Meson-meson amplitudes, constraints and new methods

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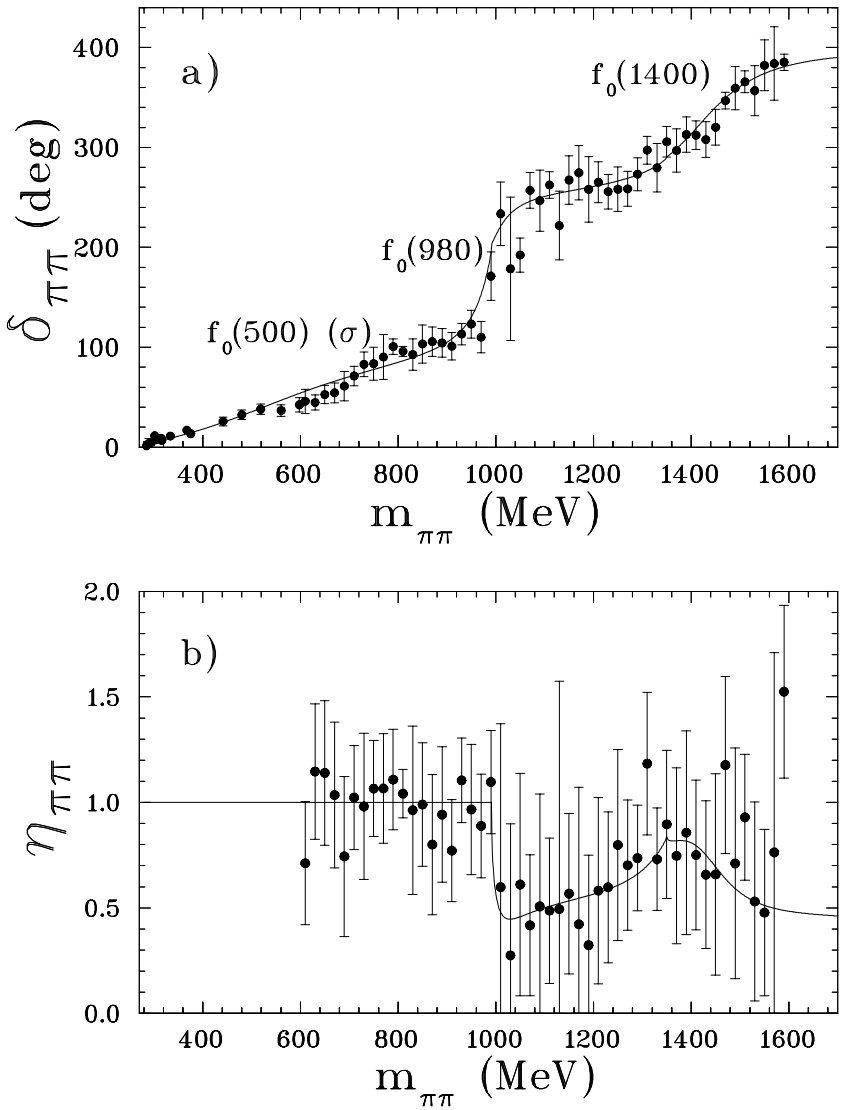
PWA 9 / ATHOS 4, Bad Honnef, III 2017

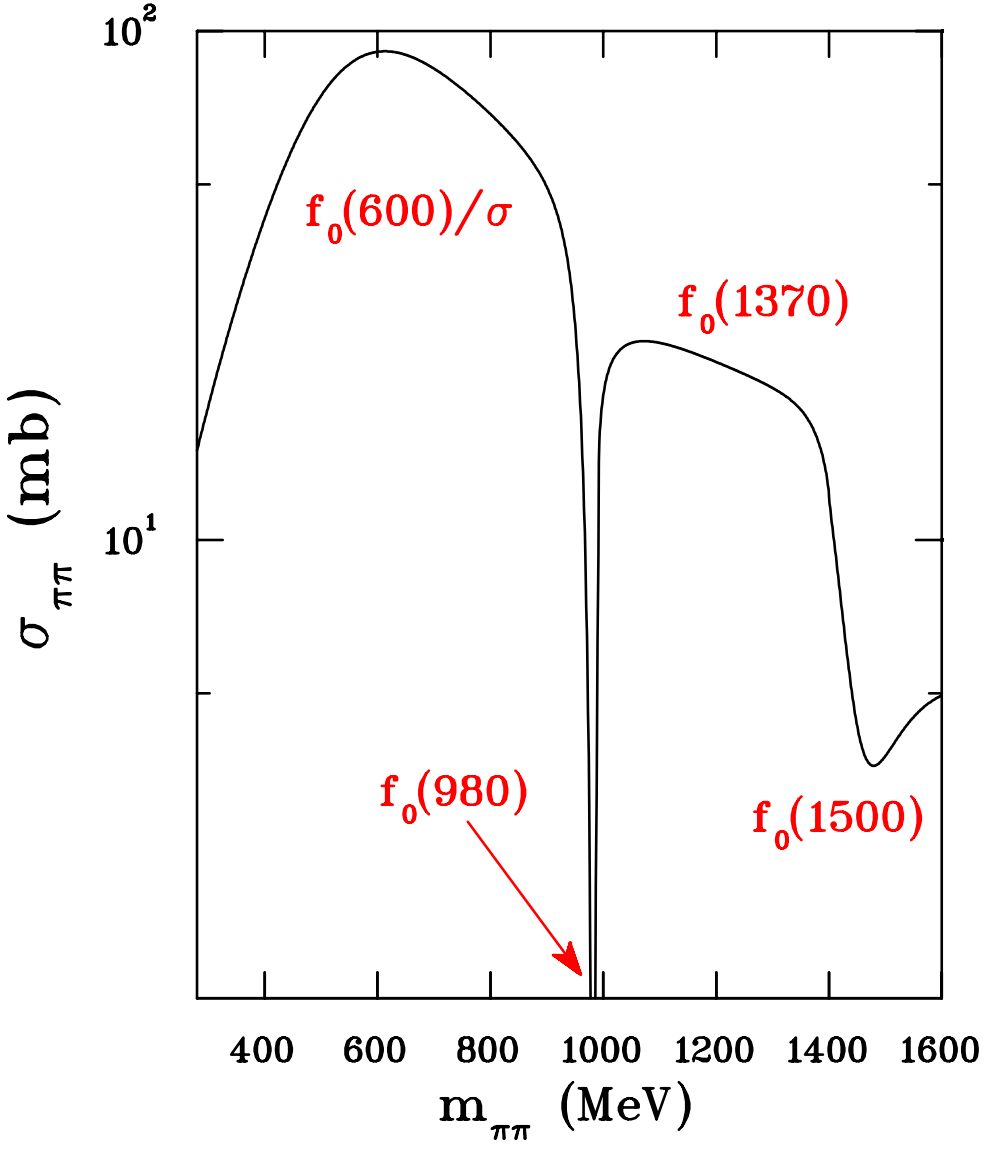
- An overview of the most effective methods used for parameterization of $\pi\pi$ amplitudes below 2 GeV...
- .... $\pi\pi$ amplitudes in all important partial waves (S, P, D and F),
- Theoretical constraints imposed on the amplitudes and requirements for amplitudes ...
- Outcome and perspectives

Best known experimentally and theoretically: $\pi\pi$ interactions in the (JI) S0 wave

data mostly from CERN-Kraków-Munich coll' 74, $\pi p \to \pi\pi p$

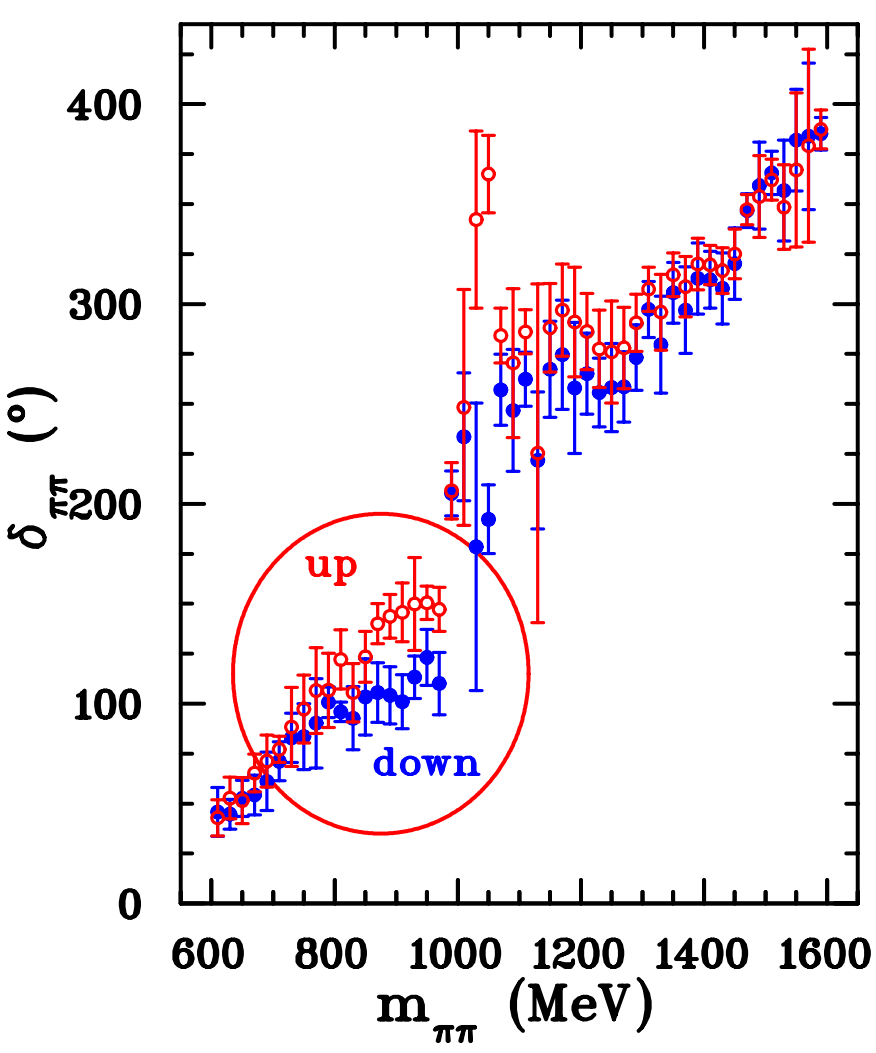


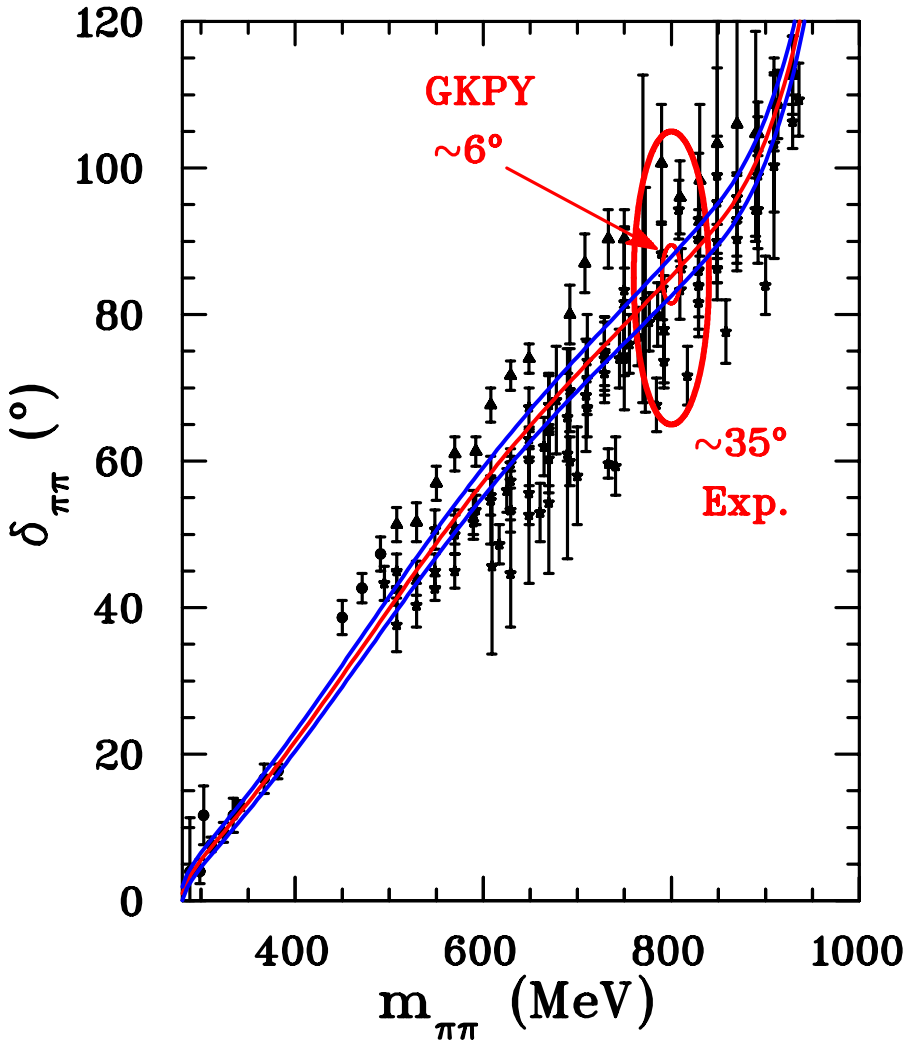
Puzzling S0 wave ππ cross section



Experimental data for the $\pi\pi$ in the S0 wave (JI)

PWA (CERN-Kraków-Munich group'74 $\pi p \to \pi\pi p$)





What to do?

- a model and spectrum via analytical continuation to the complex energy plane,
- poles as free parameters,
- theoretical constraints imposed on model or positions of poles

separable potential model, Kraków-Paris model'2000

$$< \mathbf{p} \mid V_{ij} \mid \mathbf{q} >= \sum_{\mu,\nu} \lambda_{ij}^{\mu\nu}\, g_i^{\mu}(p) g_j^{\nu}(q),$$

$$g_i^{\nu}(p) = \sqrt{\frac{4\pi}{m_i}}\, \frac{1}{p^2 + (\beta_i^{\nu})^2}.$$

$$\begin{aligned}
< \mathbf{p} \mid T_{11} \mid \mathbf{q} > &= g_1^a(p)\, t_{11}^{aa}\, g_1^a(q) + g_1^a(p)\, t_{11}^{ab}\, g_1^b(q) \\
&+ g_1^b(p)\, t_{11}^{ba}\, g_1^a(q) + g_1^b(p)\, t_{11}^{bb}\, g_1^b(q), \\
< \mathbf{p} \mid T_{22} \mid \mathbf{q} > &= g_2(p)\, t_{22}\, g_2(q), \\
< \mathbf{p} \mid T_{12} \mid \mathbf{q} > &= < \mathbf{q} \mid T_{21} \mid \mathbf{p} > = \\
&= g_1^a(p)\, t_{12}^a\, g_2(q) + g_1^b(p)\, t_{12}^b\, g_2(q).
\end{aligned}$$

$\pi\pi \to K\overline{K}$ and $K\overline{K} \to \pi\pi$

$$t = \lambda + \lambda\, I\, t.$$

$$\lambda = \begin{pmatrix} \lambda_{11}^a & 0 & \lambda_{12}^a \\ 0 & \lambda_{11}^b & \lambda_{12}^b \\ \lambda_{12}^a & \lambda_{12}^b & \lambda_{22} \end{pmatrix}, \qquad I = \begin{pmatrix} I_{11}^{aa} & I_{11}^{ab} & 0 \\ I_{11}^{ab} & I_{11}^{bb} & 0 \\ 0 & 0 & I_{22} \end{pmatrix}.$$

$$I_{11}^{\mu\nu} = \int \frac{d^3 s}{(2\pi)^3}\, g_1^{\mu}(s) G_1(s) g_1^{\nu}(s), \quad (\mu = a, b, \ \ \nu = a, b)$$

$$t = (1 - \lambda I)^{-1} \lambda.$$

$$D(E) = det(1 - \lambda I(E)),$$

$$D(k_1, k_2) = D_\pi(k_1) D_K(k_2) - C(k_1, k_2),$$

$$\begin{aligned}
D_\pi(k_1) &= 1 - \Lambda_{11}^a\, J_{11}^{aa}(k_1) - \Lambda_{11}^b\, J_{11}^{bb}(k_1) + \Lambda_{11}^a \Lambda_{11}^b\, d(k_1), \\
D_K(k_2) &= 1 - \Lambda_{22}\, J_{22}(k_2), \\
C(k_1, k_2) &= \Lambda_{12}^a\, J_{22}(k_2)[\Lambda_{12}^a\, J_{11}^{aa}(k_1) + \Lambda_{12}^b\, J_{11}^{ab}(k_1) - \Lambda_{12}^a \Lambda_{11}^b\, d(k_1)] \ + \\
&+ \ \Lambda_{11}^b\, J_{22}(k_2)[\Lambda_{12}^a\, J_{11}^{ab}(k_1) + \Lambda_{12}^b\, J_{11}^{bb}(k_1) - \Lambda_{12}^b \Lambda_{11}^a\, d(k_1)]
\end{aligned}$$

$$\begin{aligned}
S_{11} &= \frac{D(-k_1, k_2)}{D(k_1, k_2)}, \\
S_{22} &= \frac{D(k_1, -k_2)}{D(k_1, k_2)},
\end{aligned}$$

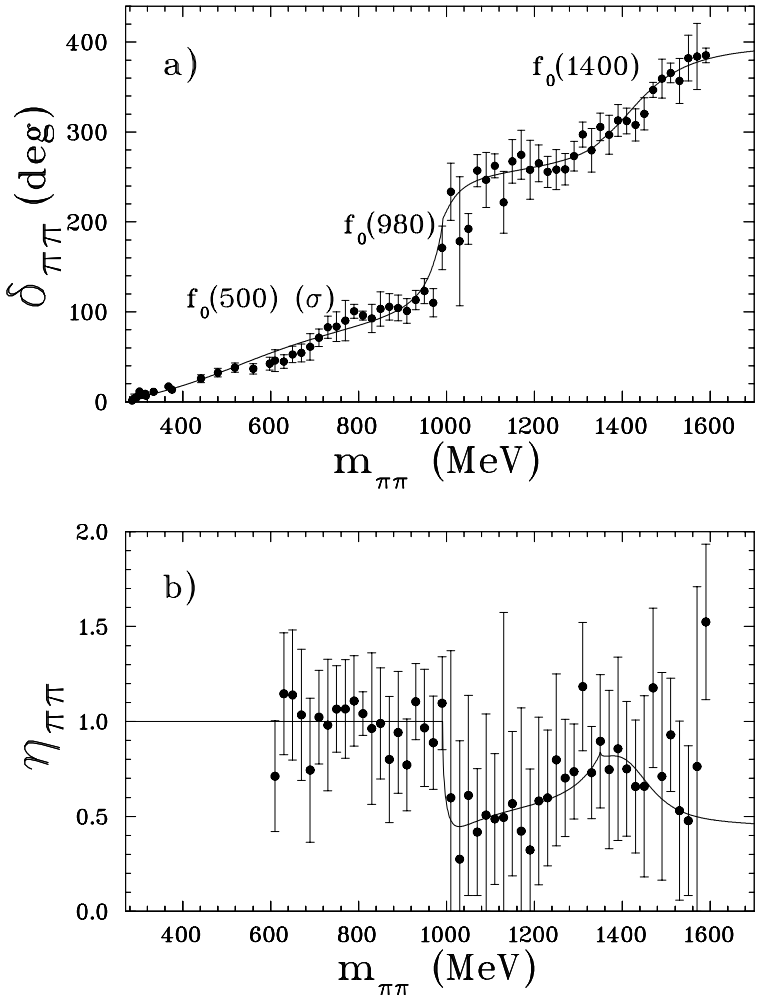
$$S_{11} S_{22} - S_{12}^2 = \frac{D(-k_1, -k_2)}{D(k_1, k_2)}.$$

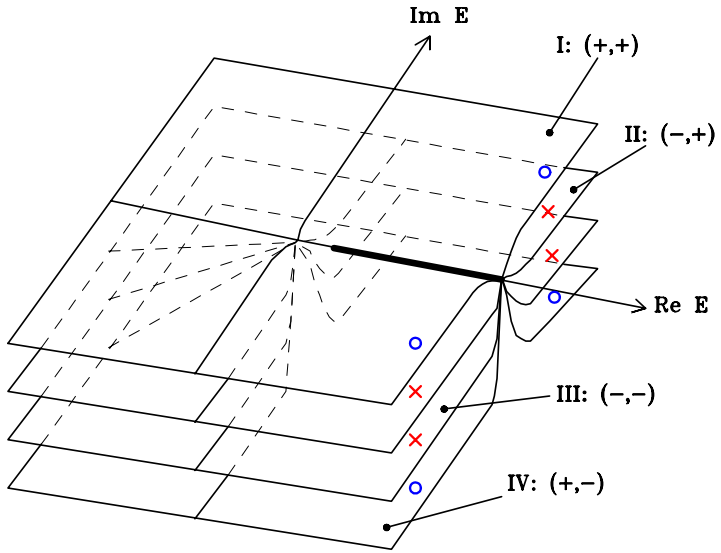
$$
\begin{aligned}
S_{11} &= 1 - \frac{ik_1 E_1(k_1)}{2\pi}\, T_{11}(k_1, k_2), \\
S_{22} &= 1 - \frac{ik_2 E_2(k_2)}{2\pi}\, T_{22}(k_1, k_2), \\
S_{12} &= S_{21} = -\frac{i}{2\pi}\sqrt{k_1 E_1(k_1) k_2 E_2(k_2)}\, T_{12}(k_1, k_2),
\end{aligned}
$$

$$
S = \begin{pmatrix} \eta e^{2i\delta_\pi} & i\sqrt{1-\eta^2}\, e^{i(\delta_\pi+\delta_K)} \\ i\sqrt{1-\eta^2}\, e^{i(\delta_\pi+\delta_K)} & \eta e^{2i\delta_K} \end{pmatrix}.
$$

Results

For three-coupled channel model up to 2 GeV:
- only - 14 free parameters,
- correct threshold behavior (not fixed!) and
- spectrum of all scalar-isoscalar mesons.





2^n Riemann sheets for n channels

channel	$C=0$		$C=1$		sign	sheet
	ReE	*ImE*	*ReE*	*ImE*	Imk_π, Imk_K, Imk_3	
$\pi\pi$	658	-607	564	-279	$-,-,-$	VI
			518	-261	$-,+,+$	II
			211	0	$-,+,-$	VII
			532	-315	$-,-,+$	III
			235	0	$+,+,-$	VIII
$\pi\pi$	1346	-275	1405	-74	$-,-,-$	VI
			1445	-116	$-,+,+$	II
			1424	-94	$-,+,-$	VII
			1456	-47	$-,-,+$	III
$K\bar{K}$	881	-498	170	0	$+,-,-$	V
			159	0	$-,-,-$	VI
			418	-10	$-,-,+$	III
			1038	-204	$-,+,-$	VII
			988	-31	$-,+,+$	II
$\sigma\sigma$	118	-2227	4741	-4688	$-,-,-$	VI
			3687	-2875	$-,+,-$	VII
			3626	-3456	$+,-,-$	V
			3533	-579	$+,+,-$	VIII

2^n Riemann sheets for n channels

channel	$C=0$		$C=1$		sign	sheet	
	ReE	*ImE*	*ReE*	*ImE*	Imk_π, Imk_K, Imk_3		
$\pi\pi$	658	-607	564	-279	$-,-,-$	VI	
			518	-261	$-,+,+$	II	$\leftarrow f_0(500)$
			211	0	$-,+,-$	VII	
			532	-315	$-,-,+$	III	
			235	0	$+,+,-$	VIII	
$\pi\pi$	1346	-275	1405	-74	$-,-,-$	VI	$\leftarrow f_0(1370)?$
			1445	-116	$-,+,+$	II	
			1424	-94	$-,+,-$	VII	
			1456	-47	$-,-,+$	III	$\leftarrow f_0(1500)?$
$K\bar{K}$	881	-498	170	0	$+,-,-$	V	
			159	0	$-,-,-$	VI	
			418	-10	$-,-,+$	III	
			1038	-204	$-,+,-$	VII	
			988	-31	$-,+,+$	II	$\leftarrow f_0(980)$
$\sigma\sigma$	118	-2227	4741	-4688	$-,-,-$	VI	
			3687	-2875	$-,+,-$	VII	
			3626	-3456	$+,-,-$	V	
			3533	-579	$+,+,-$	VIII	

Results for data: $p\pi \to p\pi\pi$

For three-coupled channel model up to 2 GeV:
- only - 14 free parameters,
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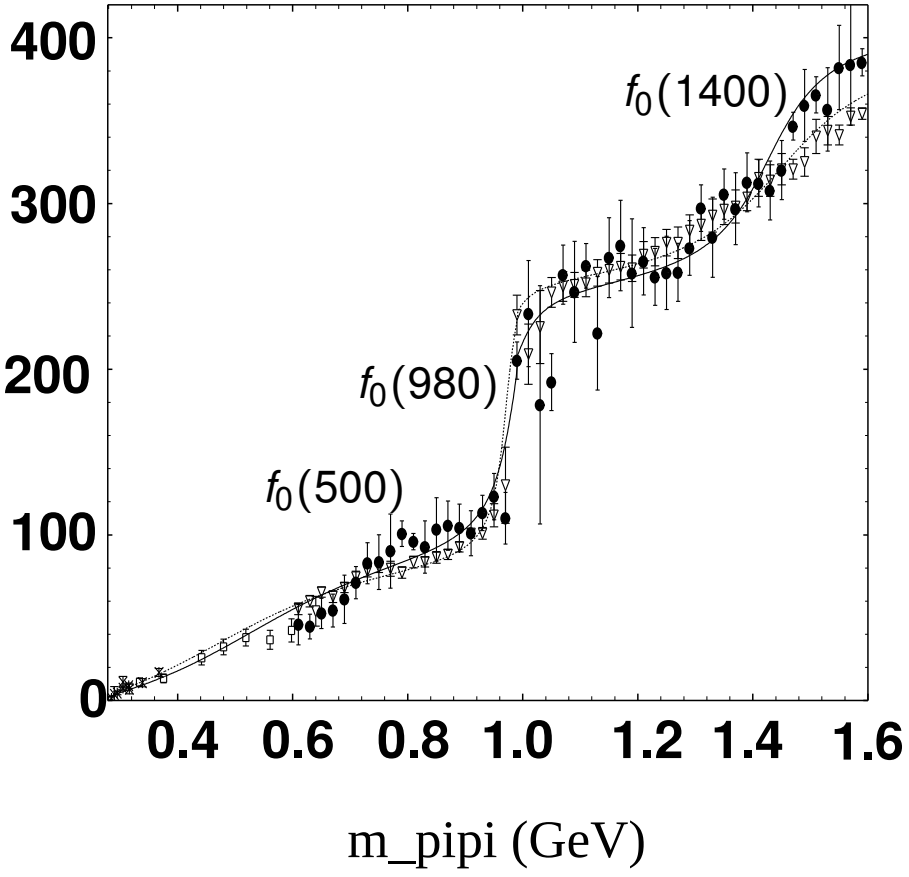
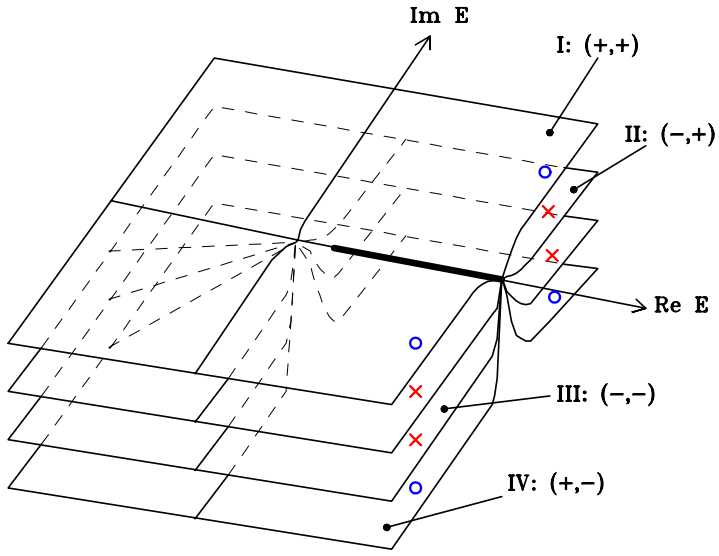
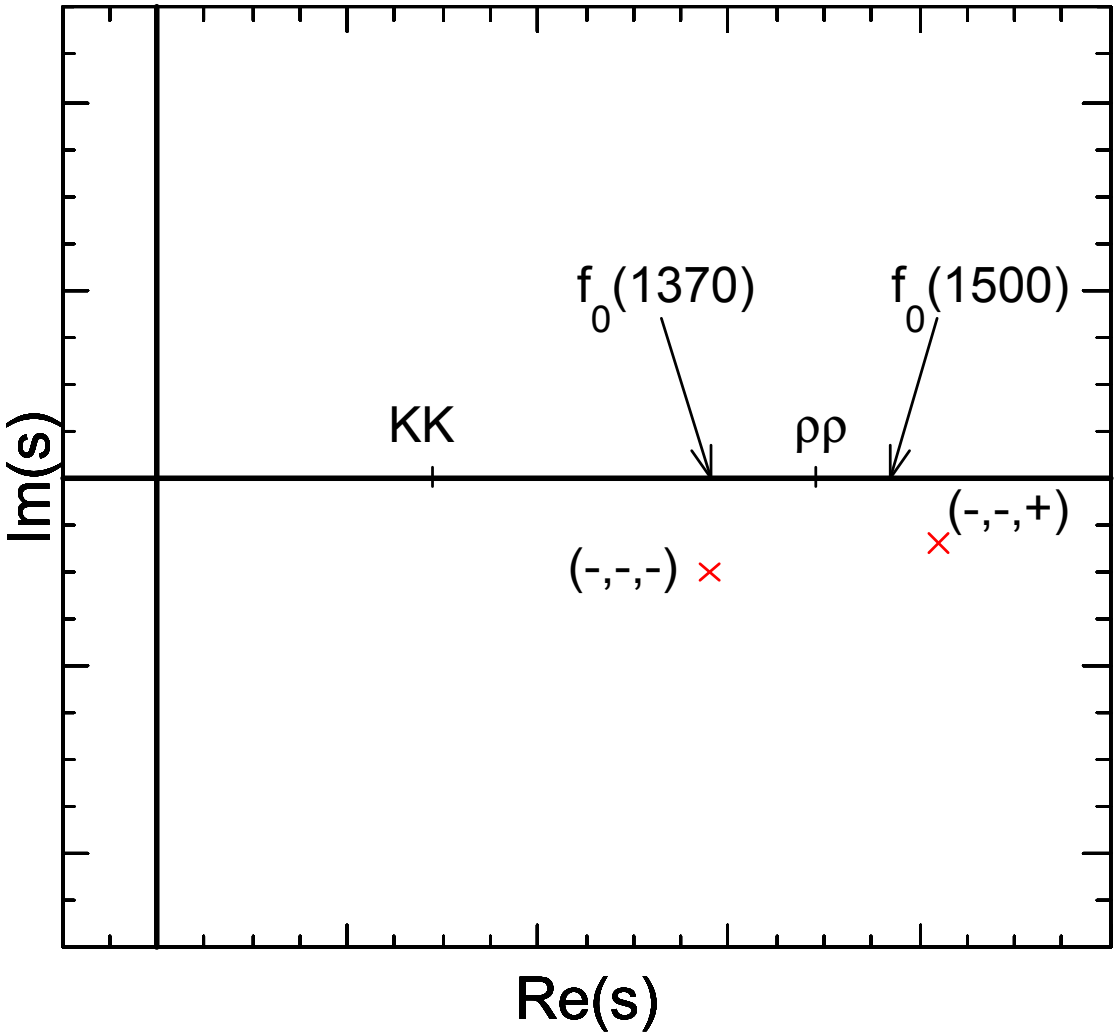
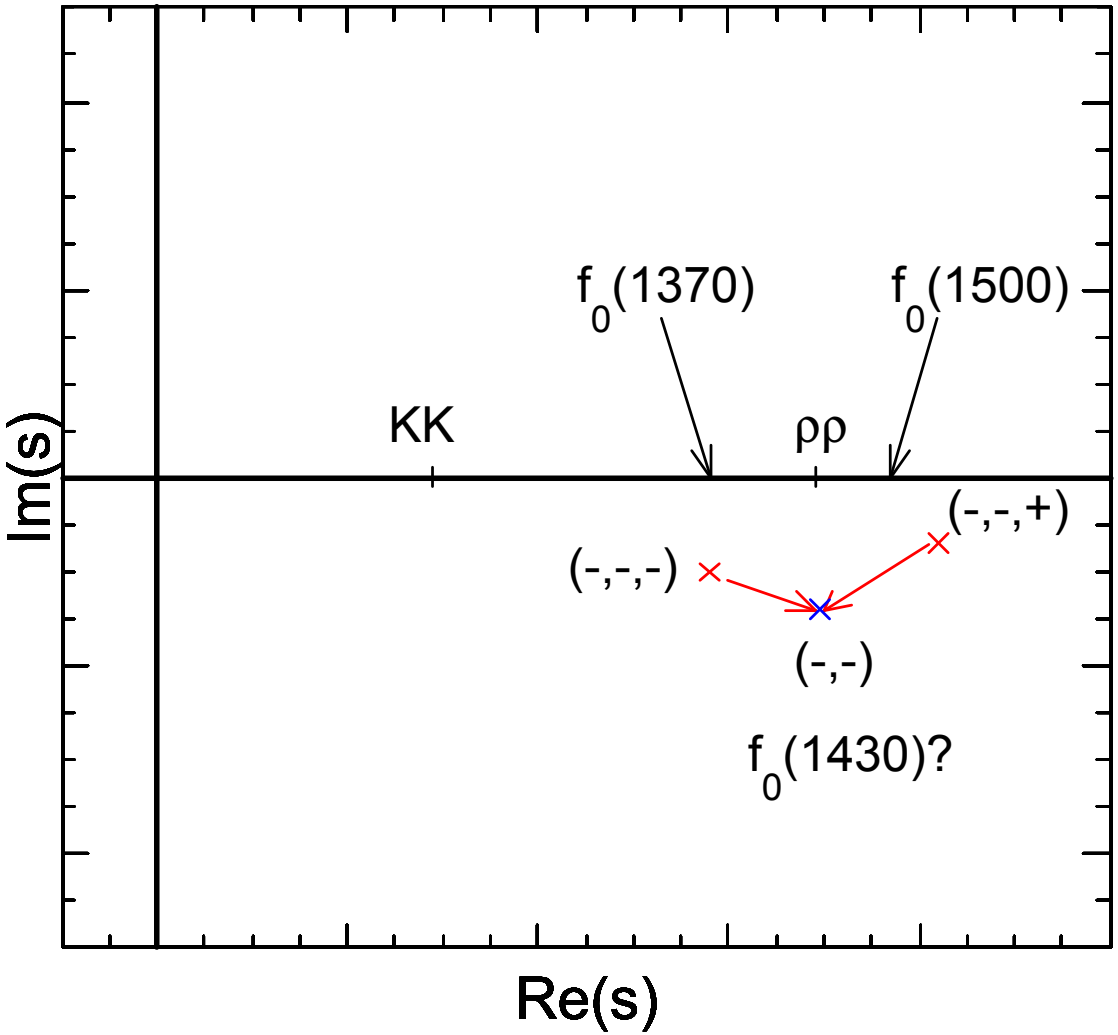




$f_0(1370)$ *and* $f_0(1500)$*: positions of poles,* $\lambda = 1$



$f_0(1370)$ *and* $f_0(1500)$*: positions of poles,* $\lambda = 0$



$\pi\pi$ amplitudes - 6 partial waves: $S0, S2, P1, D0, D2, F1$

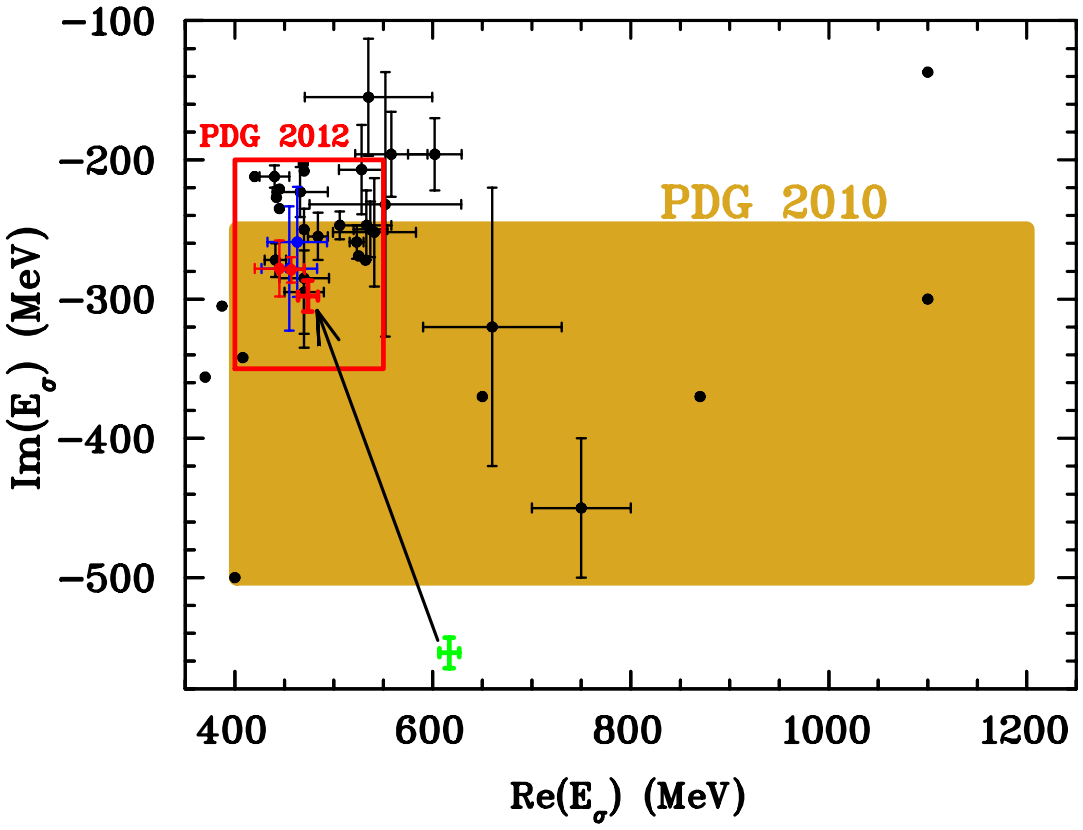
Used in Dubna-Kraków-Prague group (PRD'2014, 2016):

$$S_{\pi\pi} = \frac{d_{res}(-k_\pi, k_2, \ldots)}{d_{res}(k_\pi, k_2, \ldots)}, \qquad d_{res} = \prod_r \left[M_r^2 - s - i \sum_{j=1}^{N} \rho_{rj}^{2J+1} R_{rj} f_{rj}^2 \theta(s - s_j) \right],$$

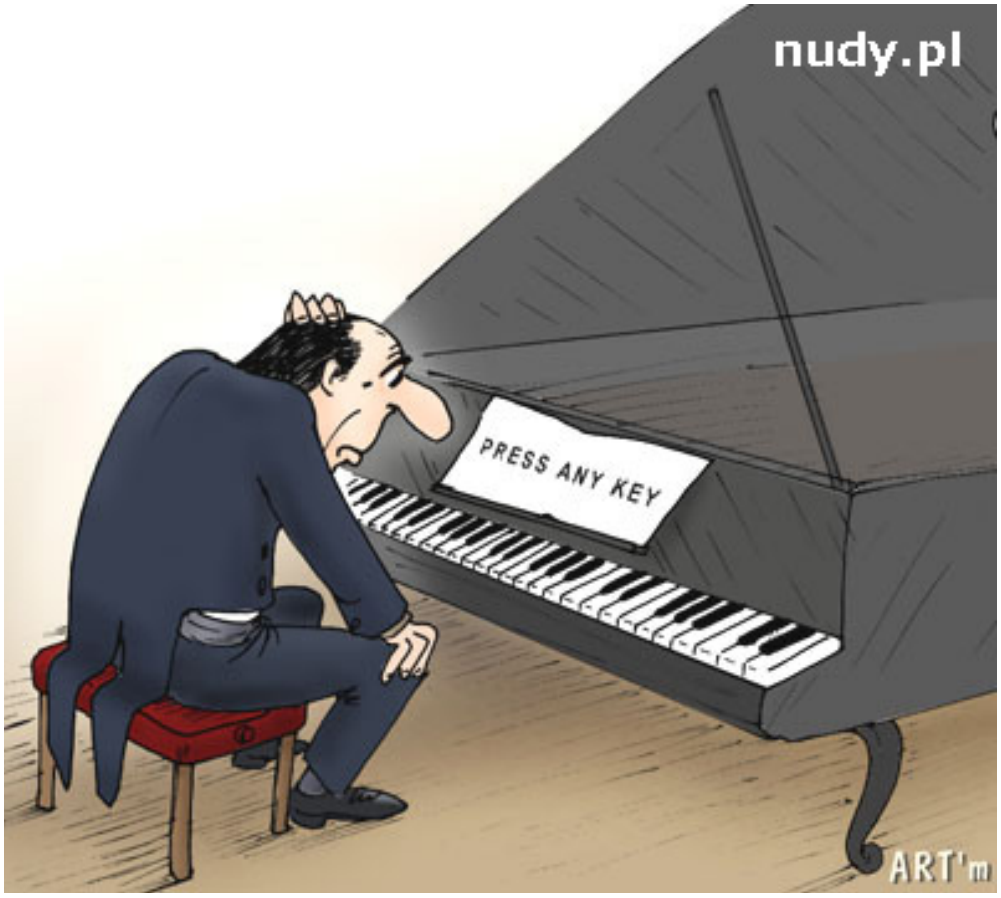
Totally $\sim$ 60 parameters

$$\chi^2_{Data}(k) = \sum_{i=1}^{N_\delta^k} \frac{(\delta_i^{exp} - \delta_i^{th})^2}{(\Delta\delta_i^{exp})^2} + \sum_{i=1}^{N_\eta^k} \frac{(\eta_i^{exp} - \eta_i^{th})^2}{(\Delta\eta_i^{exp})^2}$$

sigma pole without any additional control

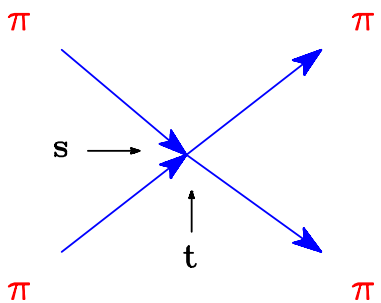


The pion-pion scattering amplitude (Roy eqs) ...



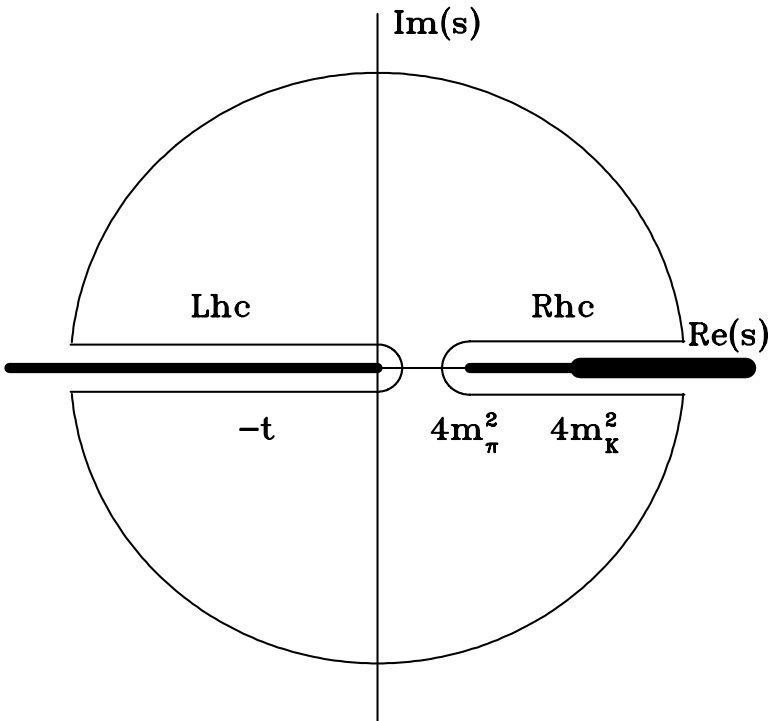
Dispersion relations with imposed crossing symmetry condition for $\pi\pi$ interactions theory $\longleftrightarrow$ experiment

crossing symmetry:



$\rightarrow \vec{T}_s(s,t) = \hat{C}_{st}\vec{T}_t(t,s)$

$\vec{T}(s,t)$ + crossing symmetry $\rightarrow$ dispersion relations for $4m_\pi^2 < s < \sim (1150\ \text{MeV})^2$

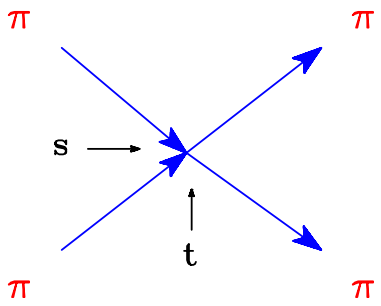


Once subtracted DR:

$$\text{Re}\,\vec{F}(s,t) = \text{Re}\,\vec{F}(s_0,t) + \frac{s-s_0}{\pi} \times \left[\int_{4m_\pi^2}^{\infty} ds' \frac{\text{Im}\,\vec{F}(s',t)}{(s'-s_0)(s'-s)} + \int_{-t}^{-\infty} ds' \frac{\text{Im}\,\vec{F}(s',t)}{(s'-s_0)(s'-s)} \right]$$

Dispersion relations with imposed crossing symmetry condition for $\pi\pi$ interactions theory $\longleftrightarrow$ experiment

crossing symmetry:



$\rightarrow \vec{T}_s(s,t) = \hat{C}_{st}\vec{T}_t(t,s)$

$\vec{T}(s,t)$ + crossing symmetry $\rightarrow$ dispersion relations for $4m_\pi^2 < s < \sim (1150\ \mathrm{MeV})^2$

Once subtracted dispersion relations ("GKPY" for the S and P waves):

$$\mathrm{Re}\ t_\ell^{I(OUT)}(s) = \sum_{I'=0}^{2} C_{st}^{II'} a_0^{I'} + \sum_{I'=0}^{2}\sum_{\ell'=0}^{4} \int_{4m_\pi^2}^{\infty} ds' K_{\ell\ell'}^{II'}(s,s')\,\mathrm{Im}\ t_{\ell'}^{I'(IN)}(s')$$

$a_0^{I'}$ - subtraction constant $= \vec{T}_s(s = 4m_\pi^2, t = 0)$ - scattering lengths from only S wave

due to $\mathrm{Re}\ t_\ell^I(k) = k^{2\ell}(a_\ell^I + b_\ell^I k^2 + O(k^4))$ $\qquad \mathrm{Re}\ t_\ell^{I(OUT)}(s) - \mathrm{Re}\ t_\ell^{I(IN)}(s) \rightarrow 0$

GKPY equations and $\pi\pi$ amplitudes

partial waves: *JI*

experiment

F1 **D2**

S0 **D0**

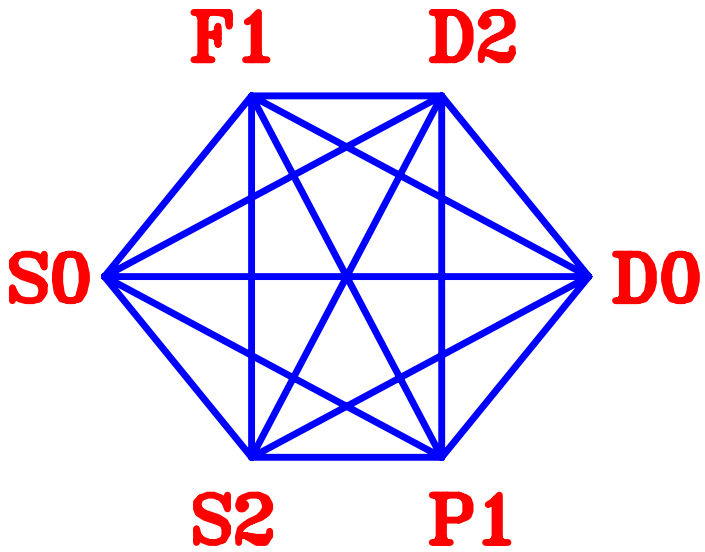
S2 **P1**

GKPY equations and $\pi\pi$ amplitudes

partial waves: *JI*

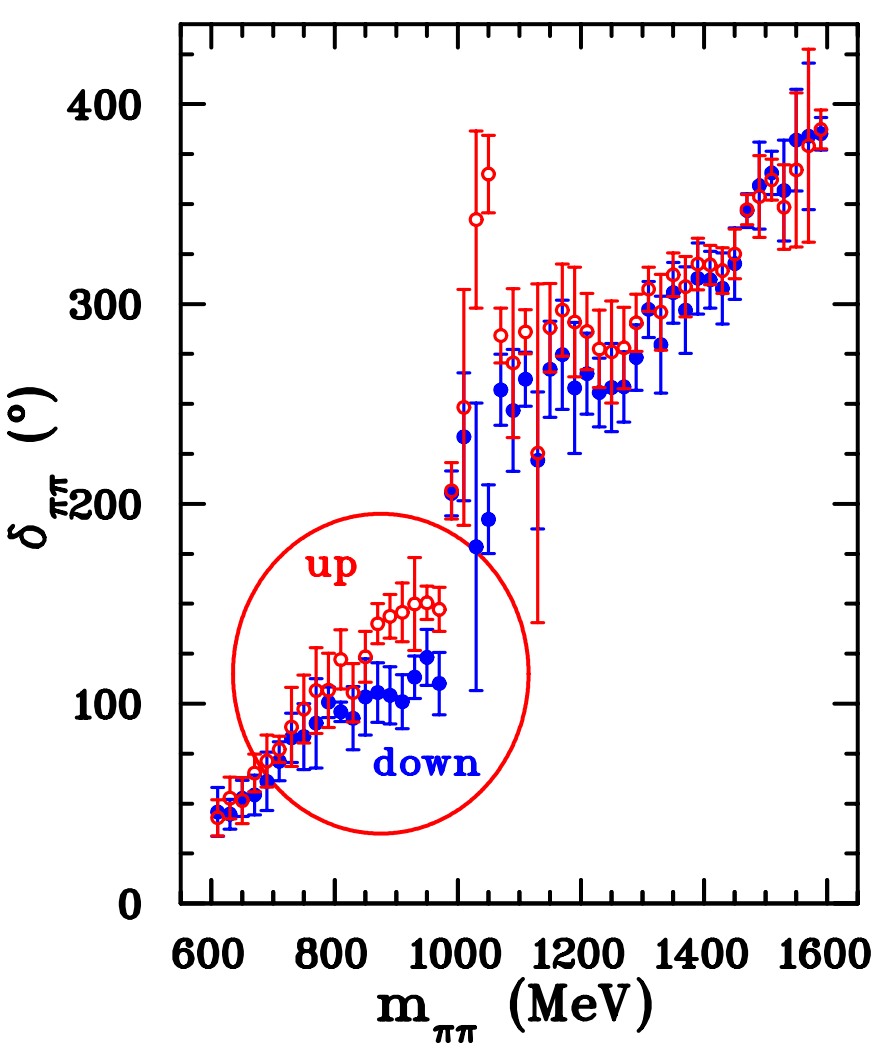
experiment + theory (GKPY)

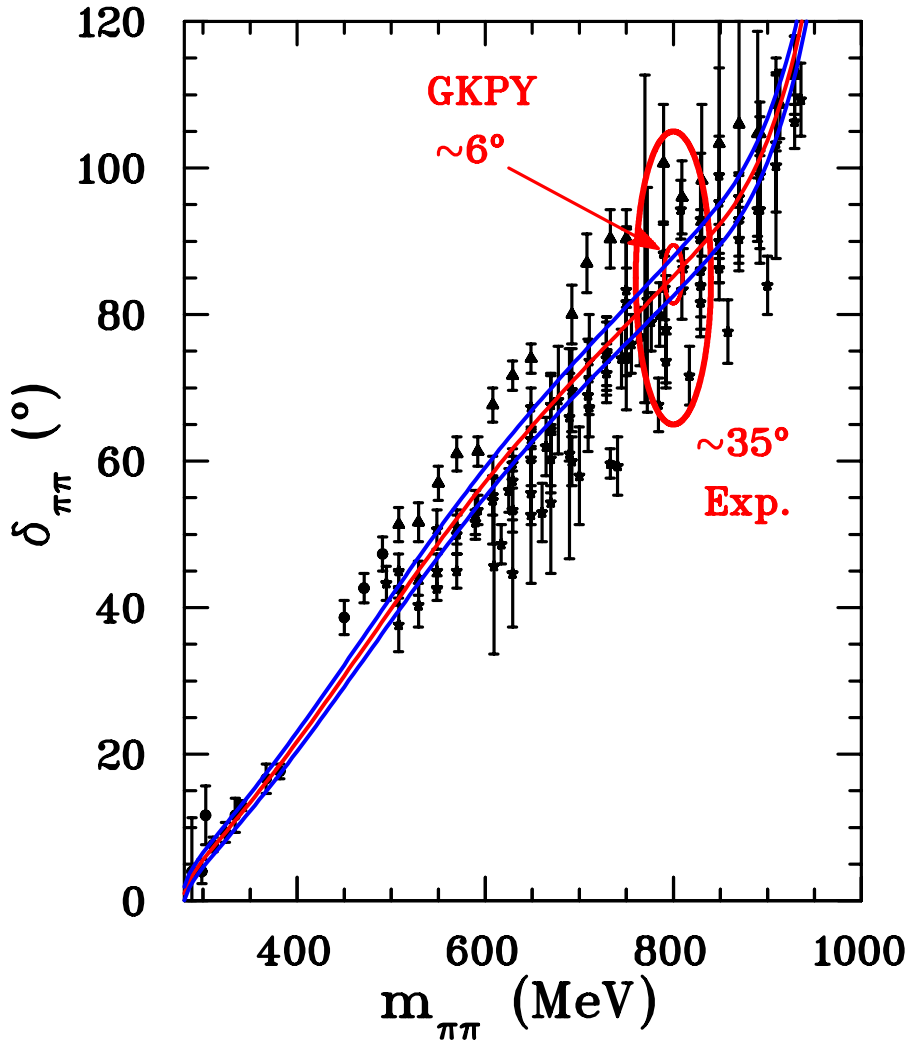
$$\int_{4m_\pi^2}^{\infty}$$



Experimental data for the $\pi\pi$ in the S0 wave (JI)

PWA (CERN-Munich group'74)





$\pi\pi$ amplitudes - 6 partial waves: $S0$, $S2$, $P1$, $D0$, $D2$, $F1$

Spin-isospin configuration: $J + I$ always even (Bose symmetry)

Used for GKPY equations: Madrid-Kraków group for the $S0$ wave (PRD,PRL'2011):

$$\cot\delta_0^{(0)}(s) = \frac{s^{1/2}}{2k}\frac{M_\pi^2}{s-\frac{1}{2}z_0^2}\times\left\{\frac{z_0^2}{M_\pi\sqrt{s}} + B_0 + B_1 w(s) + B_2 w(s)^2 + B_3 w(s)^3\right\}.$$

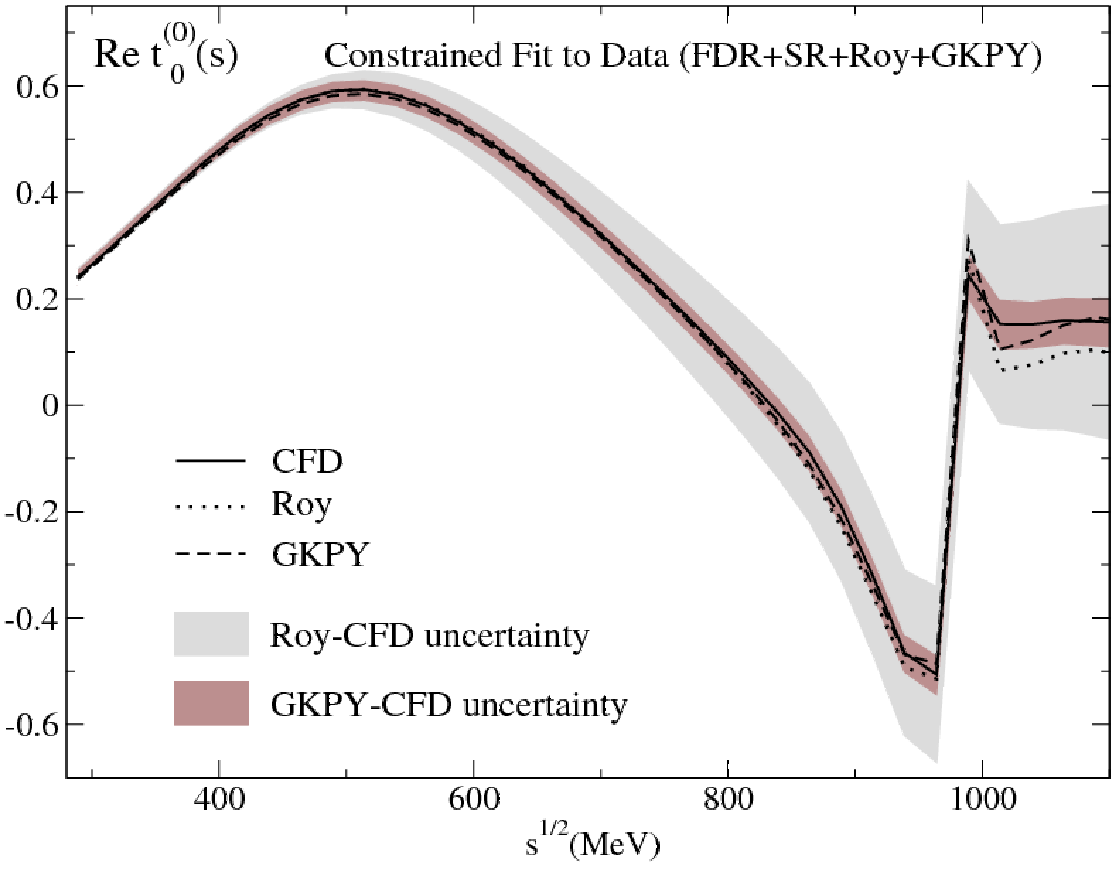
$$w(s) = \frac{\sqrt{s}-\sqrt{s_0-s}}{\sqrt{s}+\sqrt{s_0-s}}, \qquad s_0 = 4M_K^2 \sim (992\ \text{MeV})^2.$$

$$\delta_0^{(0)}(s) = \begin{cases} d_0\left(1-\frac{|k_2|}{k_M}\right)^2 + \delta_M\frac{|k_2|}{k_M}\left(2-\frac{|k_2|}{k_M}\right) + |k_2|(k_M-|k_2|)\left(8\delta'_M + c\frac{(k_M-|k_2|)}{M_K^3}\right), \\ d_0 + B\frac{k_2^2}{M_K^2} + C\frac{k_2^4}{M_K^4} + D\,\theta(s-4M_\eta^2)\frac{k_3^2}{M_\eta^2}, \end{cases}$$

Below and above 850 MeV (up to 1420 MeV - above - Regge)

Totally for all 6 waves: 52 free parameters

precision of the Roy and GKPY equations



"Precise determination of the f0(500) and f0(980) pole parameters from a dispersive data analysis",
R. Garcia-Martin, R. Kamiński, J.R. Pelaez, J. Ruiz de Elvira,
Phys. Rev. Lett. 107 (2011) 072001

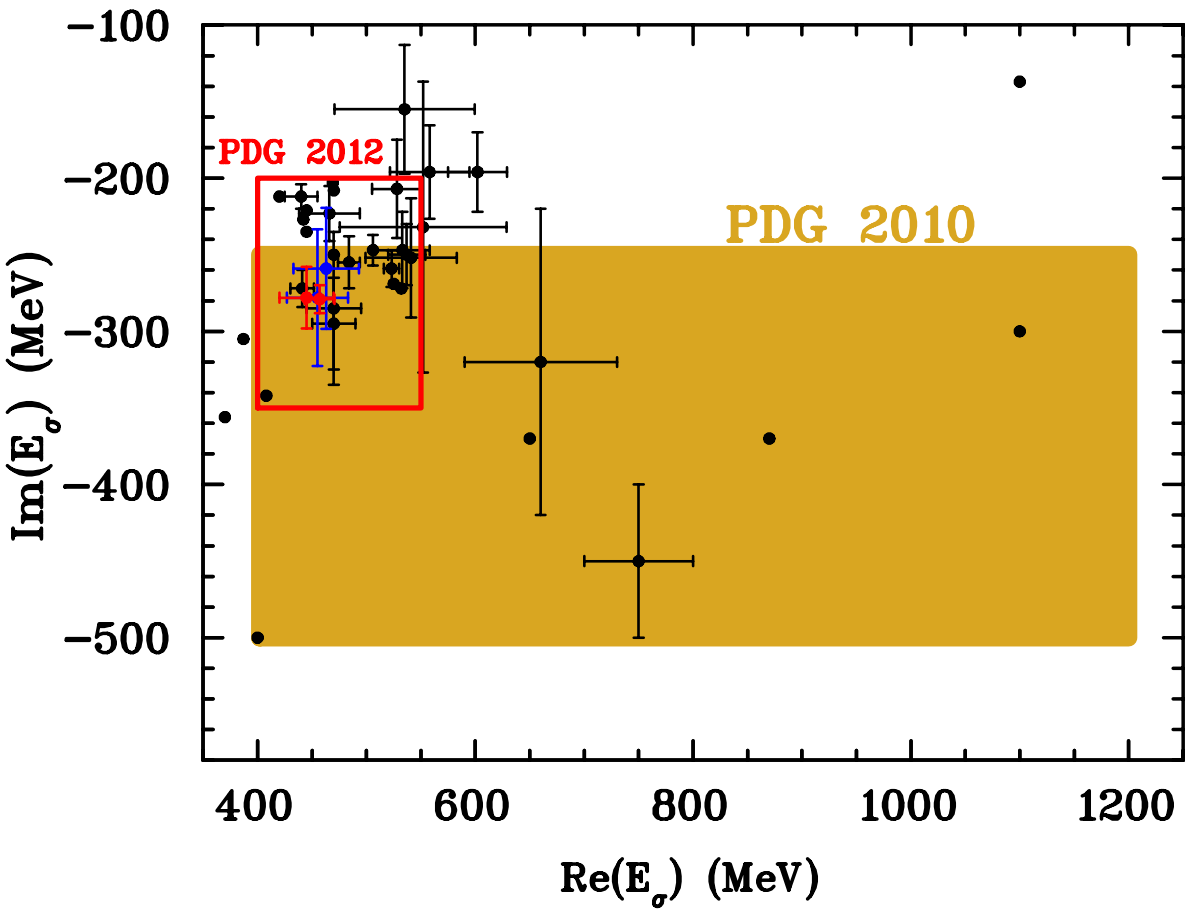
precise determination of $f_0(500)$ (σ) meson and threshold parameters

$f_0(500)$ (σ)

- PDG 2010:
  M = 400 − 1200 MeV
  $\Gamma = 2 \times (250 - 500)$ MeV
- PDG 2012:
  M = 400 − 550 MeV
  $\Gamma = 2 \times (200 - 350)$ MeV
- GKPY:
  $E_\sigma = 457 \pm 14 - i279^{+11}_{-7}$ MeV

threshold parameters, e.g. a_0^0:

- ChPT + Roy eqs (Bern group):
  $0.220 \pm 0.005\, m_\pi^{-1}$
- GKPY:
  $0.220 \pm 0.008\, m_\pi^{-1}$



Another group - "Bern" group:

H. Leutwyller, J. Gasser, G. Colangelo, I. Caprini: 2001-2011

FAQ: ... for sure your solution is not unique

The Role of the input in Roy's equations for pi pi scattering" G. Wanders, Eur. Phys. J. C17 (2000) 323-336

In the abstract:

An updated survey of known results on the dimension of the manifold of solutions is presented. The solution is unique for a low energy interval with upper end at 800 MeV. We determine its response to small variations of the input: S-wave scattering lengths and absorptive parts above 800 MeV.

I.e.:
Fixed two boundary conditions for the $\pi\pi$ amplitude:

- at the threshold ($S0$ wave scattering length) and
- at 800 MeV

$\pi\pi$ amplitudes - 6 partial waves: $S0, S2, P1, D0, D2, F1$

Used in Dubna-Kraków-Prague group (PRD'2014, 2016):

$$S_{\pi\pi} = \frac{d_{res}(-k_\pi, k_2, ...)}{d_{res}(k_\pi, k_2, ...)}, \qquad d_{res} = \prod_r \left[M_r^2 - s - i \sum_{j=1}^{N} \rho_{rj}^{2J+1} R_{rj} f_{rj}^2 \theta(s - s_j) \right],$$

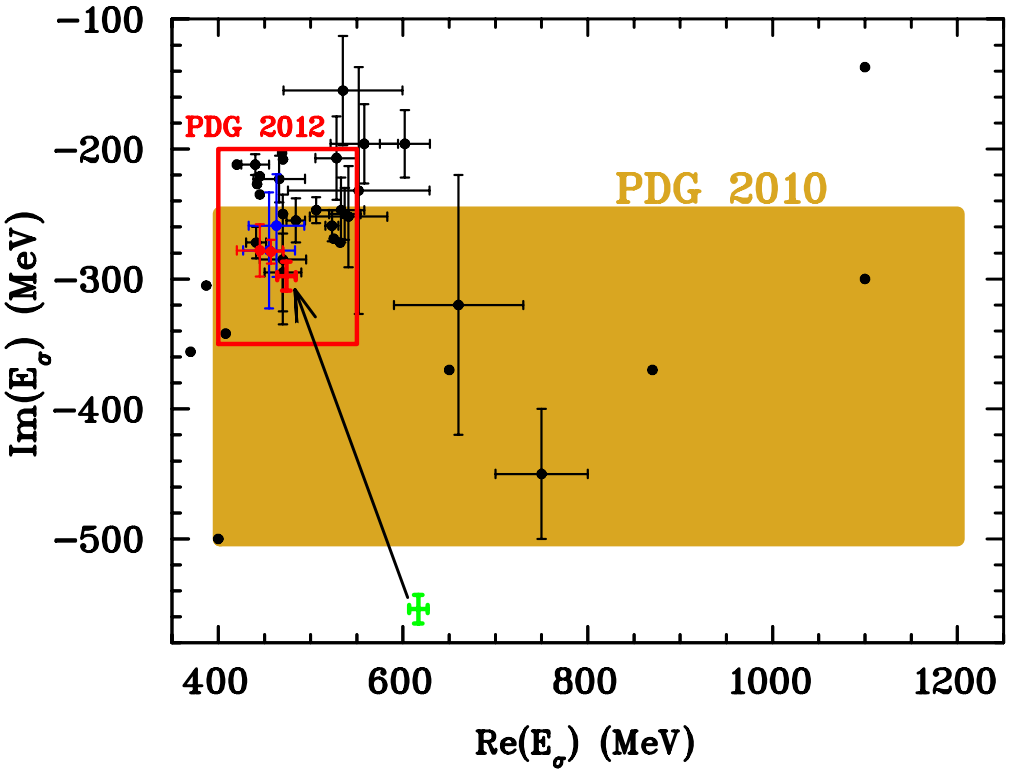
Totally $\sim$ 60 parameters

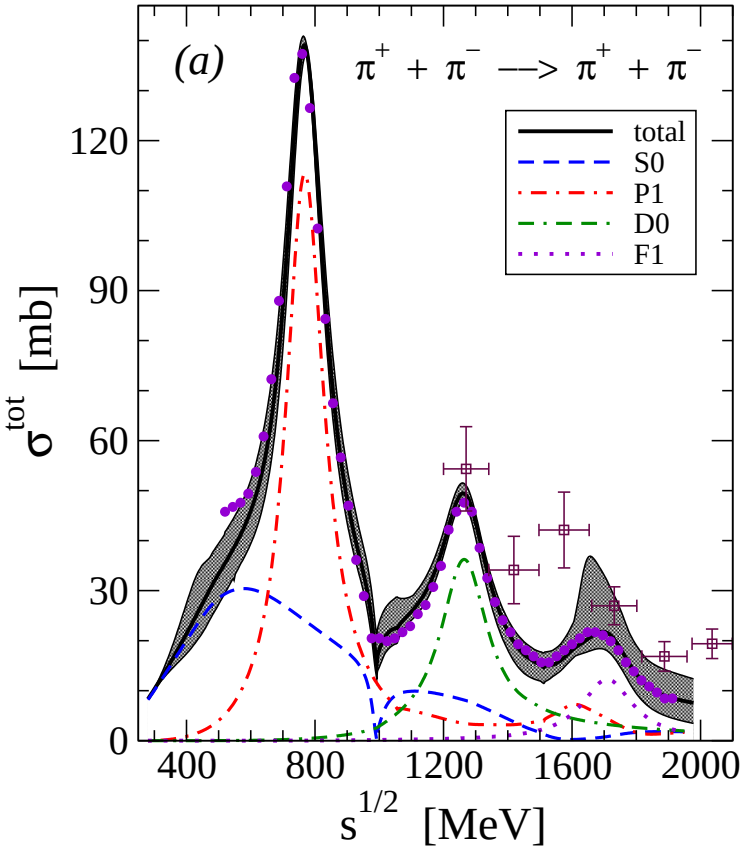
$$\chi^2_{Data}(k) = \sum_{i=1}^{N_\delta^k} \frac{(\delta_i^{exp} - \delta_i^{th})^2}{(\Delta\delta_i^{exp})^2} + \sum_{i=1}^{N_\eta^k} \frac{(\eta_i^{exp} - \eta_i^{th})^2}{(\Delta\eta_i^{exp})^2}$$

and

$$\chi^2_{DR}(k) = \sum_{i=1}^{N_{DR}} \frac{\left[\mathrm{Re\,Re}\, t_\ell^{I(IN)}(s_i) \right]^2}{\left[\Delta \mathrm{Re}\, t_\ell^{I(OUT)}(s_i) \right]^2},$$

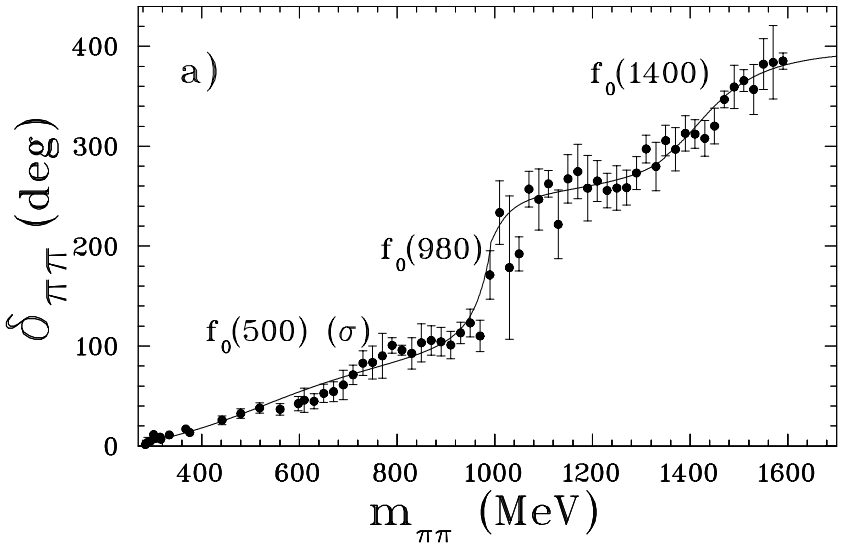
σ pole controlled by the ππ crossing symmetry, total cross section - not fitted to the data!

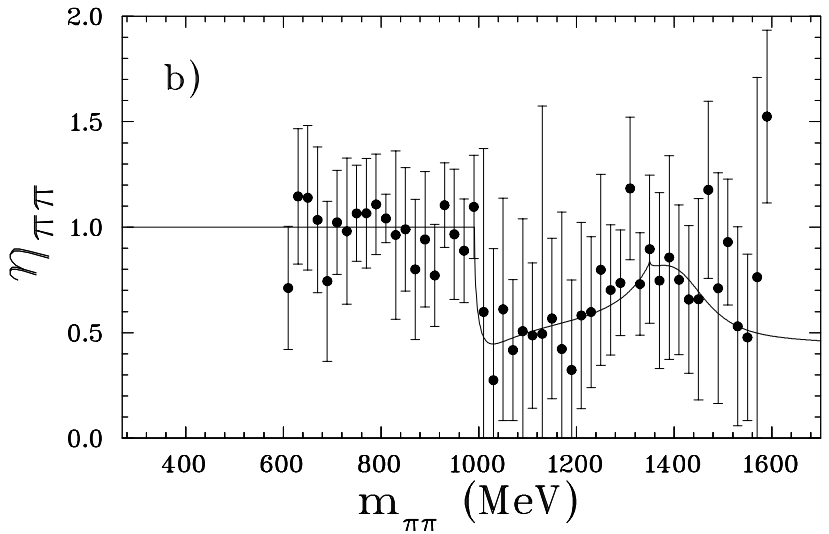




All 0^+0^{++} *resonances*

- $f_0(500)$ (σ) - now well known and for sure not pure $q\bar{q}$ state, most likely mix with $2q2\bar{q}$ (or loosely bound $\pi\pi$) state but why not gg?
- $f_0(980)$ (mass: 990 ± 20 MeV, width: 40-100 MeV) - $K\bar{K}$ threshold: 992 MeV, so possible components: $K\bar{K}$ bound state ($Br_{KK} \approx 25\%$!), $q\bar{q}$, $2q2\bar{q}$, $2q2g$, let's look at new papers by Francesco Giacosa on a_0 states,
- $f_0(1370)$ (mass: 1200-1500 MeV, width: 200-500 MeV) - most probably ordinary $q\bar{q}$ state decaying mostly into $\pi\pi$ and 4π,
- $f_0(1500)$ (mass: 1505 ± 6 MeV, width: 109 ± 7 MeV)- most probably ordinary $q\bar{q}$ state decaying into $\pi\pi$ (35%), 4π (50%), $K\bar{K}$ (9%) and $\eta\eta$ (5%),
- $f_0(1720)$ - possible glueball - lattice calculations and series of works by FG





What really σ can be?

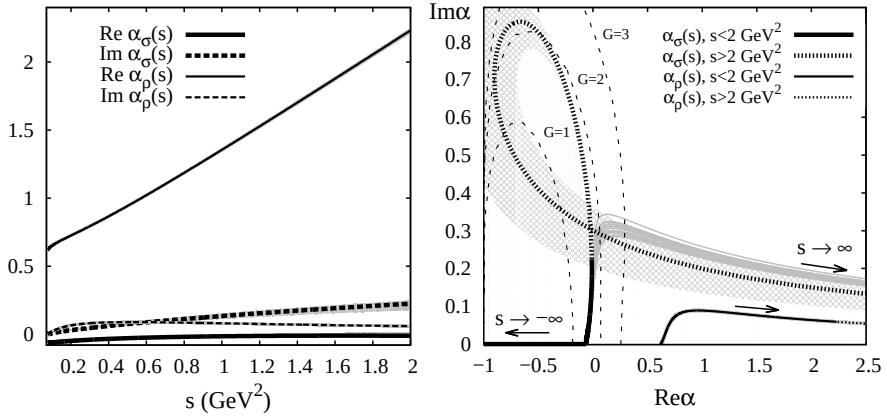


Fig. 1. (Left) $\alpha_\rho(s)$ and $\alpha_\sigma(s)$ Regge trajectories, from our constrained Regge-pole amplitudes. (Right) $\alpha_\sigma(s)$ and $\alpha_\rho(s)$ in the complex plane. At low and intermediate energies (thick continuous lines), the trajectory of the σ is similar to those of Yukawa potentials $V(r) = -Ga\exp(-r/a)/r$ [8] (thin dashed lines). Beyond 2 Gev^2 we plot our results as thick discontinuous lines because they should be considered just as extrapolations.

Furthermore, in Fig. 1 we show the striking similarities between the $f_0(500)$ trajectory and those of Yukawa potentials in non-relativistic scattering [8]. From the Yukawa G=2 curve in that plot, which lies closest to our result for the $f_0(500)$, we can estimate $a \simeq 0.5\,\mathrm{GeV}^{-1}$, following [8]. This could be compared, for instance, to the S-wave $\pi\pi$ scattering length $\simeq 1.6\,\mathrm{GeV}^{-1}$. Thus it seems that the range of a Yukawa potential that would mimic our low energy results is comparable but smaller than the $\pi\pi$ scattering length in the scalar isoscalar channel. Of course, our results are most reliable at low energies (thick continuous line) and the extrapolation should be interpreted cautiously. Nevertheless, our results suggest that the $f_0(500)$ looks more like a low-energy resonance of a short range potential, *e.g.* between pions, than a bound state of a confining force between a quark and an antiquark.

In summary, our formalism and the results for the $f_0(500)$ explains why the lightest scalar meson has to be excluded from the ordinary linear Regge fits of ordinary mesos.

"Identification of non-ordinary mesons from the dispersive connection between their poles and their Regge trajectories: The f0(500) resonance"

by

J. R. Pelaez, J. T. Londergan, J . Nebreda and A. Szczepaniak

Phys.Lett. B729 (2014)

scalar - isovector $0^{++}1^{-}$ *section -* a_0 *mesons*

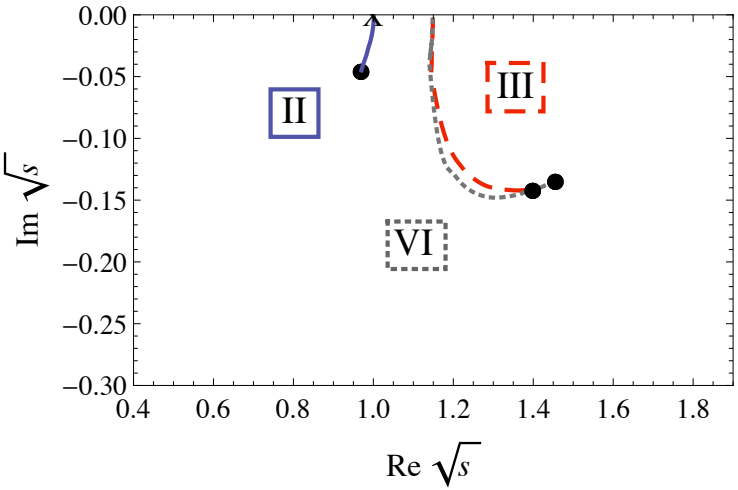
Works by Francesco Giacosa (now Kielce University)

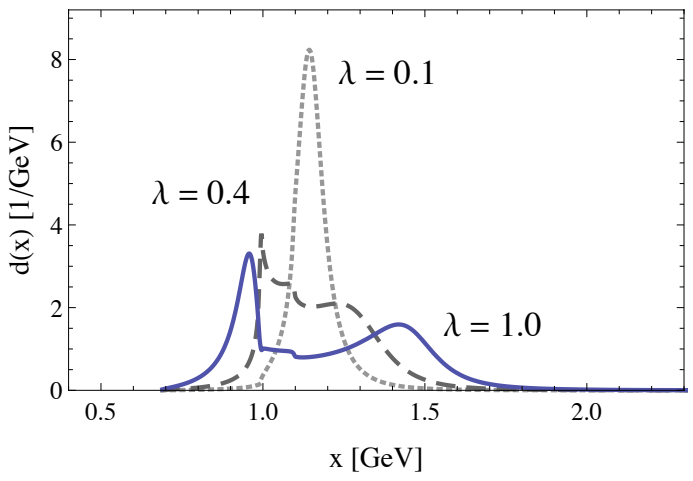
"*a*0(980) *revisited*, "Phys. Rev. D93 (2016), 014002
and proceedings:
"a_0(980) *as a companion pole of a0(1450)*"

issue:
coupling of the heavy scalar-isovector **seed $q\bar{q}$ state $a_0(1450)$** to $\pi\eta$, $K\bar{K}$ and $\pi\eta'$ is capable to **dynamically generate** the light state $a_0(980)$.

$$\Delta^{-1}(s) = s - m_0 - \Pi(s),$$

where m_0: "bare mass" of the $a_0(1450)$ and $\Pi(s) = \sum_i \Pi_i(s)$ - "self energy"





Spectral function

Theoretical and experimental constraints imposed on amplitudes

- threshold parameters: scattering length (ChPT) - good but not necessary,
- crossing symmetry (via dispersion relations): till now done for the $\pi\pi \to \pi\pi$ channel (Roy and GKPY eqs) and $\pi K \to \pi K$ channels, will be for the $\pi\pi \to KK$ channel,
- good data (fulfilling crossing symmetry and sum rules),
- pion scalar radius,
- $\pi\eta$ channel: one can create correct theoretical amplitudes without data in the scalar-isovector $\pi\eta$ channel using only: threshold parameter, scalar radius, (M. Albaladejo and B. Moussallam, *Form factors of the isovector scalar current and the $\eta\pi$ scattering phase shifts"*, Eur. Phys. J. C (2015)) and branching ratios to given channels ($\pi\eta$, $K\bar{K}$ and $\pi\eta'$).

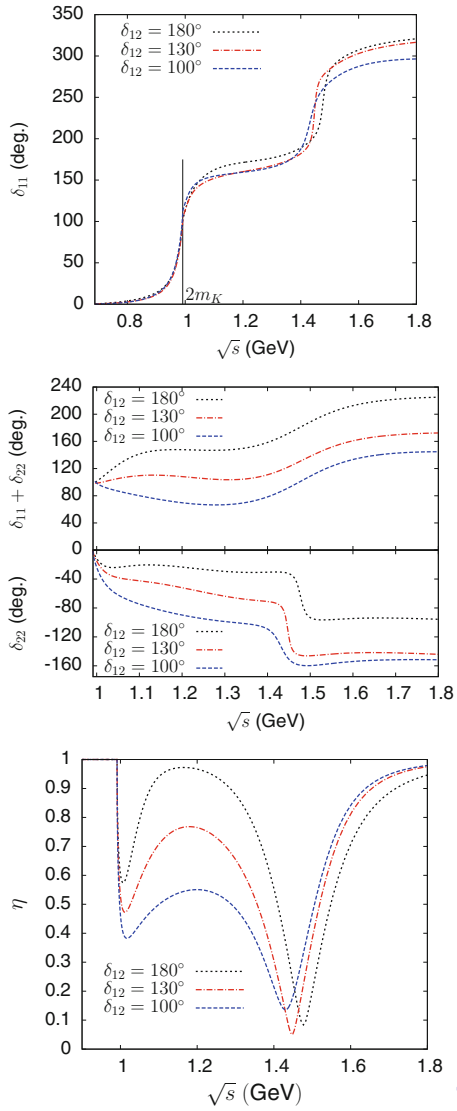
scalar-isovector $\pi\eta$ theoretical amplitude

M. Albaladejo and B. Moussallam Eur. Phys. J. C (2015)

This T-matrix model involves the phenomenological parameters: α_1, α_2, β_1, β_2, m_8, c'_d, c'_m. For simplicity, we will keep the ratio c'_m/c'_d fixed and allow only six parameters to vary. We determine them by imposing six conditions on the T-matrix:

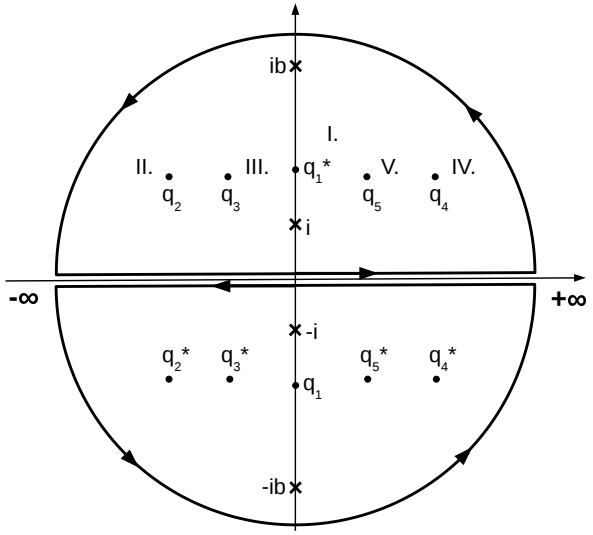
(a) As the first four conditions, we impose the requirement that the real and imaginary parts of the poles $s^{\rm II}_{a_0(980)}$ and $s^{\rm III}_{a_0(1450)}$ be reproduced.
(b) As a fifth condition, we impose the requirement that the minimum of the inelasticity parameter at $\sqrt{s} = m_{a_0(1450)}$ be close to zero (in practice, we used $\eta_{min} \approx 0.05$, as in Ref. [60]).
(c) As a final condition, we choose a value for the phase δ_{12} as defined in Eq. (47).

Within this model, having imposed the first five conditions, the value of δ_{12} is found to be bounded from above: $\delta_{12} \lesssim 205°$. In addition, consistently with our assumption that most of the phase variations should take place below 2 GeV, it seems plausible that the phase sum $\delta_{11} + \delta_{22}$ should not be smaller than its value at the mass of the $a_0(980)$, i.e., one should have $\delta_{12} \gtrsim 90°$. Figure 4 shows results from this model for the phases δ_{11}, δ_{22} and the inelasticity η as a function of energy, corresponding to several different imposed values of δ_{12}. One observes that the two phases δ_{11}, δ_{22} undergo a sharp variation, in opposite directions, close to the mass of the $a_0(1450)$ resonance. The figure illustrates a pattern where δ_{11} increases while δ_{22} decreases. However, a small modification of the phenomenological parameters which enter into the T-matrix model can lead to a pattern with a reversed behaviour (with δ_{11} decreasing and δ_{22} increasing), which would then be similar to the one obtained in Ref. [60]. In contrast, the phase sum, $\delta_{11} + \delta_{22}$, is completely stable



pion scalar radius (RK et al., PRD'2016)

$$< r^2 >_S^{\pi} = \frac{6}{\pi} \int_{4m_\pi^2}^{\infty} \frac{\delta_\Gamma(s)}{s^2} ds$$

$$\Gamma_\pi(s) = \frac{(q-q_1^*)(q-q_6^*)(q-q_7^*)}{(q-q_2)(q-q_3^*)(q-q_4^*)(q-q_5^*)} \times \frac{(i-q_2^*)(i-q_3^*)(i-q_4^*)(i-q_5^*)}{(i-q_1^*)(i-q_6^*)(i-q_7^*)}$$

$$< r^2 >_S^{\pi} = \frac{3}{4}\left[\frac{i}{i-q_2^*} + \frac{i}{i-q_3^*} + \frac{i}{i-q_4^*} + \frac{i}{i-q_5^*} - \frac{i}{i-q_1^*} - \frac{i}{i-q_6^*} - \frac{i}{i-q_7^*}\right]$$