \nu^2}}{\alpha + \sqrt{\nu_{th1}^2 - \nu^2}}, \quad z_2 = \frac{\beta - \sqrt{\nu_{th2}^2 - \nu^2}}{\beta + \sqrt{\nu_{th2}^2 - \nu^2}}.$$



# Fixed-t amplitude analysis

Coefficients  $\{b_i^{(1)}\}$ ,  $\{b_i^{(2)}\}$  are obtained by minimizing a quadratic form

$$\chi^2 = \chi_{data}^2 + \chi_{PW}^2 + \Phi$$

$\chi_{data}^2$  contains data at a given t-value.

$\chi_{PW}^2$  contains as the “data” the helicity amplitudes from the SE PWA analysis.  $\Phi$  is Pietarinen’s convergence test function.

$$\Phi = \lambda_1 \Phi_1 + \lambda_2 \Phi_2 + \lambda_3 \Phi_3 + \lambda_4 \Phi_4.$$

$$\Phi_1 = \sum_{n=0}^N (n+1)^3 (c_n^+)^2, \dots, \Phi_4 = \sum_{n=0}^N (n+1)^3 (b_n^-)^2.$$

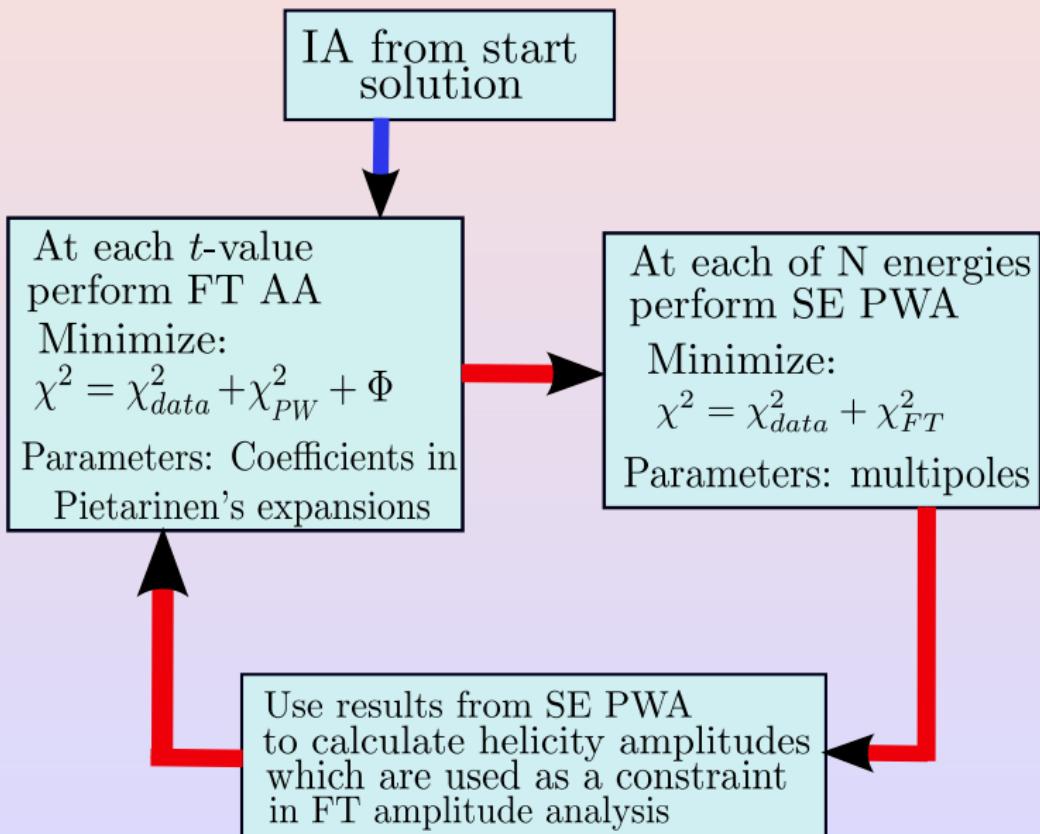
$\lambda_1$ ,  $\lambda_2$ ,  $\lambda_3$  and  $\lambda_4$  are determined in an iterative procedure.

In a first iteration helicity amplitudes are calculated from initial, already existing PW solution.

In subsequent iterations helicity amplitudes are calculated from multipoles obtained in SE PWA of the same set of experimental data.



# Fixed-t amplitude analysis

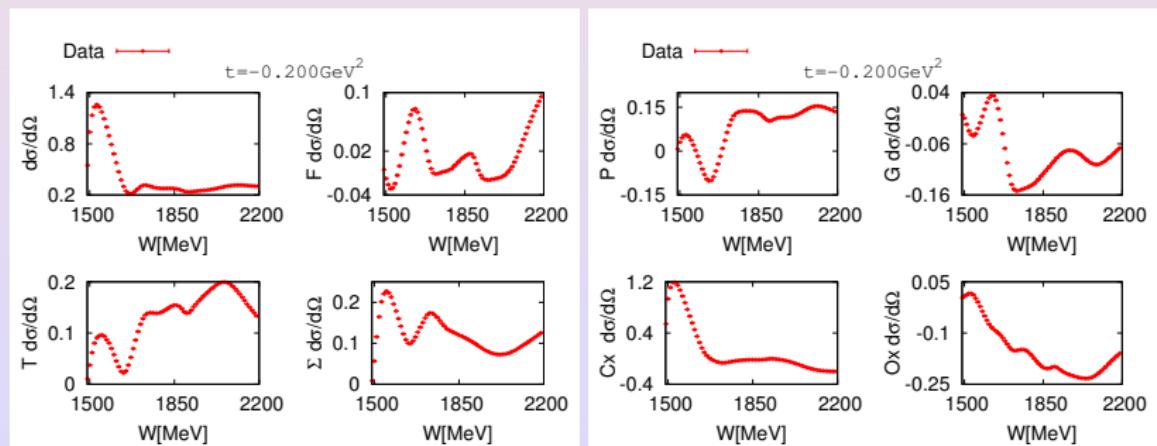


# How does it work? - testing with pseudo data

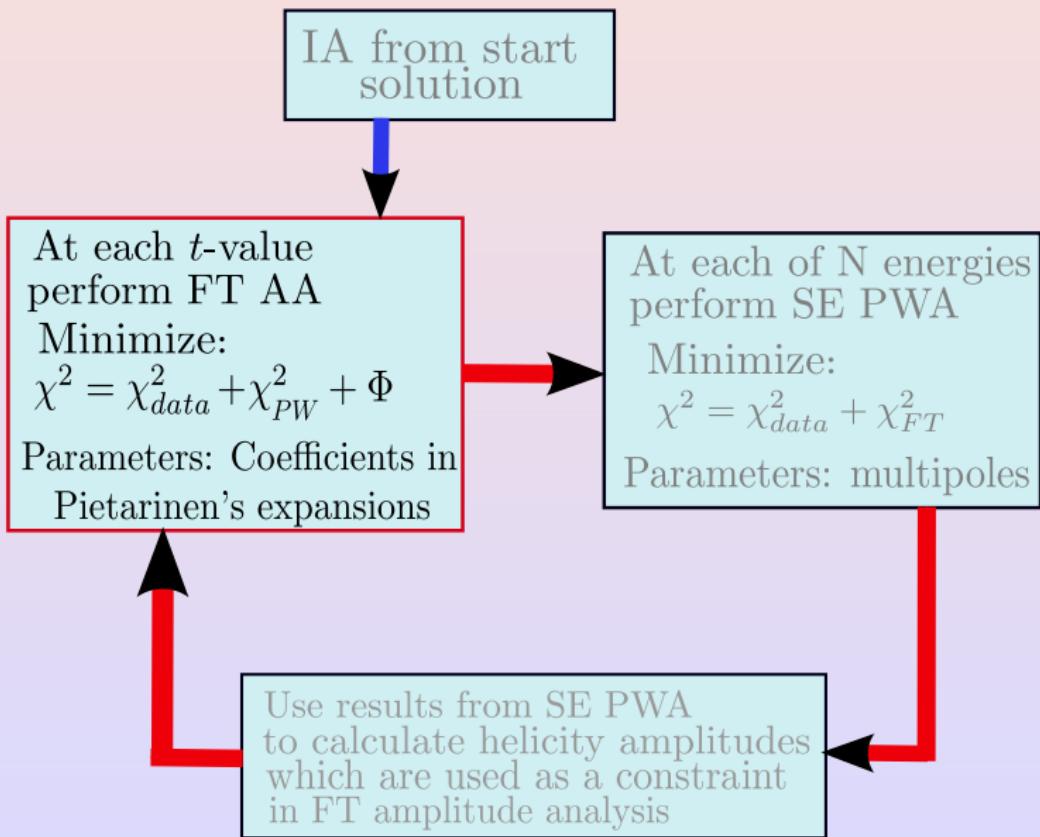
## Input data:

We used pseudo data created from Solution I with relative error 0.1%.

In present calculations we fitted 8 observables (Complete set of observables)  $\frac{d\sigma}{d\Omega}$ ,  $F \frac{d\sigma}{d\Omega}$ ,  $T \frac{d\sigma}{d\Omega}$ ,  $P \frac{d\sigma}{d\Omega}$ ,  $\Sigma \frac{d\sigma}{d\Omega}$ ,  $G \frac{d\sigma}{d\Omega}$ ,  $C_x \frac{d\sigma}{d\Omega}$  and  $O_x \frac{d\sigma}{d\Omega}$ . Observables at  $t = -0.2 \text{ GeV}^2$ .

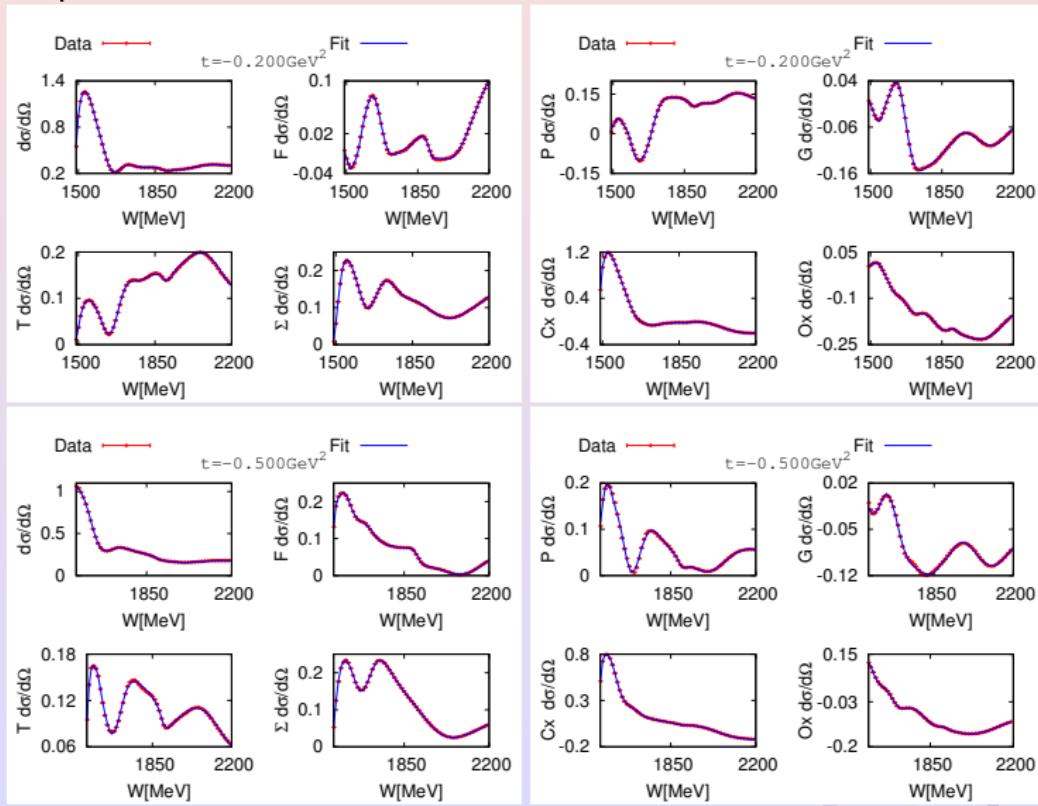


# Fixed-t amplitude analysis-pseudo data



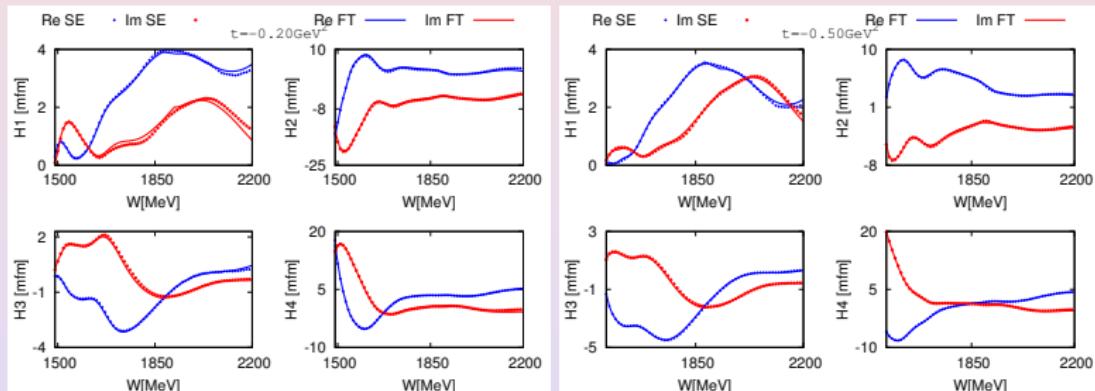
# Fixed-t amplitude analysis-pseudo data

Fit of pseudo data at  $t = -0.2\text{GeV}^2$  and  $t = -0.5\text{GeV}^2$



# Fixed - t amplitude analysis - helicity amplitudes

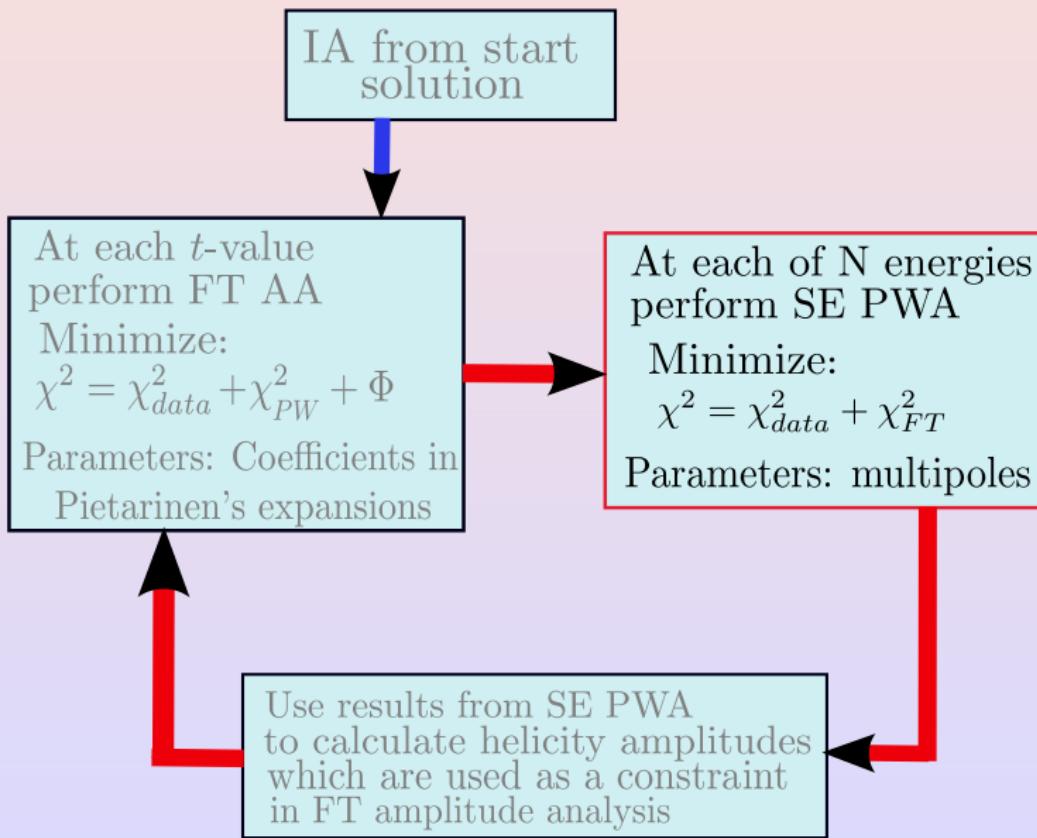
Obtained helicity amplitudes at  $t = -0.2 \text{ GeV}^2$  and  $t = -0.5 \text{ GeV}^2$ .



Full lines (Re FT and Im FT) are results from 3rd iteration. Dots (Re SE and Im SE) are results from 2nd iteration.

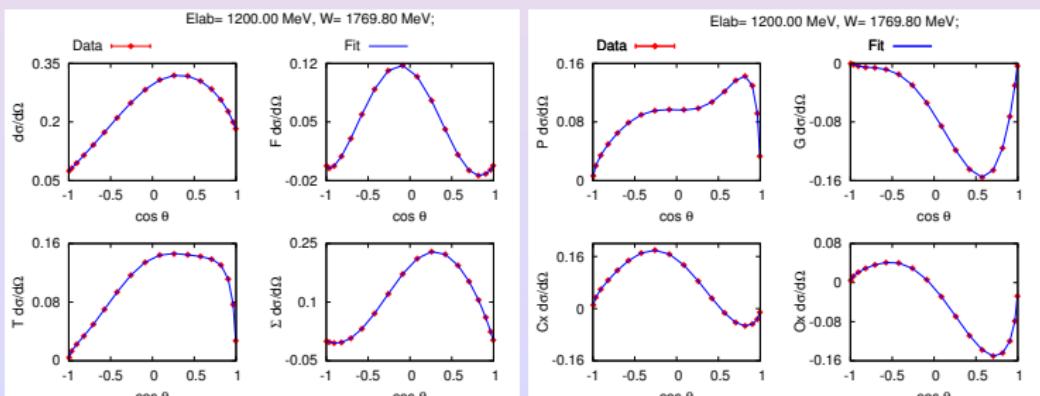
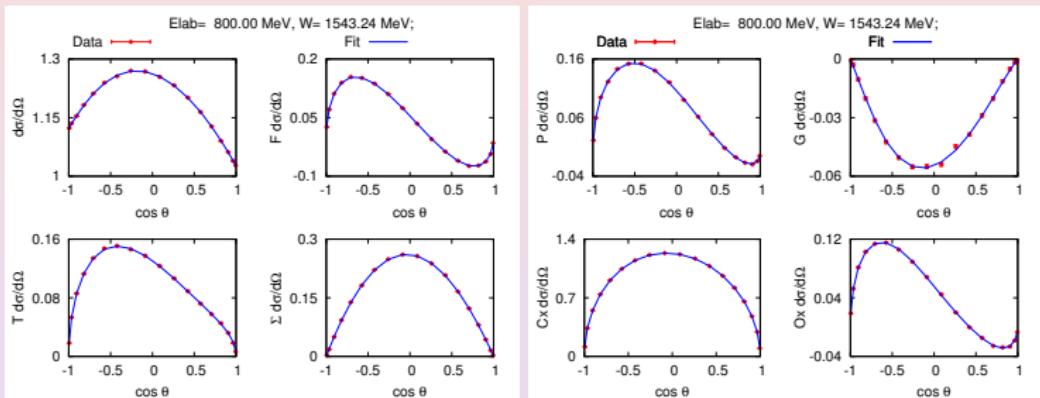


# From fixed-t to single energy PWA



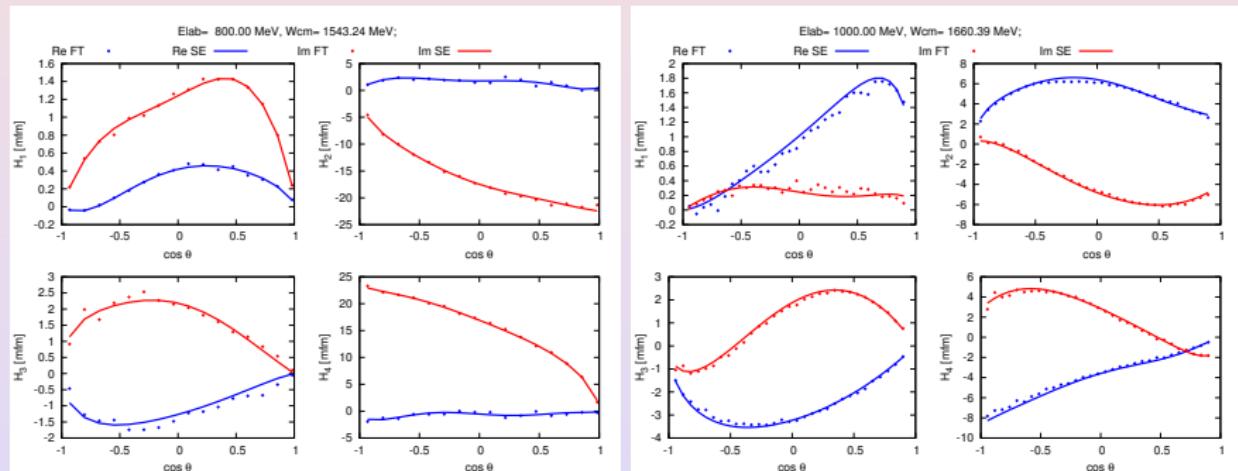
# Constrained SE PWA-pseudo data

Fit to the data at  $W = 1543.24\text{MeV}$  and  $W = 1769.8\text{MeV}$ .



# Constrained SE PWA-Helicity amplitudes

Helicity amplitudes  $H_k(W, \cos \theta)$  at  $W = 1543.24\text{MeV}$  and  $W = 1660.3\text{MeV}$ .

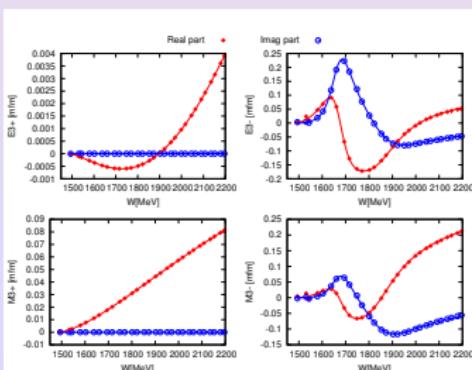
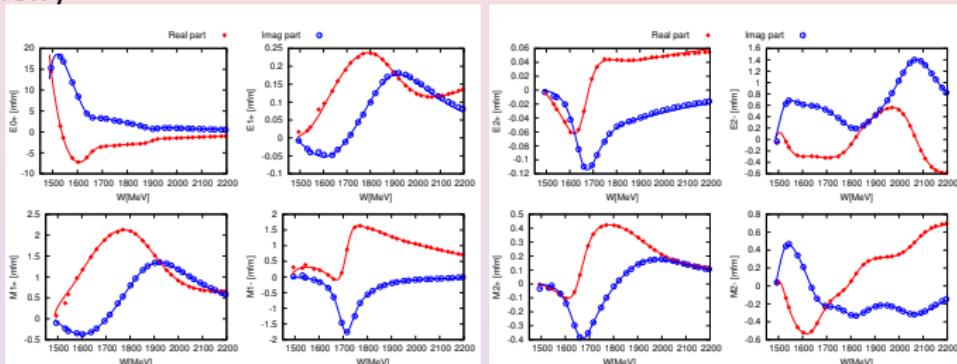


To be pointed out: Real and imaginary parts of helicity amplitudes (blue and red dots- Re FT and Im FT) are obtained from independent fixed-t AA at different  $t$ -values.



# Constrained SE PWA-Multipoles

Red and blues solid lines are multipoles from Solution I. (3rd iteration)



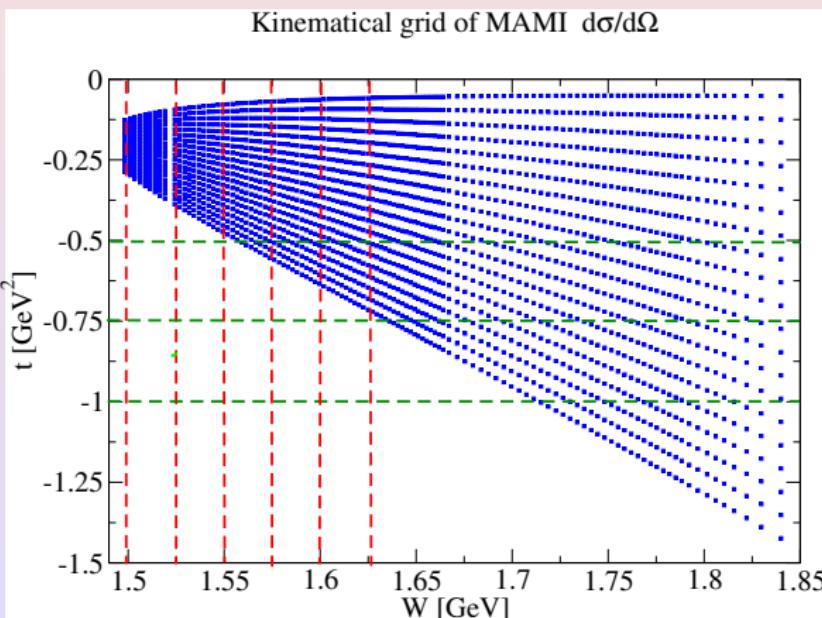
# Real case - $\eta$ photoproduction data base

Data base consists of following experimental data

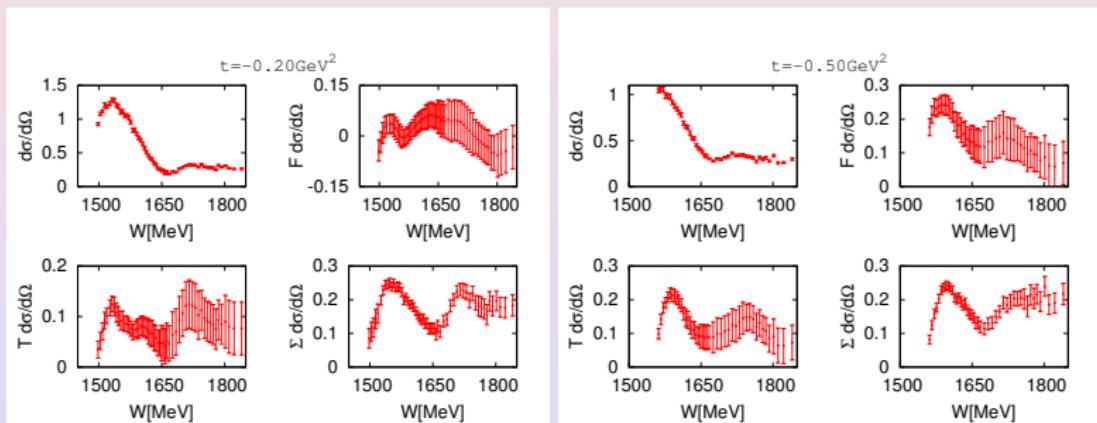
- A2 Collaboration at MAMI
  - Differential cross section  $\frac{d\sigma}{d\Omega}$   
CBall/MAMI: E.McNicoll et al., PRC 82(2010) 035208
  - Target asymmetry  $T$   
C.S. Akondi et al. (A2 Collaboration at MAMI) Phys. Rev. Lett. 113, 102001 (2014).
  - Double-polarisation asymmetry  $F$   
C.S. Akondi et al. (A2 Collaboration at MAMI) Phys. Rev. Lett. 113, 102001 (2014).
- GRAAL collaboration
  - Beam asymmetry  $\Sigma$   
O. Bartalini et al., EPJ A 33 (2007) 169



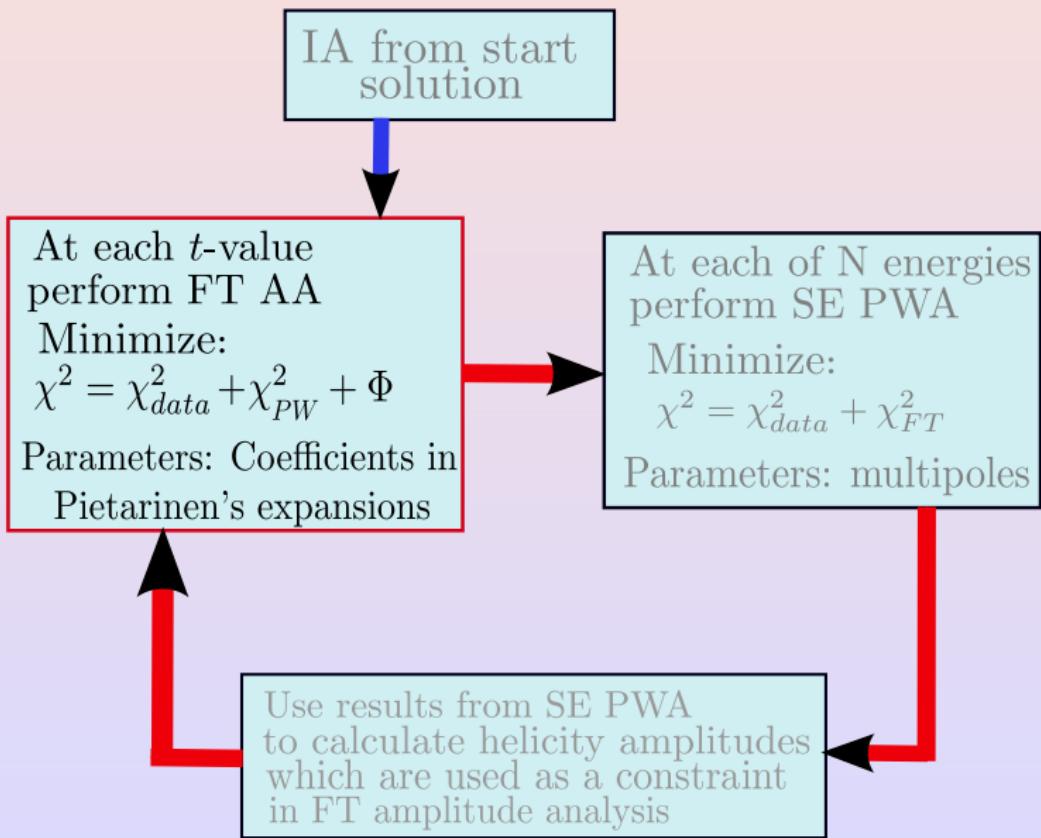
# Preparing input data



Observables at  $t = -0.2 \text{ GeV}^2$  and  $t = -0.5 \text{ GeV}^2$ .



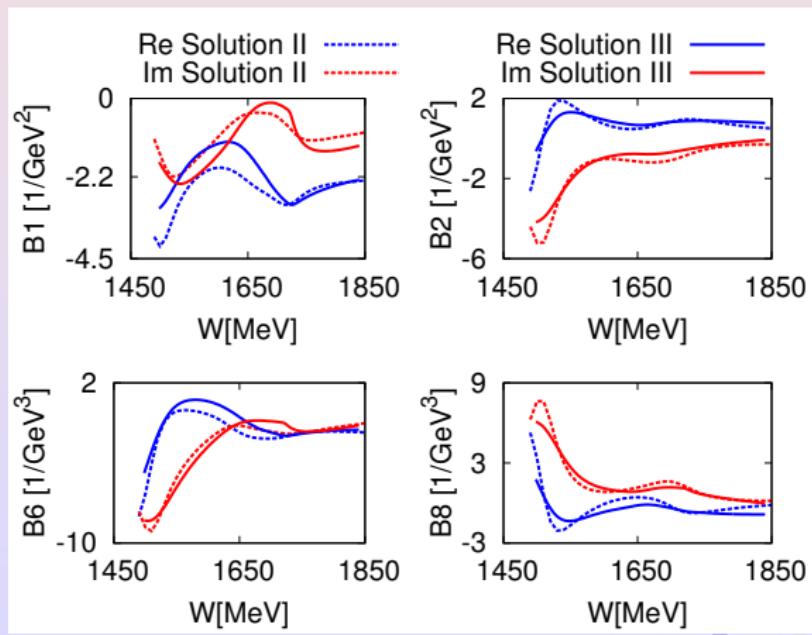
# Fixed-t amplitude analysis



# Fixed-t amplitude analysis-real data

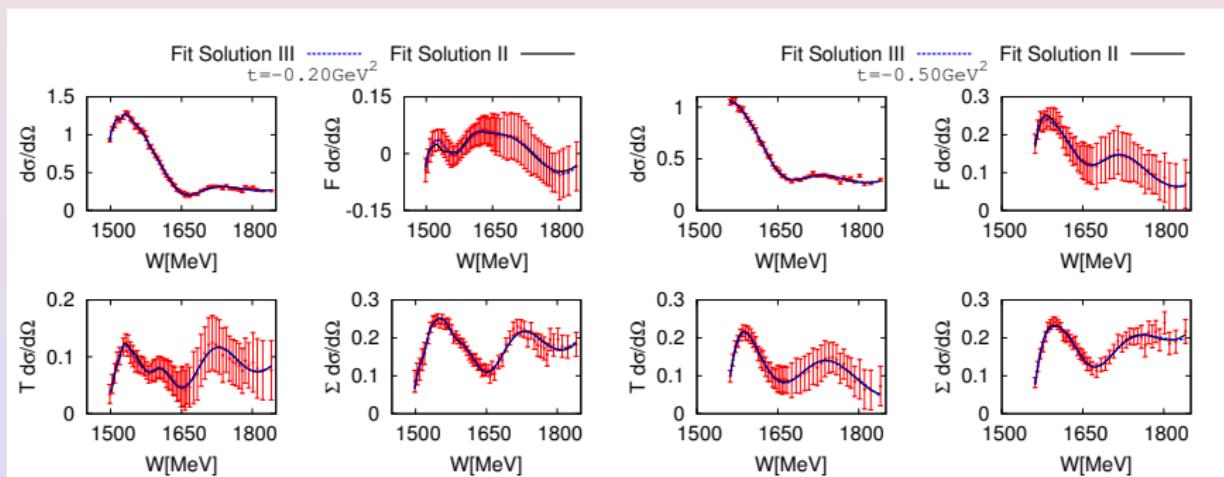
In order to explore model dependence of our solution we started with two MAID solutions and performed complete analysis (Solution II and Solution III).

Invariant amplitudes from initial Solution II and III  $t = -0.2 \text{ GeV}^2$ .

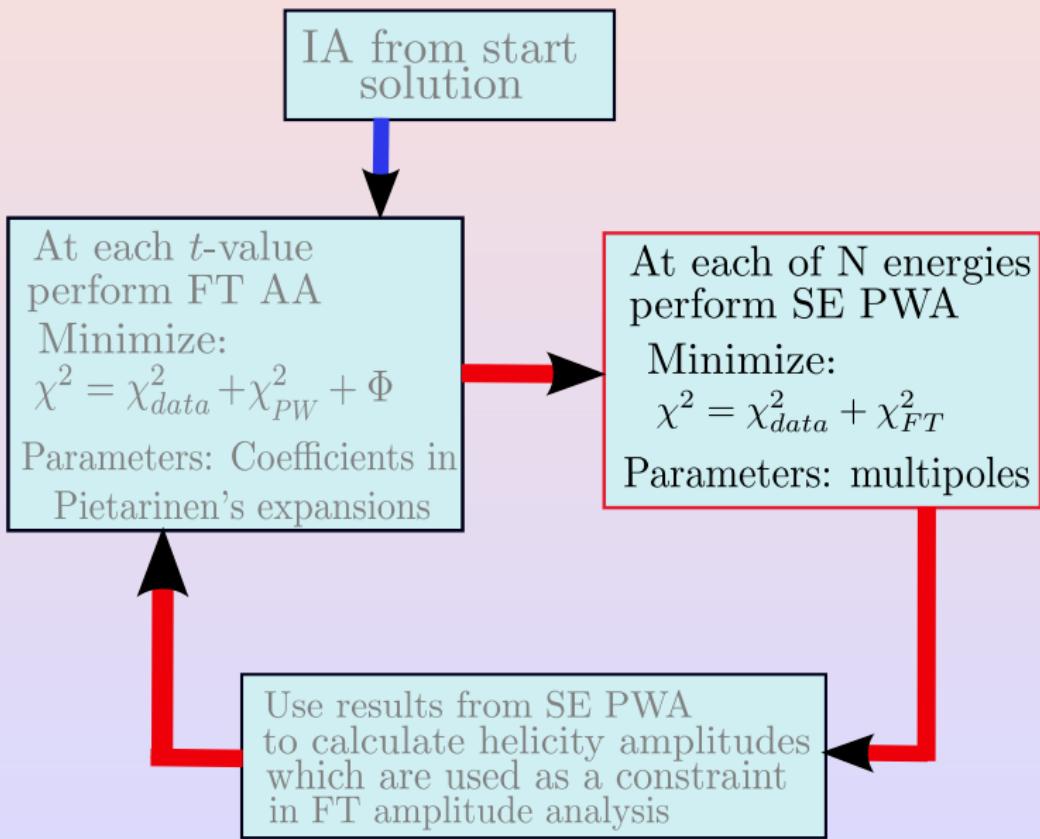


# Fixed-t amplitude analysis-real data

Fit to the data at  $t = -0.2 \text{ GeV}^2$  and  $t = -0.5 \text{ GeV}^2$ .

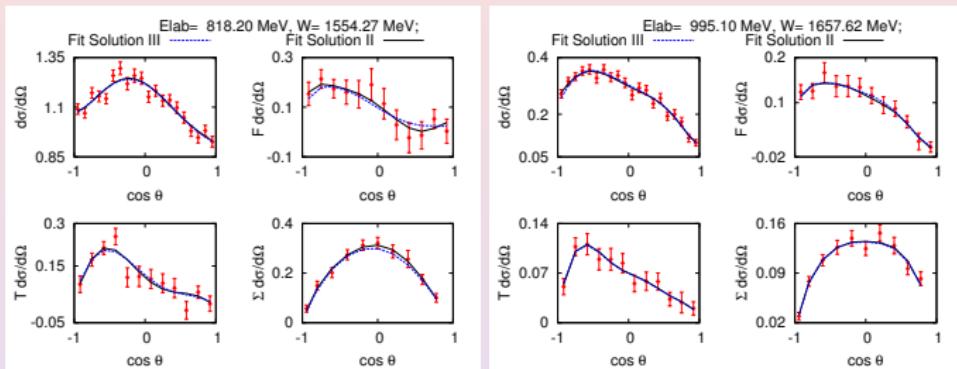


# Constrained SE PWA-real data

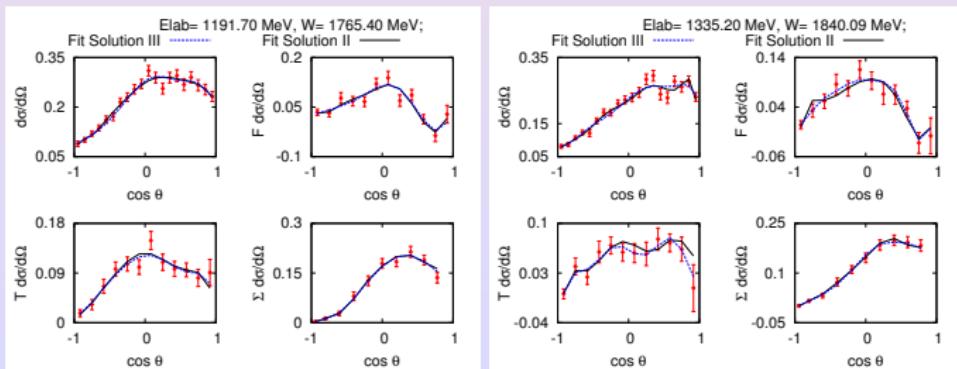


# Constrained SE PWA-real data

Fit to the data at  $W = 1554.27\text{MeV}$  and  $W = 1657.62\text{MeV}$ .

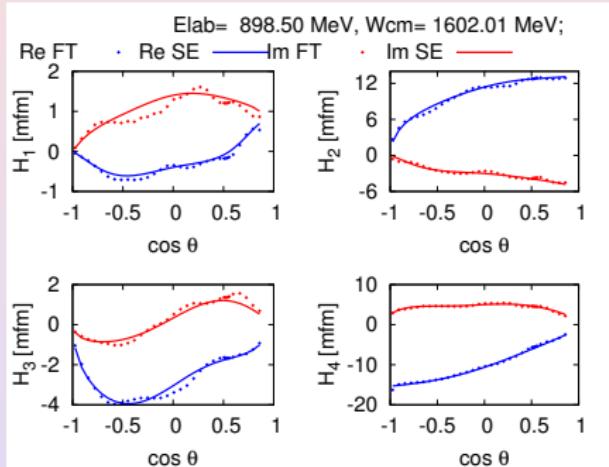


Fit to the data at  $W = 1765.4\text{MeV}$  and  $W = 1840.09\text{MeV}$ .

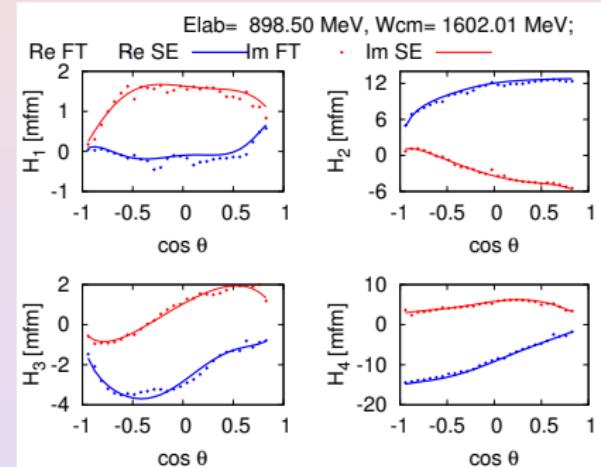


# Constrained SE PWA-helicity amplitudes

Helicity amplitudes at  $W = 1602.01\text{MeV}$ .



(a) Helicity amplitudes obtained using Solution II as constraint.



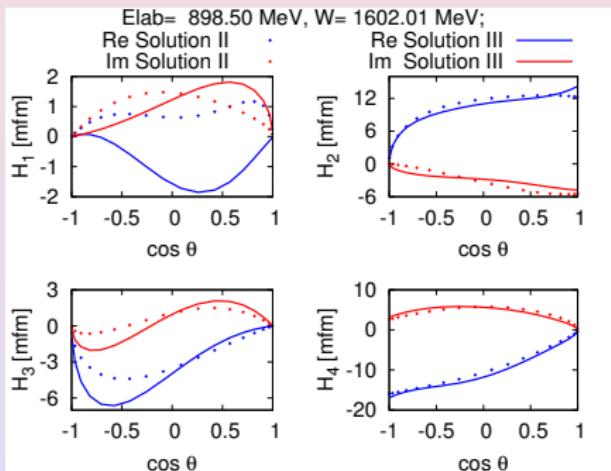
(b) Helicity amplitudes obtained using Solution III as constraint.

To be pointed out: Real and imaginary parts of helicity amplitudes (blue and red dots- Re FT and Im FT) are obtained from independent fixed-t AA at different  $t$ -values.

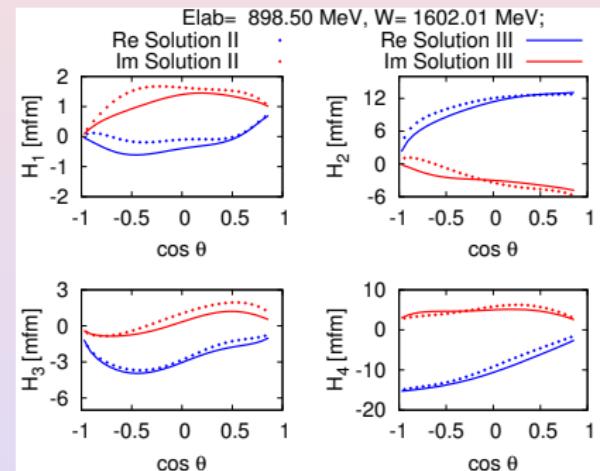


# Constrained SE PWA-helicity amplitudes

Helicity amplitudes at  $W = 1602.01 \text{ MeV}$ .



(c) Initial helicity amplitudes- solutions II and III.

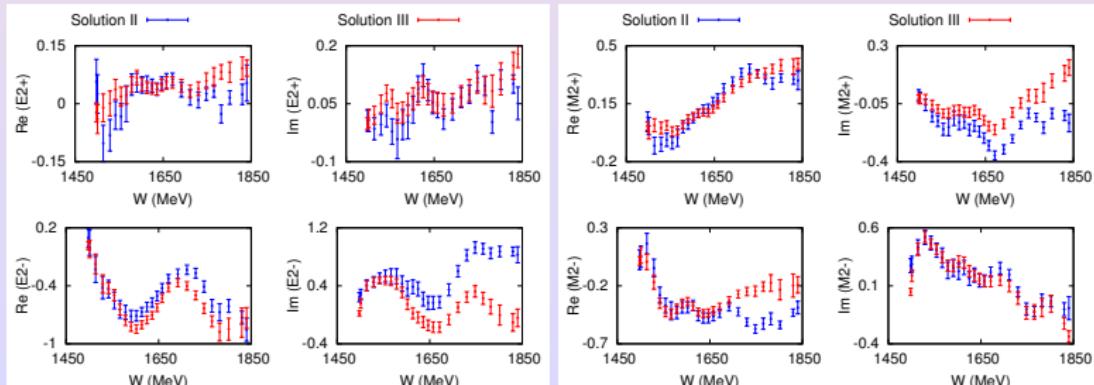
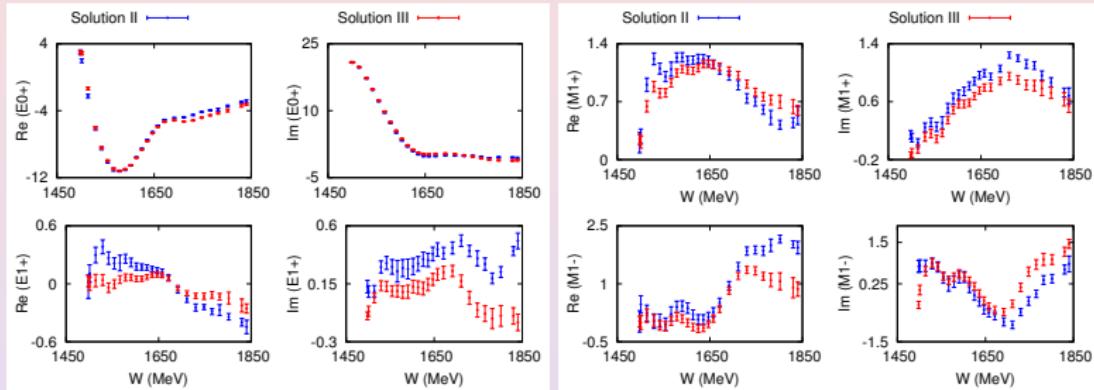


(d) Helicity amplitudes after three iterations using solutions II and III as a constraint.



# Constrained SE PWA-multipoles

$s, p, d$  waves



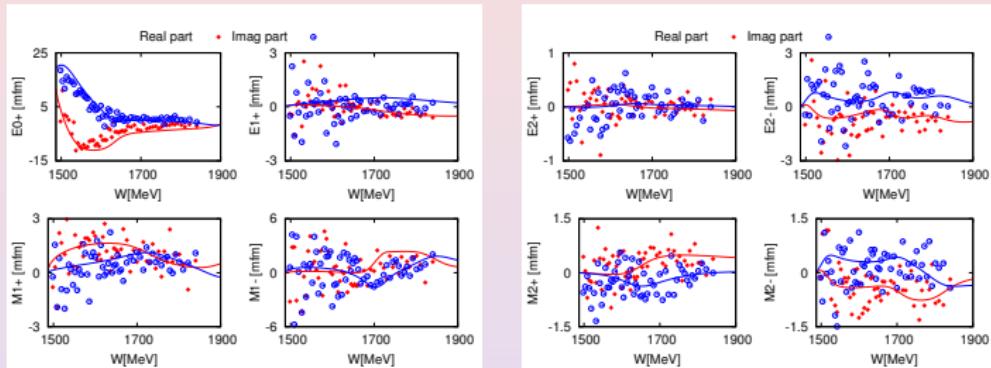
# Conclusions

- Multipoles obtained in our coupled fixed-t-single energy PWA are:
  - model independent - almost
  - consistent with fixed-s and fixed-t analyticity
  - consistent with crossing symmetry



# Conclusions

## Unconstrained single energy PWA



SE PWA with fixed-t constraint

