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IPN INSTITUT DE PHYSIQUE NUCLÉAIRE ORSAY

The $\eta \to 3\pi$, $\eta\pi \to \pi\pi$ amplitudes from the chiral expansion and an extended Khuri-Treiman formalism

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Overview:

- Introduction
- Khuri-Treiman: elastic rescattering
- Khuri-Treiman with inelastic channels
- Results for $\eta \to 3\pi$, $\eta\pi \to \pi\pi$

Introduction

- High precision data on $\eta \to 3\pi$
  [KLOE(2016)] : $4.6 \cdot 10^6$ events,
  [Layter(1976)]: 80884 events, used in the 90's

  Challenging for theory !

- Many recent results on 3-body decays
  $\eta' \to \eta\pi\pi$, $\eta' \to 3\pi$ ([BESIII]), $D \to K\pi\pi$, $D \to K\bar{K}\pi$, $D \to K\bar{K}K \cdots$ ([FOCUS, CLEO, Babar, LHCb])
  - → Higher masses: may require coupled-channel unitarity
  - → $\eta \to 3\pi$: Influence of $a_0(980)$, $f_0(980)$ resonances ?
    [Abdel-Rehim, Black, Fariborz, Schechter (2002)]

- $\eta \to 3\pi$: Two theoretical frameworks
  - → Low-energy chiral expansion [η and π are (quasi) Goldstone bosons]
  - → General analyticity +unitarity: [Khuri-treiman (1961)] equations

- Chiral expansion:
  - → Good aspect: explicit dependence on QCD+QED parameters. Determination of isospin breaking quark mass ratio

$$Q^{-2} = \frac{m_d^2 - m_u^2}{m_s^2 - m_{ud}^2}$$

  - → Bad aspect: unitarity not exact, slow convergence for FSI amplitudes

- Assume convergence in unphysical region, combine with Khuri-Treiman
  - → New approaches: fit some parameters to data, fit some others to CHPT [P. Guo, I.V. Danilkin, D. Schott, C. Fernández-Ramírez, V. Mathieu, A.P. Szczepaniak (2015), G. Colangelo, S. Lanz, H. Leutwyler, E. Passemar (2016)]
  - → Former approaches: try to predict physical amplitude from CHPT amplitude[J. Kambor, C. Wiesendanger, D. Wyler (1996), A. Anisovich, H. Leutwyler (1996)]

- Advantages of old approach:
  a) Respects chiral structure of amplitude:

$$\mathcal{T} = Q^{-2}\,\mathcal{T}_a + e^2\,\mathcal{T}_b + (m_{K^0}^2 - m_{K^+}^2)\,\mathcal{T}_c$$

  b) Probes KT formalism:
  – approx. sol. gives diff. result from exact
  – we can probe influence of 1 GeV resonances
  – also probes chiral expansion

Khuri-Treiman: single-variable amplitudes, elastic unitarity

- Start from dispersion relations
  - → Combine fixed-t, fixed-s

$$
\begin{aligned}
\mathcal{T}(s,t,u) \sim & \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} \frac{ds'}{(s'-s)} disc_s[\mathcal{T}(s',t,u')) \\
& + \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} \frac{du'}{(u'-u)} disc_u[\mathcal{T}(s',t,u')) \\
& + \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} \frac{dt'}{(t'-t)} disc_t[\mathcal{T}(s,t',u'))
\end{aligned}
$$

  - → Truncate P-W expansion inside integrand: $J \leqslant J_{max}$
  - → Cut integrations into two, $1 < \Lambda_{cut} << (1.3)^2$ (GeV^2)
  - → Higher-energy integrations approximated by polynomial.

- Single-variable amplitudes:
  - → Only *S*-wave [Khuri-Treiman (1961)], $K \to 3\pi$ applies to $\eta \to 3\pi^0$

$$\mathcal{T}^{3\pi^0}(s,t,u) = \Phi(s) + \Phi(t) + \Phi(u)$$

  - → $\Phi(z)$ analytic with right-hand cut
  - → *S*-wave + *P*-wave (+isospin I=0, I=1, I=2) [KWW(1996), AL(1996)]

$$\mathcal{T}^{\pi^+\pi^-\pi^0}(s,t,u) = -\epsilon_L\Big[M_0(s) - \frac{2}{3}M_2(s) + (s-u)M_1(t) + (s-t)M_1(u) + M_2(t) + M_2(u)\Big]$$

with

$$\epsilon_L = Q^{-2}\frac{(m_K^2 - m_\pi^2)m_K^2}{3\sqrt{3}F_\pi^2\, m_\pi^2}$$

Matching with ChPT:

- ChPT expression of amplitude has same single-variable repres. up to NNLO ($O(p^6)$)[Bijnens, Ghorbani (2007)]

- Matching at NLO:

$$\text{disc}\left[M_I(s) - \bar{M}_I(s)\Big|_{NLO}\right] = NNLO$$

  → Expand each difference as polynomial

  → Set complete polynomial=0: four matching equations

- obtain unique dispersive amplitude IF : four polynomial parameters

- Note: $s + t + u = m_\eta^2 + 3m_\pi^2$. Must compare complete amplitudes a posteriori to check consistency

Unitarity relations:

- Consider $\eta\pi \to \pi\pi$ partial-waves

$$\mathcal{T}_0^0(s) = \frac{\sqrt{6}\epsilon_L}{32\pi}\Big(M_0(s) + \hat{M}_0(s)\Big)$$
$$\mathcal{T}_1^1(s) = \frac{\epsilon_L}{48\pi}\kappa(s)\Big(M_1(s) + \hat{M}_1(s)\Big)$$
$$\mathcal{T}_0^2(s) = -\frac{\epsilon_L}{16\pi}\Big(M_2(s) + \hat{M}_2(s)\Big)$$

→ $\hat{M}_I(s)$ (left-hand cut) expressed as angular integrals in terms of $M_I(s)$

→ Elastic unitarity

$$\mathrm{disc}[\mathcal{T}_I^J(s)] = \sigma_\pi(s)\,\big(t^I_{\pi\pi,J}(s)\big)^*\;\mathcal{T}_J^I(s)$$

→ $\eta \to 3\pi$ 3-body unitarity ? [I. Aitchison, R. Pasquier, PR 152 (1966)]

→ Transform DR's (singular integral eqs.) with Muskhelishvili-Omnès:

$$M_0(w) = \Omega_0(w)\left[\alpha_0 + w\,\beta_0 + w^2\left(\gamma_0 + \hat{I}_0(w)\right)\right]$$
$$M_1(w) = \Omega_1(w)\,w\left(\beta_1 + \hat{I}_1(w)\right)$$
$$M_2(w) = \Omega_2(w)\,w^2\,\hat{I}_2(w)$$

Omnès functions

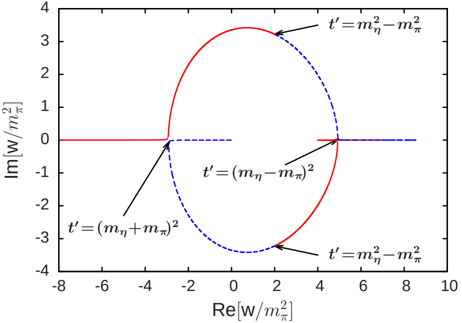
$$\Omega_I(s) = \exp\left[\frac{s}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'(s'-s)}\,\delta_I(s')\right]$$

Integrals

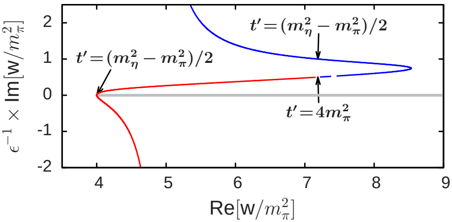
$$\hat{I}_I(w) = -\frac{1}{\pi}\int_{4m_\pi^2}^{\infty} ds'\,\frac{\mathrm{Im}\,(1/\Omega_I(s'))}{(s')^n(s'-w)}\,\hat{M}_I(s')$$

→ Evaluation necessary for proper matching w. ChPT

- Structure of left-hand cut



- Integrals $\hat{I}_l$ well-defined with proper $i\epsilon$ prescriptions



Khuri-Treiman with inelastic channels

- Include more states in unitarity:
  - → $\pi\eta \to \pi\eta$ easy to include (but relevant near $a_0(980)$)
  - → New amplitudes with $K\bar{K}$:

$K^+\bar{K}^0 \to \pi^+\eta$	isospin conserving $I=1$
$K^+\bar{K}^0 \to \pi^+\pi^0$	
$K^+K^- \to \pi^+\pi^-$	
$K^+K^- \to \pi^0\pi^0$	
$K^0\bar{K}^0 \to \pi^+\pi^-$	Isospin conserving
$K^0\bar{K}^0 \to \pi^0\pi^0$	+ isospin violating:
$K^+K^- \to \pi^0\eta$	$[I=1 \to I=2]$
$K^0\bar{K}^0 \to \pi^0\eta$	$[I=0 \to I=1]$
$K^+K^- \to K^+K^-$	
$K^+K^- \to K^0\bar{K}^0$	
$K^0\bar{K}^0 \to K^0\bar{K}^0$	

- Isospin separation: partial-waves ($J = 0$)
  - Isospin violating amplitudes

$$I = 0 \to 1 \quad \mathbf{T}^{(01)}: \begin{pmatrix} (\pi\pi)_0 \to \eta\pi & (K\overline{K})_0 \to \eta\pi \\ (\pi\pi)_0 \to (K\overline{K})_1 & (K\overline{K})_0 \to (K\overline{K})_1 \end{pmatrix}$$

$$I = 1 \to 2 \quad \mathbf{T}^{(12)}: \begin{pmatrix} \eta\pi^+ \to \pi^+\pi^0 \\ K^+\bar{K}^0 \to \pi^+\pi^0 \end{pmatrix}$$

  - Isospin conserving amplitudes

$$\mathbf{T}^{(0)}: \begin{pmatrix} \pi\pi \to \pi\pi & \pi\pi \to K\overline{K} \\ \pi\pi \to K\overline{K} & K\overline{K} \to K\overline{K} \end{pmatrix}_{I=0}$$

$$\mathbf{T}^{(1)}: \begin{pmatrix} \eta\pi \to \eta\pi & \eta\pi \to K\overline{K} \\ \eta\pi \to K\overline{K} & K\overline{K} \to K\overline{K} \end{pmatrix}_{I=1}$$

$$T^{(2)}: \pi^+\pi^0 \to \pi^+\pi^0$$

- Unitarity relation (J=0): first order in isospin breaking:

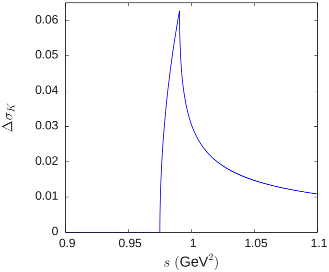
  → $\mathbf{T}^{(01)}$ relation:

$$\mathrm{Im}\,[\mathbf{T}^{(01)}] = \mathbf{T}^{(0)*}\,\Sigma^0\,\mathbf{T}^{(01)} + \mathbf{T}^{(01)*}\,\Sigma^1\,\mathbf{T}^{(1)} + \mathbf{T}^{(0)*}\begin{pmatrix}0 & 0\\ 0 & \Delta\sigma_K\end{pmatrix}\mathbf{T}^{(1)}$$

$$\Delta\sigma_K(s) = \tfrac{1}{2}(\sigma_{K^+}(s) - \sigma_{K^0}(s))$$

[Achasov,Devyanin,Shestakov, PL B88(1979)367]

with $\sigma_P(s) = \sqrt{1 - 4m_K^2/s}$



  → $\mathbf{T}^{(12)}$ relation:

$$\mathrm{Im}\,[\mathbf{T}^{(12)}] = \mathbf{T}^{(1)*}\,\Sigma^1\,\mathbf{T}^{(12)} + \mathbf{T}^{(12)*}\sigma_\pi\,T^{(2)}$$

Matrix generalisation of KT equations:

$$\mathbf{M}_0(w) = \boldsymbol{\Omega}_0(w)\left[\mathbf{P}_0(w) + w^2\left(\hat{\mathbf{I}}_A(w) + \hat{\mathbf{I}}_B(w)\right)\right]{}^t\boldsymbol{\Omega}_1(w)$$

- → $\boldsymbol{\Omega}_I$: Omnès-Muskhelishvili matrices
- → $\mathbf{P}_0$: polynomials, 12 parameters
- → "left-cut" integrals

$$\hat{\mathbf{I}}_A(w) = -\frac{1}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{(s')^2(s'-w)}\left[\mathrm{Im}\,\boldsymbol{\Omega}_0^{-1}\,\hat{\mathbf{M}}_0\,{}^t\boldsymbol{\Omega}_1^{-1} + \boldsymbol{\Omega}_0^{-1*}\,\hat{\mathbf{M}}_0\mathrm{Im}\,{}^t\boldsymbol{\Omega}_1^{-1}\right]$$

$$\hat{\mathbf{I}}_B(w) = \frac{32}{\sqrt{6}\epsilon_L}\int_{4m_\pi^2}^{\infty}\frac{ds'\,\Delta\sigma_K(s')}{(s')^2(s'-w)}\boldsymbol{\Omega}_0^{-1*}\mathbf{T}^{(0)*}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\mathbf{T}^{(1)}\,{}^t\boldsymbol{\Omega}_1^{-1}$$

Solving coupled-channel KT

- Complete determination require analogous equations in all crossed-channels
- Simple approximations: ($K\overline{K}$ amplitudes not needed to high accuracy)
  - → $K\overline{K}$ left-cut functions neglected
  - → ChPT matching of $K\overline{K}$ amplitudes at leading-order

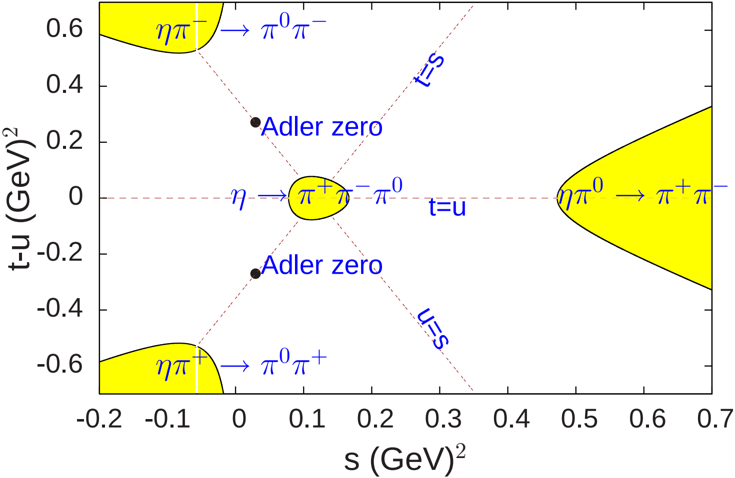
Isospin conserving T-matrices

→ $I = 0$: (reasonably) good experimental constraints

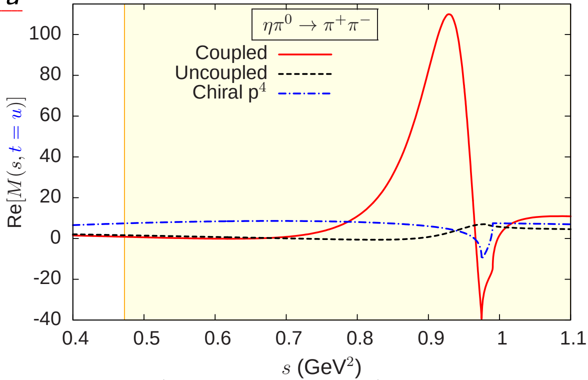
→ $I = 1$: model[M. Albaladejo, B. Moussallam (2015)]
– reproduce $a_0(980)$, $a_0(1450)$ properties
– enforce chiral constraints on $\eta\pi$ scalar radius

Results

Physical regions



- Amplitude along $t = u$ (medium energy)



- Influence of inelastic channels:

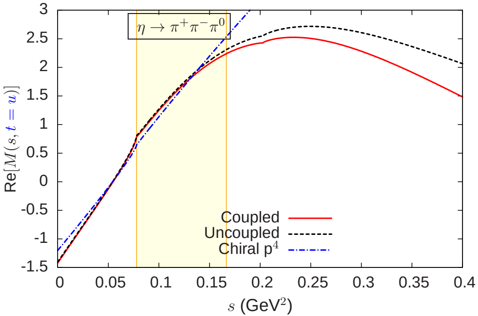
$$[\mathbf{M}_0]^{inel}_{11} \simeq (\boldsymbol{\Omega}_0)_{12} (\boldsymbol{\Omega}_1)_{11} \left(\mathbf{P}_0 + s^2 (\hat{\mathbf{I}}_a + \hat{\mathbf{I}}_b) \right)_{21}$$

and

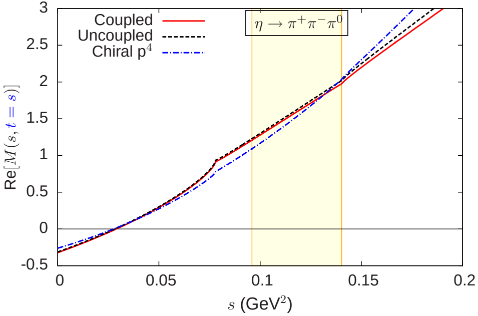
$$\mathrm{Re}\,(\boldsymbol{\Omega}_0)_{12} / \mathrm{Im}\,(\boldsymbol{\Omega}_0)_{12} = \cot \delta^0_0 \ \text{(Watson)}$$
$$(\boldsymbol{\Omega}_1)_{11} > 1$$

Amplitude at low energy: $\eta \to \pi^+\pi^-\pi^0$

Along $t = u$ line

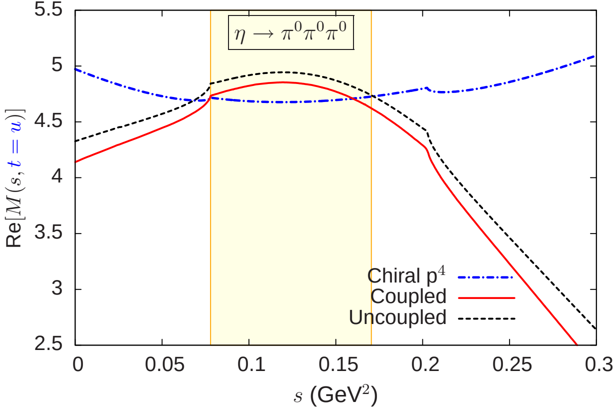


Along $t = s$ line



- Good matching with NLO ChPT

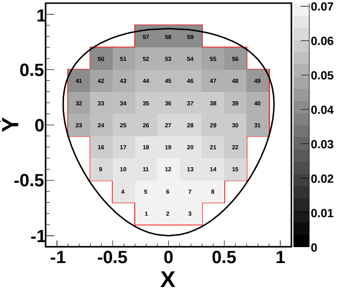
Amplitude at low energy: $\eta \to \pi^0\pi^0\pi^0$



- Substantial influence of coupled-channel rescattering at low-energy
- No good matching with NLO ChPT

Dalitz plot parameters

- Charged mode: define coordinates X,



$$X = \sqrt{3}\,\frac{E_{\pi^+} - E_{\pi^0}}{Q_c}, \quad Y = 3\,\frac{E_{\pi^0} - M_{\pi^0}}{Q_c}$$

$$Q_c = M_\eta - 2M_{\pi^+} - M_{\pi^0}$$

Description with 5 parameters

$$|\mathcal{T}_c(X,Y)|^2 = |\mathcal{T}_c(0,0)|^2 \left(1 + a\,Y + b\,Y^2 + d\,X^2 + f\,Y^3 + g\,X^2 Y\right)$$

Charge conjugation : $X \to -X$ invariance

- Neutral mode: description with one parameter

$$|\mathcal{T}_n(X,Y)|^2 = |\mathcal{T}_n(0,0)|^2 \left(1 + 2\,\alpha\,(X^2 + Y^2)\right)$$

Comparison with experiment

$\pi^+\pi^-\pi^0$	$O(p^4)$	single	coupled	KLOE	BESIII
a	-1.328	-1.154	-1.142	-1.095(4)	-1.128(15)
b	0.429	0.202	0.171	0.145(6)	0.153(17)
d	0.089	0.094	0.097	0.081(7)	0.085(16)
f	0.016	0.108	0.123	0.141(10)	0.173(28)
g	-0.081	-0.087	-0.088	-0.044(16)	–
$\pi^0\pi^0\pi^0$				PDG	
α	+0.0142	-0.0274	-0.0337	-0.0318(15)	

- b, f, α improved by KT and by coupled-channels
- g may require NNLO in matching conditions

Quark mass ratio Q^{-2}

- Remark: Q is QCD scale dependent

$$\mu_0 \frac{dm_q(\mu_0)}{d\mu_0} = -(\gamma^{QCD} + \frac{3e_q^2 e^2}{8\pi^2}) m_q(\mu_0)$$

which implies

$$\mu_0 \frac{d\epsilon_L}{d\mu_0} = \frac{e^2\, m_\pi^2}{48\pi^2\sqrt{3}F_\pi^2} + O(e^2(m_d - m_u), e^2 m_q^2)$$

compensated by scale variation of (Urech) LEC's

$$\mu_0 \frac{d}{d\mu_0}(K_9^r(\mu,\mu_0) + K_{10}^r(\mu,\mu_0)) = \frac{3}{64\pi^2}$$

→ Need K_9^r, K_{10}^r, but numerically not important.

- Q^2 adjusted to reproduce $\eta \to 3\pi$ decay width

Mode	Γ_{exp} (eV)	Q
$\pi^+\pi^-\pi^0$	299 ± 11	21.6 ± 0.2
$\pi^0\pi^0\pi^0$	427 ± 15	21.7 ± 0.2

- Error determination: assume 10% error from NNLO contribs. in parameters α_0, β_0, β_1, γ_0

$$\Delta Q = \pm 2.2$$

- Recent results from $\eta \to 3\pi$
  [Guo et al. (JPAC), 1608.01447]: $Q = 21.6 \pm 0.4$
  [Colangelo et al. ,PRL118,022001]: $Q = 22.0 \pm 0.7$
- Recent results from lattice QCD (hadron masses)
  [(QCDSF-UKQCD) JHEP 04,093] 22.9 ± 0.4
  [(BMW) PRL 117,082001] 23.4 ± 0.6

Conclusions/Outlook

- Chiral EFT is effective in unphysical region for $\eta \to \pi^+\pi^-\pi^0$, NNLO corrections less than 10%

- Applications of $\eta\pi \to \pi\pi$
  - → Vector form-factor in isospin-violating (2nd class) $\tau \to \pi\eta\nu$ related by unitarity to $\eta\pi^+ \to \pi^0\pi^+$ (P-wave)[Descotes-Genon, B.M. EPJ C74 (2014)]
  - → $a_0 - f_0$ mixing phenomenon:
    $J/\Psi \to \phi\pi^0\eta$: seen ? [BESIII, PR D83 (2011)]
    $B_s \to J/\Psi\pi^0\eta$ suggested [Wei Wang, PL B759 (2016)]
    Again related by unitarity to $\eta\pi^0 \to \pi\pi$ (some work required)