Gauge theory at strong coupling, isomonodromy problem and non-perturbative string

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Exact results in Quantum Field Theories

Going beyond perturbation theory is an important challenge in Quantum Field Theory and String Theory.

N=2 supersymmetric gauge theories in four dimensions are useful to study:

- exact non-perturbative effects (instantons)
- strong-weak coupling dualities
- confinement through breaking to N=1

systematic study of the above lead to discover rich algebraic and integrable structures, which had deep impact also in mathematics.

Exact results in Quantum Field Theories

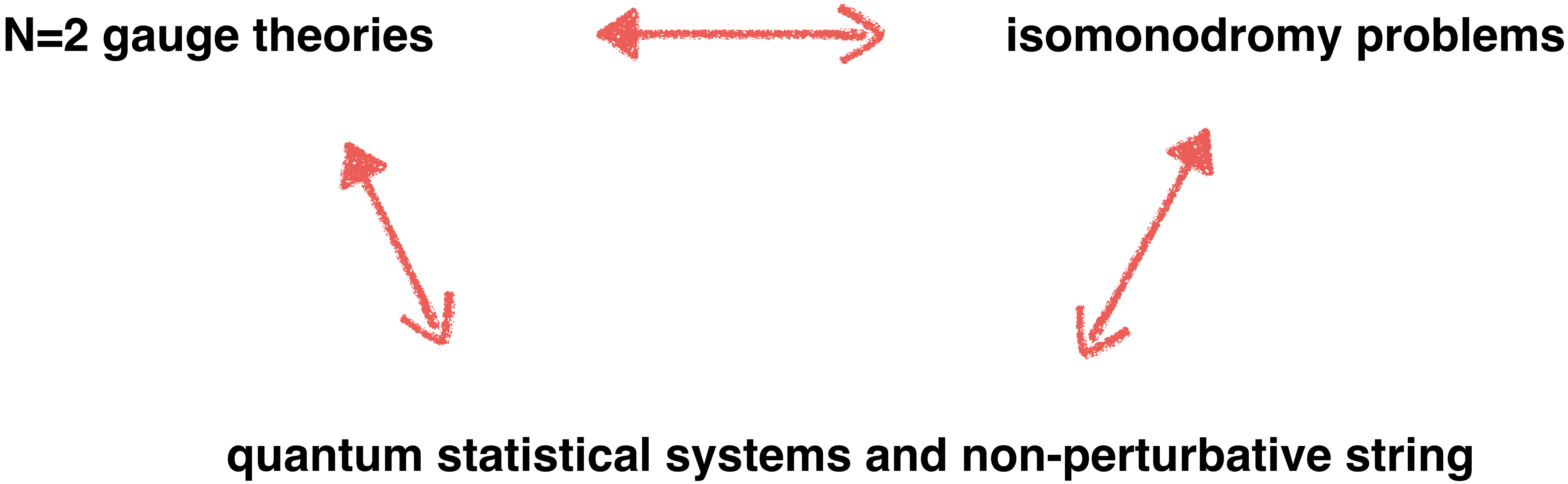
In the *weak coupling* phase the supersymmetric path integral can be reduced via *equivariant localization* to combinatorial objects - Nekrasov function, matrix models - paving the way to the link with rich algebraic structures as CFT - type algebrae (Virasoro, W - algebrae, Kac-Moody) and *quantum integrable systems.*

In the *strong coupling* phase the embedding in superstring theories provides a crucial tool to explore S-duality and to calculate protected gauge theory quantities - e.g. *holomorphic anomaly equations, topological recursion.*

Exact results in Quantum Field Theories

In this talk we will show how to compute a class of gauge theory amplitudes in strongly coupled phases by reducing them to the solutions of non-linear ODEs . These are associated to a set of *isomonodromy problems* and display a deep link with the non-perturbative completion of topological string proposed by Grassi, Hatsuda and Marino.

Today's talk: gauge theory partition functions and BPS correlators
as τ - functions of isomonodromy problems



N=2 gauge theories

isomonodromy problems

quantum statistical systems and non-perturbative string

new tools to explore strongly coupled phases/
sectors of gauge theory

based on:

G. Bonelli, O. Lisovyy, K. Maruyoshi, A. Sciarappa, A.T. 1612.06235

G. Bonelli, A. Grassi, A.T. arXiv:1603.01174, 1704.01517 & to appear

N=2 supersymmetric gauge theories in 4d

class S: M-theory compactification from 6d naturally hints a relation with 2d theory

description in terms of *Hitchin's algebraic integrable system*

Renormalization group equations of SU(2) asymptotically free gauge theories are systematically described by ***Painleve' equations***

Spectral determinant presentation of Painleve' tau-function hints to a ***new relation between gauge theory and quantum statistical systems*** *via geometric engineering limit of* ***non-perturbative string***

Generalisation to higher rank: ***new class of Matrix models*** *describing the* ***magnetic phase*** *of pure* ***SU(N) Super Yang-Mills*** *theories.*

Seiberg-Witten curve from M-theory

[Klemm-Lerche-Mayr-Vafa-Warner,
Witten, Gaiotto]

Consider M-theory on $\mathbb{R}^7 \times Q$

holomorphic symplectic two-fold

decoupling gravity ⟶ local geometry $\mathbb{R}^4 \times T^*\mathcal{C} \times \mathbb{R}^3$

r M5 branes on $\mathbb{R}^4 \times \mathcal{C}$ in the **Coulomb branch**

$$y^r + \sum_{k=2}^{r} \phi_k(z) y^{r-k} = 0$$

holomorphic k-differential

S^1 compactification gives rise to U(r) Super Yang-Mills theory in 5d on $\mathbb{R}^3 \times \mathcal{C}$. BPS vacua invariant under Super-Poincare' of $\mathbb{R}^3$ satisfy **Hitchin's equations**

$$
\begin{aligned}
F + R^2[\varphi, \bar{\varphi}] &= 0, \\
\partial_{\bar{z}}\varphi + [A_{\bar{z}}, \varphi] &= 0, \\
\partial_z\bar{\varphi} + [A_z, \bar{\varphi}] &= 0,
\end{aligned}
$$

these are equivalent to the flatness of the $SL(r, \mathbb{C})$ connection

$$\mathcal{A} = \frac{R}{\zeta}\varphi + A + R\zeta\overline{\varphi};$$

Seiberg-Witten curve and Hitchin's system

$\mathcal{M}$ is HyperKahler ⟶ twistor parameter $\zeta \in \mathbb{CP}^1$

$$\mathcal{A} = \frac{R}{\zeta}\varphi + A + R\zeta\overline{\varphi};$$

for $\zeta \to 0$ base space $\mathcal{U}$ described by

$$\phi_k(z) = \mathrm{tr}\ \varphi^k$$

SW curve is the **spectral curve**

$$\det(y - \varphi) = 0$$

Seiberg-Witten curve and Hitchin’s system

Hitchin’s complex algebraic integrable system:

$$\begin{tikzcd}\mathcal{M}_{\mathbf{R}^3\times\mathbf{S}^1} \arrow[d, swap, "\text{Lagrangian fibration}"] \\ \mathcal{M}_{\mathbf{R}^4}\equiv\mathcal{U}\end{tikzcd}$$

phase space (dim=2r)
(HyperKahler)

base (dim = r)
(Special Kahler)

Coulomb branch of 4d gauge theory

Polar structure of the k-differentials

M5 brane on $\mathbb{R}^4 \times \text{fiber}$ $\longrightarrow$ **simple pole** for the k-diff.

matter sector

residue: mass

Collision of poles with suitable rescalings of the residues

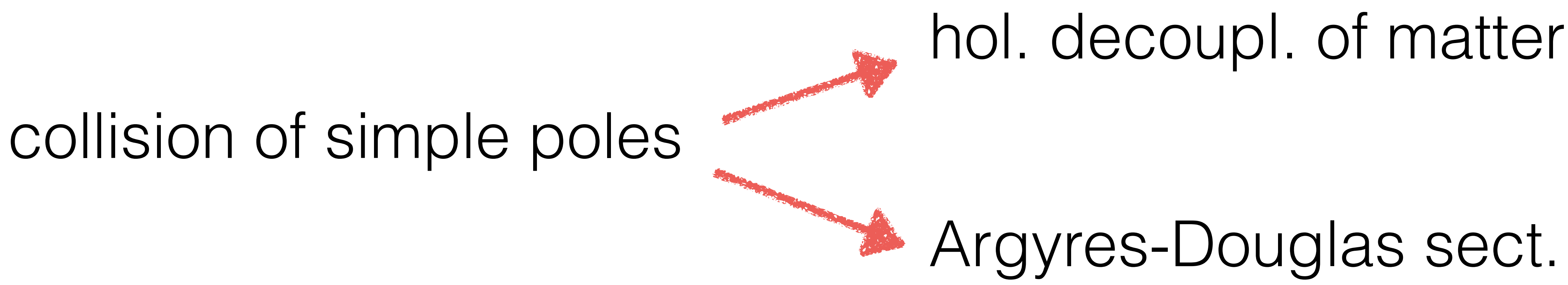
changes the matter sector

for $SU(2)$ gauge theory

quadratic differentials on a Riemann sphere with at most

four simple poles:

$N_f = 4$ *superconformal gauge theory*



collision of simple poles

hol. decoupl. of matter

Argyres-Douglas sect.

holomorphic decoupling: Lagrangian theories

$$
\begin{aligned}
N_f &= 0: \ \frac{\Lambda^2}{z^3} + \frac{2u}{z^2} + \frac{\Lambda^2}{z}, \\
N_f &= 1: \ \frac{\Lambda^2}{z^3} + \frac{3u}{z^2} + \frac{2\Lambda m}{z} + \Lambda^2, \\
N_f &= 2: \ \frac{\Lambda^2}{z^4} + \frac{2\Lambda m_1}{z^3} + \frac{4u}{z^2} + \frac{2\Lambda m_2}{z} + \Lambda^2, \qquad \text{(first realization)} \\
N_f &= 2: \ \frac{m_+^2}{z^2} + \frac{m_-^2}{(z-1)^2} + \frac{\Lambda^2 + u}{2z} + \frac{\Lambda^2 - u}{2(z-1)}, \qquad \text{(second realization)} \\
N_f &= 3: \ \frac{m_+^2}{z^2} + \frac{m_-^2}{(z-1)^2} + \frac{2\Lambda m + u}{2z} + \frac{2\Lambda m - u}{2(z-1)} + \Lambda^2.
\end{aligned}
$$

Argyres-Douglas sectors:

$$
\begin{aligned}
H_0 &: \; z^3 - 3cz + u \\
H_1 &: \; z^4 + 4cz^2 + 2mz + u, \quad \text{(first realization)} \\
H_1 &: \; z + c + \frac{u}{z} + \frac{m^2}{z^2}, \quad \text{(second realization)} \\
H_2 &: \; z^2 + 2cz + (2\widetilde{m} + c^2) + \frac{u + 2cm_-}{z} + \frac{m_-^2}{z^2}
\end{aligned}
$$

$$\begin{tikzcd}
 & & N_f = 2 \ (\mathrm{I}) \arrow[r] \arrow[dr] & N_f = 1 \arrow[r] \arrow[dr] & N_f = 0 \\
N_f = 4 \arrow[r] & N_f = 3 \arrow[ur] \arrow[r] \arrow[dr] & N_f = 2 \ (\mathrm{II}) \arrow[ur] \arrow[dr] & H_1 \ (\mathrm{I}) \arrow[r] & H_0 \\
 & & H_2 \arrow[ur] \arrow[r] & H_1 \ (\mathrm{II}) \arrow[ur] &
\end{tikzcd}$$

Painleve’ coalescence diagram

Classification problem for ODEs

Painleve’ property: ODE with only movable *poles*

$$\ddot{q} = F(q, \dot{q}; t),$$

F rational function of $q, \dot{q}$ analytic in t : **full classification**

$$\begin{tikzcd}
 & & PIII_1 \arrow[r] \arrow[dr] & PIII_2 \arrow[r] \arrow[dr] & PIII_3 \\
PVI \arrow[r] & PV \arrow[r] \arrow[ur] \arrow[dr] & PV_{deg} \arrow[ur] \arrow[dr] & PII_{JM} \arrow[r] & P_I \\
 & & PIV \arrow[r] \arrow[ur] & PII_{FN} \arrow[ur] &
\end{tikzcd}$$

Painleve' and isomonodromy deformations

Linear system: $$(\kappa\,\partial_z - \mathbf{A}(z))\Psi(z) = 0$$

$$\mathbf{A}(z) = \sum_{\nu=1}^{n} \frac{A^{(\nu)}(z)}{(z - z_\nu)^{r_\nu+1}}\,, \qquad A^{(\nu)}(z) = \sum_{i=0}^{r_\nu} A_i^{(\nu)}(z - z_\nu)^{r_\nu - i}$$

$\mathbf{A}(z) \in sl(2,\mathbb{C})$, z affine coordinate on punctured sphere

isomonodromy: family $\mathbf{A}(z;\{\vec{t}\})$ of flat $SL(2,\mathbb{C})$ conn.

four punctures → one parameter t

$$\begin{cases} \frac{d}{dz}\Psi(z,t) = \mathbf{A}(z,t)\Psi(z,t) \\ \frac{d}{dt}\Psi(z,t) = \mathbf{B}(z,t)\Psi(z,t) \end{cases}$$

Compatibility condition $\Psi_{zt}(z,t) = \Psi_{tz}(z,t)$:

$$\mathbf{A}_t(z,t) = \mathbf{B}_z(z,t) + [\mathbf{B}(z,t), \mathbf{A}(z,t)]$$

namely variation of flat conn. is infinitesimal gauge transf.

yields **Painleve' equations.** $\mathbf{A}, \mathbf{B}$: ***Lax pair***

$\kappa \to 0$ **isospectral deformations,** leave invariant

$$\det(y - \mathbf{A}) = 0\,, \qquad \Sigma \in T^*\mathcal{C}_{0,n}$$

Seiberg-Witten curve ! Upon ident. $\mathbf{A} = \varphi$

example: **Painleve' I vs H_0 Argyres-Douglas**

$$A = A_0 + zA_1 + z^2 A_2 = \begin{pmatrix} -p & q^2 + zq + z^2 + t/2 \\ 4z - 4q & p \end{pmatrix}$$

$$B = B_0 + zB_1 = \begin{pmatrix} 0 & q + z/2 \\ 2 & 0 \end{pmatrix}$$

compatibility

$$\begin{cases} \dot{q} = p \\ \dot{p} = 6q^2 + t \end{cases} \longrightarrow \qquad \ddot{q} = 6q^2 + t$$

quad. different.

$$\frac{1}{2}\mathrm{Tr}\, A^2 = 4z^3 + 2tz + 2\sigma_I(t) \qquad \sigma_I(t) = \frac{1}{2}p^2 - 2q^3 - qt$$

coincide with SW curve upon ident. $t \sim c\,, \quad \sigma_I \sim u$

source of relevant deformation

vev of relevant deformation

Painlevé isomonodromy problem	$\mathcal{N}=2$ **theory Hitchin system**
punctured Riemann sphere $\mathcal{C}_{0,n}[z]$	Gaiotto surface $\mathcal{C}_{0,n}[z]$
connection $\kappa\,\partial_z - \mathbf{A}(z)$	holomorphic $\zeta D_z + R\varphi_z(z)$ (gauge $A_z = 0$)
isomonodromic deformations	Whitham deformations
compatibility condition	gauge transformation
overall scale κ	parameter ζ of complex structure J of $\mathcal{M}$
isospectral limit $\kappa \to 0$ (Higgs bundle)	limit $\zeta \to 0$ to complex structure I (Higgs bundle)
Painlevé time t	gauge coupling Λ, c
Painlevé σ-function (Hamiltonian)	Coulomb branch parameter u
Painlevé free parameters	masses $\mathcal{N}=2$ theory

$$\begin{tikzcd}
 & & N_f = 2\ (\mathrm{I}) \arrow[r] \arrow[dr, red] & N_f = 1 \arrow[r] \arrow[dr, red] & N_f = 0 \\
N_f = 4 \arrow[r] & N_f = 3 \arrow[ur] \arrow[r] \arrow[dr, red] & N_f = 2\ (\mathrm{II}) \arrow[ur] \arrow[dr, red] & H_1\ (\mathrm{I}) \arrow[r] & H_0 \\
 & & H_2 \arrow[ur] \arrow[r] & H_1\ (\mathrm{II}) \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & PIII_1 \arrow[r] \arrow[dr] & PIII_2 \arrow[r] \arrow[dr] & PIII_3 \\
PVI \arrow[r] & PV \arrow[ur] \arrow[r] \arrow[dr] & PV_{deg} \arrow[ur] \arrow[dr] & PII_{JM} \arrow[r] & P_I \\
 & & PIV \arrow[ur] \arrow[r] & PII_{FN} \arrow[ur] &
\end{tikzcd}$$

VI	V	III_1	V_{deg}	III_2	III_3	IV	$\mathrm{II}_{\mathrm{JM}}$	$\mathrm{II}_{\mathrm{FN}}$	I
$N_f = 4$	$N_f = 3$	$N_f = 2\,(1st)$	$N_f = 2\,(2nd)$	$N_f = 1$	$N_f = 0$	H_2	$H_1\,(1st)$	$H_1\,(2nd)$	H_0

Painleve’ “transcendents" and gauge theory dual partition function

We can extend the correspondence to the full gauge theory in the so-called

self-dual Ω **- background**

calculation of the p.f.
via **localization**

relation to **string theory**

$$Z_{\mathrm{Nek}}(a,m;\Lambda,\epsilon)=\exp\Big[-\sum_{g}\epsilon^{2g-2}\mathcal{F}_g(a,m;\Lambda)\Big]$$

Nekrasov-Okounkov dual partition function

$$Z_{\mathrm{NO}}(a/\epsilon, m; \Lambda, \eta) = \sum_{n\in\mathbb{Z}} e^{4\pi i n\eta} Z_{\mathrm{Nek}}(a + n\epsilon, m; \Lambda)$$

Theorem: **this is the tau-function for Painleve' equations**
[Gamayun-Iorgov-Lisovyy, Bershtein-Shedchin, Iorgov-Lisovyy-Teschner]

$(a/\epsilon = \sigma, \eta)$ initial conditions, $\{m\}$ monodromy param.

RGE scale Λ : Painleve' time - **short time expansion corresponds to weakly coupled *electric frame***

$$\tau(t) \sim \sum e^{4\pi i n\eta} Z_{\mathrm{Nek}}(\sigma + n, t)$$

$$Z_{\mathrm{Nek}}(\sigma, t) = C(\sigma)\left[1 + \sum_{k=1}^{\infty} B_k(\sigma) t^k\right]$$

Strongly coupled sectors from long time expansion of Painleve'

RGE scale Λ : Painleve' time → **strong coupling from long time exp.**

Relation with gauge theories suggests a *new expansion* of Painleve' transcendents for long times:

$$\tau(s) \sim \sum_{n\in\mathbb{Z}} e^{in\rho}\mathcal{G}(\nu+n,s), \qquad \mathcal{G}(\nu,s)=C(\nu,s)\left[1+\sum_{k=1}^{\infty}\frac{D_k(\nu)}{s^k}\right]$$

conjecture [Bonelli, Lisovyy, Maruyoshi, Sciarappa, A.T.]

$$\boxed{\ln\left[\mathcal{G}\left(\frac{\nu}{\epsilon},\frac{s}{\epsilon}\right)\right] = \sum_{g\geqslant 0}\epsilon^{2g-2}\mathcal{F}_g(\nu,s),}$$

S-dual prepotential $\nu = a_D$

CHECKS: genus zero from special geometry on the Coulomb branch

$$a = \frac{\partial \mathcal{F}_0^D}{\partial a_D} = \oint_A \lambda(u) \ , \quad a_D = \oint_B \lambda(u)$$

$$\lambda(u) = \sqrt{\phi_2}$$ Seiberg-Witten differential.

Higher genera from **holomorphic anomaly equations** of **B-model** topological strings on the **SW curve** in terms of **modular forms** [Huang-Klemm]**:**

$$\begin{aligned} \partial_a \partial_{\bar{a}} F^{(1)} &= \tfrac{1}{2} C_{aaa} C^{aa}_{\bar{a}}, \\ \bar{\partial}_{\bar{a}} F^{(g)} &= \tfrac{1}{2} C^{aa}_{\bar{a}} \left(D_a D_a F^{g-1} + \textstyle\sum_{g=1}^{g-1} D_a F^{(g-h)} F^{(h)} \right), \qquad \text{for } g > 1 \ . \end{aligned}$$

we performed various checks in the magnetic, dyonic and Argyres-Douglas points.

what do we gain ?

new viewpoint on strongly coupled amplitudes

systematic approach to their computations in terms of ODEs

new long time expansions of Painleve’ tau-functions

link to quantum statistical systems and non-perturbative string

Matrix model for gauge theory at strong coupling

[Bonelli-Grassi-A.T.]

Consider pure SU(2) Super Yang-Mills in 4d : this corresponds to $PIII_3$

τ_{PIII_3} admits a **spectral determinant** presentation [Zamolodchikov]

$$\tau_{PIII_3} = \sum_{M\geq 0} \kappa^M Z_M(\Lambda)$$

$$Z_M(\Lambda) = \frac{1}{M!}\int \prod_{i=1}^{M} \frac{\mathrm{d}x_i}{4\pi} \mathrm{e}^{-\frac{2\Lambda}{\pi^2\epsilon}\cosh x_i} \prod_{i<j} \tanh\left(\frac{x_i - x_j}{2}\right)^2$$

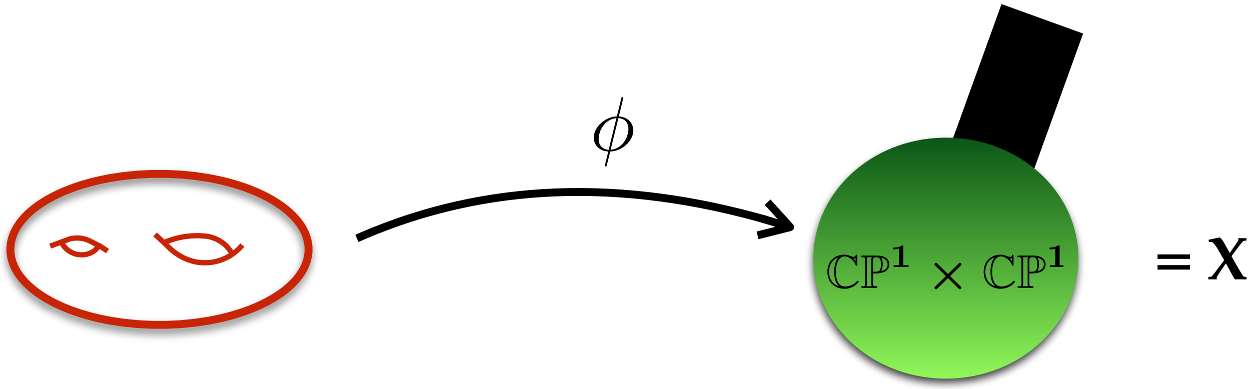
By using the Painleve'/gauge theory correspondence we can relate the 't Hooft expansion of the above matrix model with the **genus expansion of the dual Seiberg-Witten prepotential** ($\Lambda = 1$)

$$\log Z_M = \sum_{g>0} \epsilon^{2g-2} F_g^D(a_D)$$

$\epsilon^{-1}, M \to \infty$, $M\epsilon = a_D$ fixed

Spectral determinants and topological strings

topological strings on local $\mathbb{P}^1 \times \mathbb{P}^1$



Riemann surface of genus g

mirror curve:

$$e^x + e^p + e^{-p} + e^{-x} = -\kappa$$

[Witten,
Aganagic-Dijkgraaf-Klemm-Marino-Vafa,
Nekrasov-Shatashvili]

$\Big\downarrow$ **quantization**

$$[\hat{x}, \hat{p}] = i\hbar.$$

$$\mathrm{O}_X = e^{\hat{x}} + e^{\hat{p}} + e^{-\hat{p}} + e^{-\hat{x}}$$

Spectral determinants and topological strings

The operator $\rho_X = \mathrm{O}_X^{-1}$ admits an **analytic spectral determinant**

[Grassi-Hatsuda-Marino, Kashaev-Marino, Laptaev-Schimmer-Takhtajan]

$$\Xi(\kappa, \hbar) = \det\left(1 + \kappa \rho_X(\hbar)\right) = \sum_{N \geq 0} \kappa^N \boxed{Z_X^\rho(N, \hbar)}$$

ρ_X density matrix of a Fermi gas with spectral traces

$$Z_X^\rho(N, \hbar) = \frac{1}{N!} \sum_{\sigma \in S_N} (-1)^\sigma \int \mathrm{d}^N x \rho_X(x_i, x_{\sigma(i)})$$

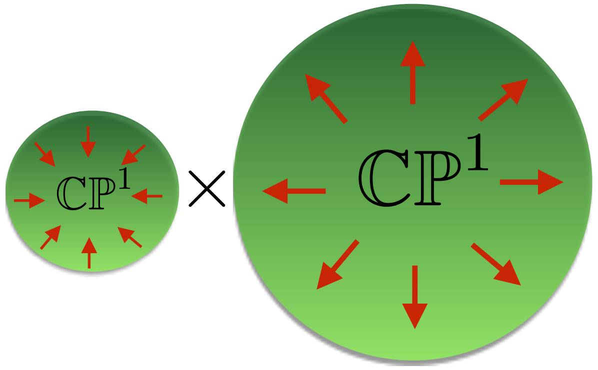
TS/ST conjecture [Grassi-Hatsuda-Marino]

$$\boxed{Z_X^\rho = Z_X^{\text{top}}}$$

Topological strings and gauge theory in the magnetic phase

[Bonelli-Grassi-A.T.]

Four-dimensional pure SU(2) gauge theory in the **dual magnetic phase** is described by type IIA superstring on local $\mathbb{P}^1 \times \mathbb{P}^1$ in the limit



$$t_F/\hbar \to 0 \qquad t_B/\hbar \to \infty \qquad \hbar \to \infty$$

$1/\hbar \sim g_s$ v.e.v. of self-dual graviphoton field strength, a.k.a. topological string coupling

remark: it is a rescaled version of geometric engineering limit of Katz-Klemm-Vafa

$$t_F \to 0 \qquad t_B \to \infty$$

this makes an important difference at the level of **quantum operators** associated to the mirror curve:

the standard geometric engineering limit makes contact with **NS quantisation** of the underlying Seiberg-Witten curve, in this case **quantum Toda chain**

$\epsilon_1 = \hbar, \epsilon_2 = 0$ [Hatsuda-Marino]

the rescaled 4d limit gives instead a **Fermi gas formulation** of Seiberg-Witten theory in a **self-dual** Ω **- background**

$\epsilon_1 = -\epsilon_2 = \epsilon$ [Bonelli-Grassi-A.T.]

Topological strings and gauge theory in the magnetic phase

[Bonelli-Grassi-A.T.]

Fermionic spectral traces for local $\mathbb{P}^1 \times \mathbb{P}^1$ computed by [Kashaev-Marino-Zakany]

in the **conifold frame.** In the rescaled 4d limit, via Cauchy identity, become

$$Z^{\rho}_{4\mathrm{D}} = \frac{1}{M!} \sum_{\sigma \in S_M} (-1)^{\sigma} \int \mathrm{d}^M x \rho_{4\mathrm{D}}(x_i, x_{\sigma(i)})$$

$$= \frac{1}{M!} \int \prod_{i=1}^{M} \frac{\mathrm{d}x_i}{4\pi} \mathrm{e}^{-2T \cosh x_i} \prod_{i<j} \tanh\left(\frac{x_i - x_j}{2}\right)^2$$

coincides with Painleve' upon $T \sim \Lambda/\epsilon$

density matrix

$$\rho_{4\mathrm{D}} = \mathrm{e}^{-T\cosh \hat{x}} 2\cosh^{-1}\left(\frac{\hat{p}}{2}\right) \mathrm{e}^{-T\cosh\hat{x}}$$

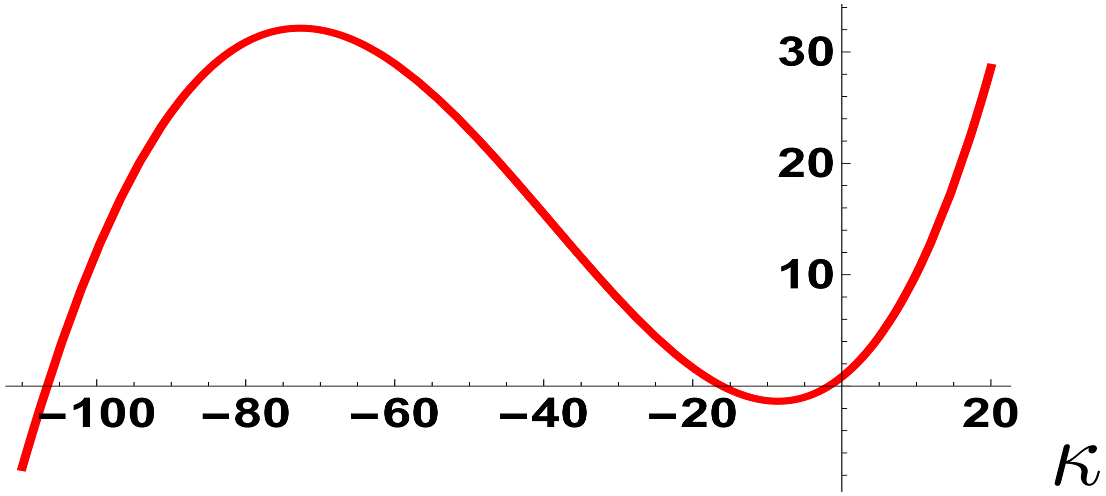
classical system

$$H^{\mathrm{cl}}_{4\mathrm{d}}(x,p) = \log\left(2\cosh p/2\right) + 2T\cosh x$$

The zeroes of the Painleve' III3 tau-function give the spectrum of the quantum Fermi gas :

$$\{E_n\}_{n=0,1,\ldots} = \left\{ \log\left[\frac{1}{2\pi}\cosh(2\pi\sigma_r^{(n)})\right] : \Xi_S^{4\mathrm{d}}(T, \frac{1}{2} + \mathrm{i}\sigma_r^{(n)}) = 0 \right\}.$$

$$\tau(21/\pi, \kappa) = Z^{\mathrm{NO}}(\kappa, 21/\pi)$$



The zeros of the NO partition function give the spectrum of $\rho_{4\mathrm{d}}$

quantum statistical system associated to gauge theory in the self-dual Omega background

Topological strings and SU(N) gauge theories

Analogous scaling limit of topological strings describing SU(N) gauge theories gives the matrix model [Bonelli-Grassi-A.T.] :

$$Z^{\mathrm{SYM}}_{4\mathrm{d}}(M_1,\cdots,M_{N-1}) = \frac{1}{M_1!\cdots M_{N-1}!}\int \frac{\mathrm{d}^M x}{(2\pi)^M}\prod_{j=1}^{N-1}\prod_{i_j\in I_j} \mathrm{e}^{-NT\sin\left(\frac{\pi j}{N}\right)\cosh(x_{i_j})}$$

$$\times \frac{\prod_{1\le i<j\le M} 2\sinh\left(\frac{x_i-x_j}{2}+\frac{1}{2}(d_i-d_j)\right)2\sinh\left(\frac{x_i-x_j}{2}+\frac{1}{2}(f_i-f_j)\right)}{\prod_{i,j=1}^{M} 2\cosh\left(\frac{x_i-x_j}{2}+\frac{1}{2}(d_i-f_j)\right)}$$

N-1 cuts, d_i, f_i are N-dependent phase shifts.

We checked this to reproduce the **dual Seiberg-Witten prepotential** with

$$a_{D_i} = M_i T^{-1}$$

$F_0^D(a_D^i)$ [Klemm-Lerche-Theisen,D'Hoker-Phong,Edelstein-Mas,Edelstein-Gomez-Reino-Marino,Douglas-Shenker]

$F_g^D(a_D^i)$ $, g \geq 2$ **very hard to compute by other methods!**

Perturbing the massless monopole point

$a_D^i = 0$ massless monopole point

One cut matrix model, with e.g. $a_D^1 \neq 0$

$$Z_{4\mathrm{d}}^{(1)}(M) = \frac{1}{M!} \int \frac{\mathrm{d}^M x}{(2\pi)^M} \prod_{i=1}^{M} \mathrm{e}^{-\frac{1}{g_s} \cosh(x_i)} \frac{\prod_{i<j} 4 \sinh\left(\frac{x_i - x_j}{2}\right)^2}{\prod_{i,j} 2 \cosh\left(\frac{x_i - x_j}{2} + \mathrm{i}\pi\beta,\right)}, \qquad \beta = \frac{2-N}{2N}.$$

fermionic spectral traces of

$$\mathrm{K}(x,y) = \mathrm{e}^{-\cosh(x)/2g_s} \frac{1}{4\pi \cosh\left(\frac{x-y}{2} - \mathrm{i}\pi \frac{(N-2)}{2N}\right)} \mathrm{e}^{-\cosh(y)/2g_s}, \qquad x, y \in \mathbb{R},$$

spectral determinant satisfies the equations of $\hat{A}_{N-1}$ Toda chain:

$$q_\ell'' + \frac{1}{t} q_\ell' = \mathrm{e}^{q_\ell - q_{\ell-1}} - \mathrm{e}^{q_{\ell+1} - q_\ell},$$

corresponds to **isomonodromic deformations** of $SL(N, \mathbb{C})$ flat connection on the cylinder with regular singularities at 0 and ∞ [Cecotti-Vafa, Guest-Its-Lin].

q-Painleve’, five dimensional gauge theories and topological strings

The 4d Renormalization Group diagram is part of a bigger picture which comes from the embedding of gauge theories in string theory.

This also has a counterpart in Painleve’ theory by going to the **multiplicative** case.

$SU(2)$ five dimensional gauge theories with N_f fundamental hypers

are relevant deformations in the IR of strongly coupled SCFTs with exceptional global symmetries

Seiberg's classification:

$$E_8 \to E_7 \to E_6 \to E_5 \to E_4 \to E_3 \to E_2 \to E_1$$

N_f from 7 to 0 via holomorphic decoupling of masses

reduction to four dimensions: $\mathbb{R}^4 \times S^1$

S^1 - radius going to zero

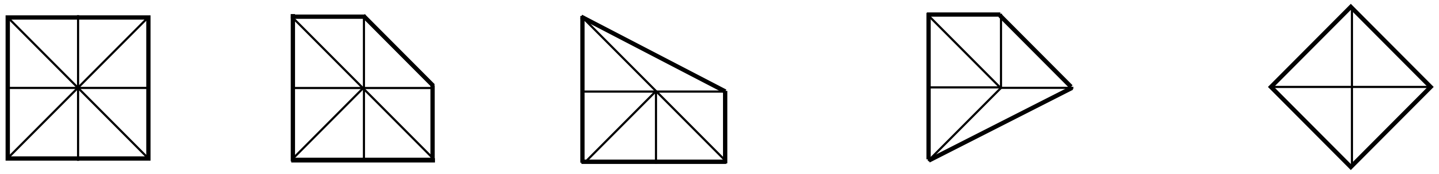
Sakai’s classification of q-Painleve’ :

$$\begin{tikzcd}
 & & & & & & & A_1^{(1)} \arrow[dr] & \\
E_8^{(1)} \arrow[r] & E_7^{(1)} \arrow[r] & E_6^{(1)} \arrow[r] & D_5^{(1)} \arrow[r] \arrow[dr] & A_4^{(1)} \arrow[r] \arrow[dr] & (A_1+A_2)^{(1)} \arrow[r] \arrow[dr] & (A_1+A_1)^{(1)} \arrow[r] \arrow[ur] \arrow[dr] & A_1^{(1)} \arrow[dr] & - \\
 & & & & D_4^{(1)} \arrow[r] & A_3^{(1)} \arrow[r] \arrow[dr] & (A_1+A_1)^{(1)} \arrow[r] \arrow[dr] & A_1^{(1)} \arrow[r] & - \\
 & & & & & & A_2^{(1)} \arrow[r] & A_1^{(1)} \arrow[r] & -
\end{tikzcd}$$

Backlund symmetries table: Weyl group of the **affinization** of **flavour symmetry group** of the corresponding 5d/4d gauge theory.

$SU(2)$ gauge theories on $\mathbb{R}^4 \times S^1$ are geometrically engineered by **topological strings** on local del Pezzo surfaces.

e.g. up to $N_f = 4$:



topological strings on local Hirzebruch and their blow-ups provide conjectural tau-functions for q-Painleve'

for toric geometries non-perturbative topological string would provide a candidate *spectral determinant* for those tau-functions.

it works for q-PIII3 tau function [Bonelli-Grassi-A.T., to appear]

Concluding remarks

gauge theory partition functions/BPS correlators and isomonodromy:

window on strongly coupled sectors of gauge and string theories

a new class of matrix models for magnetic phase

quantum statistical systems capturing self-dual Omega background

Some open problems

direct derivation of matrix model from gauge theory

including matter fields

breaking to N=1, confinement and condensates

non-perturbative string: frame dependence, relation to isomonodromy

THANKS!!