

Gauge theory at strong coupling, isomonodromy problem and non-perturbative string

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Exact results in Quantum Field Theories

Going beyond perturbation theory is an important challenge in

Quantum Field Theory and String Theory.

N=2 supersymmetric gauge theories in four dimensions are useful to study:

- exact non-perturbative effects (instantons)
- strong-weak coupling dualities
- confinement through breaking to $N=1$

systematic study of the above lead to discover rich algebraic and integrable

structures, which had deep impact also in mathematics.

Exact results in Quantum Field Theories

In the *weak coupling* phase the supersymmetric path integral can be reduced via *equivariant localization* to combinatorial objects - Nekrasov function, matrix models - paving the way to the link with rich algebraic structures as CFT - type algebrae (Virasoro, W - algebrae, Kac-Moody) and *quantum integrable systems.*

In the *strong coupling* phase the embedding in superstring theories provides a crucial tool to explore S-duality and to calculate protected gauge theory quantities - e.g. *holomorphic anomaly equations, topological recursion.*

Exact results in Quantum Field Theories

In this talk we will show how to compute a class of gauge theory amplitudes in strongly coupled phases by reducing them to the solutions of non-linear ODEs . These are associated to a set of *isomonodromy problems* and display a deep link with the non-perturbative completion of topological string proposed by Grassi, Hatsuda and Marino.

Today's talk: gauge theory partition functions and BPS correlators as - functions of isomonodromy problems

quantum statistical systems and non-perturbative string

new tools to explore strongly coupled phases/ sectors of gauge theory

based on:

G. Bonelli, O. Lisovyy, K. Maruyoshi, A. Sciarappa, A.T. 1612.06235

G. Bonelli, A. Grassi, A.T. arXiv:1603.01174, 1704.01517 & to appear

N=2 supersymmetric gauge theories in 4d

class S: M-theory compactification from 6d naturally hints a relation with 2d theory

description in terms of *Hitchin's algebraic integrable system*

Renormalization group equations of SU(2) asymptotically free gauge theories are systematically described by Painleve' equations

Spectral determinant presentation of Painleve' tau-function hints to a new relation between gauge theory and quantum statistical systems via geometric engineering limit of non-perturbative string

Generalisation to higher rank: new class of Matrix models describing the magnetic phase of pure SU(N) Super Yang-Mills theories.

Seiberg-Witten curve from M-theory

[Klemm-Lerche-Mayr-Vafa-Warner, Witten, Gaiotto]

Consider M-theory on $\mathbb{R}^7 \times Q$ we holomorphic symplectic two-fold

decoupling gravity \longrightarrow local geometry $\mathbb{R}^4 \times T^*C \times \mathbb{R}^3$

r M5 branes on $\mathbb{R}^4 \times C$ in the **Coulomb branch**

$$
y^r + \sum_{k=2}^r \phi_k(z) y^{r-k} = 0
$$

holomorphic k-differential

satisfy **Hitchin's equations** $S¹$ compactification gives rise to U(r) Super Yang-Mills theory in 5d on $\mathbb{R}^3 \times \mathcal{C}$ BPS vacua invariant under Super-Poincare' of \mathbb{R}^3 The subsequent compactification of the (twisted) *d* = 5 super-Yang-Mills theory on *C* then leads to BPS equations which are well known to be the Hitchin equations. In particular, if @*z*'*^z* + [*Az,* '*z*]=0*,* @*z*'*^z* + [*Az,* '*z*]=0*,*

same e↵ective theory in three dimensions by first compactifying on *S*¹ and then on *C*. The

first compactification on *S*¹ leads to a five-dimensional supersymmetric Yang-Mills theory.

$$
F + R^2[\varphi, \bar{\varphi}] = 0,
$$

\n
$$
\partial_{\bar{z}} \varphi + [A_{\bar{z}}, \varphi] = 0,
$$

\n
$$
\partial_z \bar{\varphi} + [A_z, \bar{\varphi}] = 0,
$$

(*E, D,* ')

these are equivalent to the flatness of the $\ SL(r, \mathbb{C})$ connection

$$
\mathcal{A} = \frac{R}{\zeta}\varphi + A + R\zeta\overline{\varphi};
$$

r
1970 - Johann Barnett
1970 - Johann Barnett, frysk kampan

Seiberg-Witten curve and Hitchin's system

 $\mathcal M$ is HyperKahler \longrightarrow twistor parameter *JM* is HyperKahler \longrightarrow twistor parameter ζ

$$
\mathcal{A} = \frac{R}{\zeta}\varphi + A + R\zeta\overline{\varphi};
$$

for $\zeta \to 0$ base space $\mathcal U$ described by $\phi_k(z) = \text{tr } \varphi^k$

Example 19 Interpretation value SW curve is the **spectral curve**

(*E, D,* ')

$$
\det(y-\varphi)=0
$$

Seiberg-Witten curve and Hitchin's system

s complex algebraic integrable system. Hitchin's complex algebraic integrable system:

Coulomb branch of 4d gauge theory

Polar structure of the k-differentials

Collision of poles with suitable rescalings of the residues

changes the matter sector

for $SU(2)$ gauge theory

quadratic differentials on a Riemann sphere with at most

four simple poles:

 $N_f = 4$ superconformal gauge theory

collision of simple poles hol. decoupl. of matter Argyres-Douglas sect.

a $\mathcal{L}=\mathcal{L}$ \exists Tr('² **holomorphic decoupling**: Lagrangian theories

$$
N_f = 0: \frac{\Lambda^2}{z^3} + \frac{2u}{z^2} + \frac{\Lambda^2}{z},
$$

\n
$$
N_f = 1: \frac{\Lambda^2}{z^3} + \frac{3u}{z^2} + \frac{2\Lambda m}{z} + \Lambda^2,
$$

\n
$$
N_f = 2: \frac{\Lambda^2}{z^4} + \frac{2\Lambda m_1}{z^3} + \frac{4u}{z^2} + \frac{2\Lambda m_2}{z} + \Lambda^2,
$$
 (first realization)
\n
$$
N_f = 2: \frac{m_+^2}{z^2} + \frac{m_-^2}{(z-1)^2} + \frac{\Lambda^2 + u}{2z} + \frac{\Lambda^2 - u}{2(z-1)},
$$
 (second realization)
\n
$$
N_f = 3: \frac{m_+^2}{z^2} + \frac{m_-^2}{(z-1)^2} + \frac{2\Lambda m + u}{2z} + \frac{2\Lambda m - u}{2(z-1)} + \Lambda^2.
$$

ttti alan a

Argyres-Douglas sectors:

(*^z* 1)² ⁺

*^z*² ⁺

N^f =3:

$$
H_0: z^3 - 3cz + u
$$

\n
$$
H_1: z^4 + 4cz^2 + 2mz + u, \text{ (first realization)}
$$

\n
$$
H_1: z + c + \frac{u}{z} + \frac{m^2}{z^2}, \text{ (second realization)}
$$

\n
$$
H_2: z^2 + 2cz + (2\tilde{m} + c^2) + \frac{u + 2cm_-}{z} + \frac{m^2_-}{z^2}
$$

 $\frac{1}{2}$

2⇤*m u*

2(*^z* 1) ⁺ ⇤2*.*

Painleve' coalescence diagram complication is the singularities of the singularities, such as essential singular points or natural boundaries, α be solved in the solvening of the terms of the problem is simplified by the p fact that movable singular points of the 1st order equations can only be poles or algebraic

 L fuchs in 1884, who showed that such equations are either reducible to linear ones or cannot reduce to linear ones or cann

be solved in terms of elliptic functions. The treatment of the problem is simplified by the

fact that movable singular points of the 1st order equations of the 1st order equations can only be poles or a

Classification problem for ODEs and the problem becomes much more involved. Nevertheless in 1900-1910 P. Painlevé and B. Gambanon propion to classified and attenpt of movement of the degree 1 2nd order of movement of movable of mov branch points. When the order of an ODE is 200 i complication propierities, such as essential singular points or natural boundaries, such as essential boundari

branch points. Such equations have the form

Painlevé equations PI–PVI1.

Painleve' property: ODE with only movable poles

Gambier undertook an attempt of classifying the degree 1 2nd order ODEs free of movable

–3–

The classification of algebraic 1st order $\mathcal{O}(D)$ and $\mathcal{O}(D)$ and $\mathcal{O}(D)$ and $\mathcal{O}(D)$

L. Fuchs in 1884, who showed that such equations are either reducible to linear ones or can

 $\ddot{q} = F(q, \dot{q}; t),$

in t. The outcome of Painlevé and Gambier studies was a list of 50 equivalence classes

F rational function of q, \dot{q} analytic in t : **full classification**

Painleve' and isomonodromy deformations *i*=0 (*z*) = X *A*(⌫) \overline{a} \overline{a} \overline{b} \overline{c} *r*e' and ad *ien* (*^z ^z*⌫)*r*⌫+1 *, A*(⌫) (*z*) = X α *deform*

*^z ^C*0*,n ⁿ* CP¹ ^A(*z*) ² *sl*(2*,* ^C) ² ⇥ ²

Linear system:
$$
(\kappa \partial_z - \mathbf{A}(z))\Psi(z) = 0
$$

$$
\mathbf{A}(z) = \sum_{\nu=1}^{n} \frac{A^{(\nu)}(z)}{(z - z_{\nu})^{r_{\nu}+1}}, \qquad \qquad A^{(\nu)}(z) = \sum_{i=0}^{r_{\nu}} A_i^{(\nu)}(z - z_{\nu})^{r_{\nu} - i}
$$

z⌫ *r*⌫ = 0 *z*⌫ $A(z) \in sl(2,\mathbb{C}),$ *z* affine coordinate on punctured sphere (*z*) (*z*) 2 *GL*(2*,* C) isomonodromy: family $\mathbf{A}(z; {\{\vec{t}\}})$ of flat $SL(2,\mathbb{C})$ conn. A(*z*) (*z*) $\mathbf{A}(z)$ *{*~*t} SL*(2*,* ^C) $\left(\begin{array}{cc} \text{if } t & \text{if } (0, t) \end{array} \right)$ $\left(4, \infty\right)$ σ

four punctures \longrightarrow one parameter t

(*z*) = X

(*z*)

(*z*) *z*⌫ *r*⌫ > 1

(*^z ^z*⌫)*r*⌫+1 *, A*(⌫)

A(*z*) (*z*)

t (*z, t*)

i=0

t A(*z, t*)

z⌫ *r*⌫ = 0 *z*⌫

8

d

$$
\begin{cases}\n\frac{d}{dz}\Psi(z,t) = \mathbf{A}(z,t)\Psi(z,t) \\
\frac{d}{dt}\Psi(z,t) = \mathbf{B}(z,t)\Psi(z,t)\n\end{cases}
$$

Compatibility condition $\Psi_{zt}(z,t) = \Psi_{tz}(z,t)$:

A B

becomes

condition

implies the equation

$$
\mathbf{A}_t(z,t) = \mathbf{B}_z(z,t) + [\mathbf{B}(z,t), \mathbf{A}(z,t)]
$$

B(*z, t*) A(*z, t*)

where B(*z, t*) can determined in terms of the matrices appearing in A(*z, t*) by the theory

of isomonodromic deformations. This is an overdetermined system whose compatibility

namely variation of flat conn. is infinitesimal gauge transf. which yields a system of \mathcal{L} and \mathcal{L} are \mathcal{L} . The \mathcal{L} is the matrices of matrices \mathcal{L} A, B are known as the communication of the associated Pairs of the associated Paint (2008).

yields Painleve' equations. A, B: Lax pair Isomonodromic deformations admit an interesting limit to isospectral deformation

 isospectral deformations, leave invariant $\kappa \to 0$ isospectral deformations, leave invariant

$$
\det(y - \mathbf{A}) = 0, \qquad \Sigma \in T^* \mathcal{C}_{0,n}
$$

rameters of the problem in order to explicitly introduce a "Planck constant" so that (2.4)

dynamics around *T*0) the connection @*z*A(*z*) reduces to a one-form A 2 ⌦(*C*0*,n, sl*(2*,* C)),

that is a Higgs field in terms of Hitchin integrable systems; moreover in this limit the

(@*^z* A(*z*)) (*z*)=0 (2.9)

T **0** \blacksquare *N* **itten curve !** Upon ident \blacksquare \blacksquare \blacksquare \lozenge **Seiberg-Witten curve !** Upon ident. $\mathbf{A} = \varphi$

example: **Painleve' I vs** H_0 **Argyres-Douglas** *A* = *A*⁰ + *zA*¹ + *z*2*A*² = $\frac{1}{2}$ $\frac{1}{2}$ ainleve'l vs H_0 Argyre 0 *q* + *z/*2 2 0 ! nple: Painleve' I H_{\circ} Arc H_0 Argyr

$$
A = A_0 + zA_1 + z^2 A_2 = \begin{pmatrix} -p & q^2 + zq + z^2 + t/2 \\ 4z - 4q & p \end{pmatrix}
$$

$$
B = B_0 + zB_1 = \begin{pmatrix} 0 & q + z/2 \\ 2 & 0 \end{pmatrix}
$$

⁴*^z* ⁴*q p* !

⁴*^z* ⁴*q p* !

0 *q* + *z/*2 2 0 !

⌧*^I* (*t*)

B = *B*⁰ + *zB*¹ =

compatibility

$$
\begin{cases} \dot{q} = p \\ \dot{p} = 6q^2 + t \end{cases} \qquad \qquad \dot{q} = 6q^2 + t
$$

⌧*^I* (*t*)

quad. different.

2

*y*² =

$$
\frac{1}{2} \text{Tr} A^2 = 4z^3 + 2tz + 2\sigma_I(t) \qquad \sigma_I(t) = \frac{1}{2}p^2 - 2q^3 - qt
$$

^I (*t*) = ¹ *^p*² ²*q*³ *qt* **I** $\frac{1}{2}$ $\frac{1}{2}$ Ĭ. *<u>Incide</u>* with SW curve upon ide *Diecome in the levant deformation* vertically coincide with SW curve upon ident. *p* $t \sim$ σ ^{*t*} $\sim c$, σ _{*I*} \sim ⌧*^I* (*t*) source of relevant deformation vev of relevant deformation

u
Li

$$
N_f = 4 \longrightarrow N_f = 3 \longrightarrow N_f = 2 (I) \longrightarrow N_f = 1 \longrightarrow N_f = 0
$$
\n
$$
N_f = 2 (II) \longrightarrow H_1 (I) \longrightarrow H_0
$$
\n
$$
H_2 \longrightarrow H_1 (II) \longrightarrow H_0
$$
\n
$$
PII_1 \longrightarrow PII_2 \longrightarrow PII_3
$$
\n
$$
PVI \longrightarrow PV \longrightarrow PV_{deg} \longrightarrow PII_{JM} \longrightarrow P_I
$$
\n
$$
PIV \longrightarrow PII_{FN}
$$

A(*z*)

Painleve' "transcendents" and gauge theory dual partition function

We can extend the correspondence to the full gauge theory in the so-called

$$
Z_{\text{Nek}}(a, m; \Lambda, \epsilon) = \exp \left[- \sum_{g} \epsilon^{2g-2} \mathcal{F}_{g}(a, m; \Lambda) \right]
$$

Nekrasov-Okounkov dual partition function

$$
Z_{\rm NO}(a/\epsilon,m;\Lambda,\eta)=\sum_{n\in\mathbb{Z}}e^{4\pi in\eta}Z_{\rm Nek}(a+n\epsilon,m;\Lambda)
$$

Theorem: this is the tau-function for Painleve' equations [Gamayun-Iorgov-Lisovyy, Bershtein-Shedchin, Iorgov-Lisovyy-Teschner]

$$
(a/\epsilon = \sigma, \eta)
$$
 initial conditions, $\{m\}$ monodromy param.

RGE scale Λ : Painleve' time - **short time expansion corresponds to weakly coupled** *electric frame*

$$
\tau(t) \sim \sum e^{4\pi i n \eta} Z_{\rm Nek}(\sigma + n, t)
$$

$$
Z_{\text{Nek}}(\sigma, t) = C(\sigma) \left[1 + \sum_{k=1}^{\infty} B_k(\sigma) t^k \right]
$$

Strongly coupled sectors from long time expansion of Painleve' t_{max} the corresponding as Pational tematical computers in the corrections of the corre tematically compute subleading corrections, we indeed observe a periodic pattern (2.29) mansion of Painleve In order to later the sectors from the sectors f **Frederical sequence in an and group terms with the symmetry with the same power of Painleve'**

system defined by (3.1) are parameterized by two complex quantities giving a pair of PI

integrals of motion. The leading behavior of q(t) as t → ∞ along the canonical rays

with G(1 μ in Barnes G-functions would correspond to the 1-loop particle correspon

^D1(ν) = [−]iν(94ν² + 17)

⁴⁸^s [−] ⁷⁷¹⁷ν⁴

⁹⁶ ,

⁴⁶⁰⁸s² ⁺ ^O(s−3),

 5×5 R >0 has been described in terms of S for S parameters in $[53]$. Determining in $[53]$

 $5 + 8$

² G(1 + ν),

RGE scale Λ : Painleve' time **strong coupling from long time exp.**
Relation with gauge theories suggests a new expansion of Painleve' transcendents on the rays argued the rays arg
The rays argued the rays argu ⁵ , [±]^π ν (−1) +# me exp

Relation with gauge theories suggests a *new expansion* of Painleve' transcendents
. for long times: 4 2 e' tra necondonte

$$
\tau(s) \sim \sum_{n \in \mathbb{Z}} e^{in\rho} \mathcal{G}(\nu + n, s), \qquad \mathcal{G}(\nu, s) = C(\nu, s) \left[1 + \sum_{k=1}^{\infty} \frac{D_k(\nu)}{s^k} \right]
$$

• [Bonelli, Lisovyy, Maruyoshi, Sciarappa sk , **conjecture** [Bonelli, Lisovyy, Maruyoshi, Sciarappa, A.T.] **(** γ, s) = (2π)
2π) = (2π) = (2
2π) = (2π) = Coniecture [Bonelli Lisovyy Maruvoshi Sciaranna A T] $\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{j=1}^{n}$

2

series for the tau function.

^F0(ν, s) = ^s²

⁴⁵ ⁺

5

By using

$$
\ln\left[\mathcal{G}\left(\frac{\nu}{\epsilon},\frac{s}{\epsilon}\right)\right] = \sum_{g\geqslant 0} \epsilon^{2g-2} \mathcal{F}_g(\nu,s),
$$

S-dual prepotential $\nu = a_D$

 44.4

contribution to the partition function; they can be computed recursively, and the first few

CHECKS: genus zero from special geometry on the Coulomb branch 2.1 Topological string amplitude F(g) as a polynomial of E

$$
a = \frac{\partial \mathcal{F}_0^D}{\partial a_D} = \oint_A \lambda(u) \; , \quad a_D = \oint_B \lambda(u)
$$

direct integration method of the holomorphic anomaly equations to the SU(2) gauge

In the next section we will review the direct integration we will review the direct integration approach for solving Γ

ˆ

$$
\lambda(u) = \sqrt{\phi_2} \quad \text{Seiberg-Witten differential.}
$$

theory completely using the gap completely using the gap condition. The point is to show the po might genera from The magnetic and control more control the main forms \mathfrak{g} . Higher genera from **holomorphic anomaly equations** of **B-model** topological strings on the **SW curve** in terms of **modular forms** [Huang-Klemm]**:**

$$
\partial_a \partial_{\bar{a}} F^{(1)} = \frac{1}{2} C_{aaa} C_{\bar{a}}^{aa},
$$

\n
$$
\bar{\partial}_{\bar{a}} F^{(g)} = \frac{1}{2} C_{\bar{a}}^{aa} \left(D_a D_a F^{g-1} + \sum_{g=1}^{g-1} D_a F^{(g-h)} F^{(h)} \right), \quad \text{for } g > 1.
$$

 ϵ we use the coordinate a introduced introduced in the last section, but the equation, but the equation, but the equations of ϵ are performed vancus cheens in the magnetic, ayonic and Augyres Douglas
Mointe connection Da, whose calculation from the solutions of the Picard Fuchs equation are we performed various checks in the magnetic, dyonic and Argyres-Douglas points.

what do we gain?

new viewpoint on strongly coupled amplitudes

systematic approach to their computations in terms of ODEs

new long time expansions of Painleve' tau-functions

link to quantum statistical systems and non-perturbative string

Matrix model for gauge theory at strong coupling

ZX(*N*) is the non-perturbative partition function of topological string on *X*. More precisely, [Bonelli-Grassi-A.T.]

Consider pure SU(2) Super Yang-Mills in 4d : this corresponds to **by Consider pure SU(2) Super Yang-Mills in 4d · this corresponds to** $PIII_2$ function of this this this this theory in the conifold frame \mathcal{I}_1 frame \mathcal{I}_2 frame \mathcal{I}_3 frame \mathcal{I}_4 frame \mathcal{I}_5 frame \mathcal{I}_6 frame \mathcal{I}_7 frame \mathcal{I}_8 frame \mathcal{I}_9 frame \mathcal{I}_9 frame

 τ_{PIII_3} admits a **spectral determinant** presentation [Zamolodchikov]

well known *O*(2) model

in the magnetic frame.

$$
\tau_{PIII_3} = \sum_{M \geq 0} \kappa^M Z_M(\Lambda)
$$

$$
Z_M(\Lambda) = \frac{1}{M!} \int \prod_{i=1}^M \frac{\mathrm{d} x_i}{4\pi} \mathrm{e}^{-\frac{2\Lambda}{\pi^2 \epsilon} \cosh x_i} \prod_{i
$$

By using the Painleve'/gauge theory correspondence we can relate the 't Hooft expansion of the above matrix model with the **genus expansion of the dual Seiberg-Witten prepotential** $(\Lambda = 1)$

$$
\log Z_M = \sum_{g \ge 0} \epsilon^{2g-2} F_g^D(a_D)
$$

interpretation of the self-dual α background, it gives Fredholm determinant representation for the self-dua
The self-dual α background, it gives Fredholm determinant representation for the self-dual self-dual self-d

$$
\epsilon^{-1}, M \to \infty \quad , \qquad M \epsilon = a_D \quad {\rm fixed}
$$

Spectral determinants and topological strings Opecual determinants

topological strings on local $\phi: \Sigma_g \to X$ \Box Cte^lpolog¹

Spectral determinants and topological strings Operator theory and mirror curves al determinants and topological string
 $\Xi_X(\kappa,\hbar)=\det(1+\kappa\rho_X)=\prod_{i=1}^N(1+\kappa{\rm e}^{\imath\omega_i})$

AG, Hatsuda,Marino

Laptev,Schimmer,Takhtajan

 Kashaev,Marino

The operator $\rho_X = \mathcal{O}_X^{-1}$ admits an **analytic spectral determinant** $n \geq 0$

discrete $\{e^{-E_n}\}_{n\geq 0}$

 $\begin{equation} \mathcal{K} \hspace{1.5cm} = \{ \mathrm{e}^{-E_n} \}_{n \geq 0} \hspace{1.5cm} \text{Latsw} \end{equation}$ analy-Schimmer-Takhtaian l Laptaev-Schimmer-Takhtajan]

AG, Hatsuda,Mariño

$$
\Xi(\kappa,\hbar) = \det(1 + \kappa \rho_X(\hbar)) = \sum_{N \ge 0} \kappa^N \left(\frac{\rho}{X}(N,\hbar) \right)
$$
\ndensity matrix of X is given as with spectral traces.

 $n\geq0$ ρ_X density matrix of a Fermi gas with spectral traces

$$
Z_X^{\rho}(N,\hbar) = \frac{1}{N!} \sum_{\sigma \in S_N} (-1)^{\sigma} \int d^N x \rho_X(x_i, x_{\sigma(i)})
$$

\n**TS/ST copjecture** [Graspi-Hatsuda-Marinoj]
\n
$$
\kappa
$$

\n
$$
\kappa
$$

\n
$$
Z_X^{\rho}(2,\hbar) = \frac{1}{2} \left((\text{Tr} \frac{Z_X^{\rho}}{X} = \frac{Z_X^{\text{top}}}{X}) \right)
$$

\n
$$
\Xi(\kappa,\hbar) = \det (1 + \kappa \rho_X(\hbar)) = \sum_{\kappa} \kappa^N \overline{Z_X^{\rho}}(N,\hbar)
$$

 $\rm 30$

 $N\geq 0$

 $\mathcal{L}(\kappa, n) = \det(1 + \kappa p X(n)) - \lambda \sum_{\mathbf{X} \geq \mathbf{0}} \kappa \sum_{\mathbf{X} \geq \mathbf{0}} n \sum_{\mathbf{X} \geq \mathbf{0}}$

completion for the topological string partition function on X:

Topological strings and gauge theory in the magnetic phase engineered by using topological string. Katz,Klemm,Vafa — Iqbal, Kashani Poor …

[Bonelli-Grassi-A.T.]

Four-dimensional pure SU(2) gauge theory in the **dual magnetic phase** is described by type IIA superstring on local $\mathbb{P}^1 \times \mathbb{P}^1$ in the limit **In our example we see that the second state we see the second state we see that the second state we see the s**

 $t_F/\hbar \to 0$ $t_B/\hbar \to \infty$ $\hbar \to \infty$

Next: we will see that in this limit the previous non-perturbative formulation provides

Note: this is a rescaled version of the standard geometric engineering limit

 $1/\hbar \sim g_s$ v.e.v. of self-dual graviphoton field strength, a.k.a. topological string coupling

remark: it is a rescaled version of geometric engineering limit of Katz-Klemm-Vafa $t_F \to 0 \quad t_B \to \infty$

 $t_F \rightarrow 0$ $t_B \rightarrow \infty$

this makes an important difference at the level of **quantum operators** associated to the mirror curve:

the standard geometric engineering limit makes contact with **NS quantisation** of the underlying Seiberg-Witten curve, in this case **quantum Toda chain**

 $\epsilon_1=\hbar, \epsilon_2=0$ [Hatsuda-Marino]

the rescaled 4d limit gives instead a **Fermi gas formulation** of Seiberg-Witten theory in a **self-dual** Ω **- background**

 $\epsilon_1 = -\epsilon_2 = \epsilon$ [Bonelli-Grassi-A.T.]

Topological strings and gauge theory in the magnetic phase

[Bonelli-Grassi-A.T.]

···

 $\overline{2}$

in the **conifold frame.** In the rescaled 4d limit, via Cauchy identity, become **By using the Cauchy identity we have the Cauchy identity we have the Cauchy identity we have the cauchy in th** Fermionic spectral traces for local $\mathbb{P}^1 \times \mathbb{P}^1$ computed by [Kashaev-Marino-Zakany]

$$
Z_{4D}^{\rho} = \frac{1}{M!} \sum_{\sigma \in S_M} (-1)^{\sigma} \int d^M x \rho_{4D}(x_i, x_{\sigma(i)})
$$

=
$$
\frac{1}{M!} \int \prod_{i=1}^M \frac{dx_i}{4\pi} e^{-2T \cosh x_i} \prod_{i < j} \tanh\left(\frac{x_i - x_j}{2}\right)^2
$$

 τ cides with Painleve' ưpờn $T\sim\Lambda/\epsilon$ ρ_{4D} ρ_{4D} coincides with Painleve' upon $T \sim \Lambda/\epsilon$

density matrix
$$
\rho_{4D} = e^{-T \cosh \hat{x}} 2 \cosh^{-1} \left(\frac{\hat{p}}{2}\right) e^{-T \cosh \hat{x}}
$$

$$
\rho_{4D} = e^{-T \cosh \hat{x}} 2 \cosh^{-1} \left(\frac{\hat{p}}{2}\right) e^{-T \cosh \hat{x}}
$$

classical system **The classical limit:**

$$
H_{\text{4d}}^{\text{cl}}(x, p) = \log (2 \cosh p/2) + 2T \cosh x
$$

\n
$$
H_{\text{4d}}^{\text{cl}}(x, p) = \log (2 \cosh p/2) + 2T \cosh x
$$

\n
$$
H_{\text{4d}}^{\text{cl}}(x, p) = \log (2 \cosh p/2) + 2T \cosh x
$$

\n
$$
E_1
$$

potential kinetic term

The zeroes of the Painleve' III3 tau-function give the spectrum of the **state of the set of the set of the set of the By using the computationary we can consider the spectrum of** $\tau(T, \kappa) = e^{4VT} \det (1 + \kappa \rho_{4D})$ **quantum Fermi gas :** $_{\tau(T,\kappa) = e^{4\sqrt{T}} \det(1 + \kappa \rho_{4D})}$

$$
\{E_n\}_{n=0,1,...} = \left\{ \log \left[\frac{1}{2\pi} \cosh(2\pi \sigma_r^{(n)}) \right] : \Xi_S^{4d}(T, \frac{1}{2} + i \sigma_r^{(n)}) = 0 \right\}.
$$

In Table 1 we compare the numerical spectrum of the operator (4.2) with the zeros of (3.26): we

Therefore the region which is interesting from the spectral theory point of view is parameterized by $\mathcal{L}_\mathcal{F}$

$$
\tau(21/\pi,\kappa)=Z^{\rm NO}(\kappa,21/\pi)
$$

of the Fermi gas described by (4.2). More precisely we have

 \mathcal{L}

where

zo^{\uparrow} (4.11) (4 **The zeros of the NO partition function give**

, (4.12)

⇤4 **quantum statistical system associated to gauge theory in the self-dual** ²4✏⁴ *.* (4.13) **Omega background**

$$
N-1) = \frac{\text{Gives the matrix space with a given number of vertices. More precisely, the average of the number of vertices. More precisely, the average of the number of vertices. The second set is the number of vertices. The second set is
$$

where K denotes the instanton counting parameter in gauge theory. The shifts f_i, d_i are given in zulys v_a Oh the Bauge of OHD ever alsw used
on the range group of the rank of the research engange group. We also used $\text{counting parameter in gauge theory.}$ The shifts f_i, a_i are given in gauge theory. The shifts f_i, a_i are given in depend on the rank *N* of the gauge group.
the instanton counting parameter in gauge $(Y.1)$ the instanton counting parameter in gauge theory. The shifts f_i, d_i are given in

 \mathcal{H}_0 , \mathcal{H}_0 , \mathcal{H}_i , f_i , $M = 2$, $\sum M_i$. *II*j0 − $\overline{\Psi}$ **NF** *j*1 *s*=0 *Ms,* **ZA** *j s*=1 *M^s* # $N_{\rm max}^{\rm max}$ and N_{\rm **DULCE the Tual Seiberg-Witten** pre (1.2) jest
1 *j*1 $\overline{=}$ Wetchecked this to reproduce the **dual Seiberg-Witten prepotential** with \mathcal{H}_d , $\sum_{i=1}^{n} M_i$ $I_j =$ $\sqrt{1}$ KL α _lepend on the rank N of the gauge group N also used *s*=0 *Ms,* $\mathbf{R}^{\prime\prime}$ *j M^s* $\overline{\mathrm{d}}$ \Box N, $M_0 H_1$ H_1 M_3 , $\frac{1}{2}$ *N* $\sum_{i=1}^{N-1}$ **EXEC this to reproduce the rel** ⁄∦ \vec{k} gh *j*1 **Fed this to reproduce the all**

 $\mathbb R$ s léle pereqlie nue ve have at spicon foldeter four dimension that Nekrasovion function associated to dattesions function hassness as $\sum_{r=1}^{i=1}$ her matriced to the orien valuated in 4.2. We t and determinant representation for the four dimensional Nekrasov-to the Hitchin's system description of the relevant SW composition of section associated to the contract of section we
Strestated to the partition of the component of section function of the section of the section of the section o *I^j* = $\frac{3}{2}$ *Ms, s*=1 $\sum_{\mathbf{e}\in\mathbf{f}}M_i$ (1, $(1, 1)$ *i*=1 **1**
w dimensional Notracov. of the isomonodromy problem associated to the Hitchin's system Φ we have a spectral seconsequence we have a spectral determination second. Next as $\delta \mathbf{R}$ we have a spectral determinant representation for the four dimensional Nekrasovon fu**nction** associated to the relevant SW cheories as shown fire creation 4.2. We $I_i =$ $\tilde{s}=0$ *Ms, s*=1 \widetilde{M}_s , \cap N, $\widetilde{M}_0 = 1$, $\widetilde{M} = 1$ *i*=1 M_i (1.2) the τ -function of the isomonodromy problem associated to the Hitchin's system $\mu = 0$
 $\mu = 0$
 $\mu = 0$
 $\mu = 1$ **[KIEIHA) LEGICOE SPICE19EIY LEPPROTE-Phong,Edelstein-Mas,Edelstein-Gomez-Reino-Marino,Douglas-Shenker]**

 $\mathcal{R}_{\mathbf{S}}$ se \mathbf{R} elvent ing the $\mathrm{i} \delta W$ (2). case 2.2 FMP $\,\mathrm{SW}(2)$ case Re cheming the 160 (2).

very hard to compute by other methods!

35 The TS/ST duality [15] has led to various exact results in topological string and in spectral theory which allows us to explore all range of the couplings in both side of the duality. We denotes by unge of the couplings in Doun.side of the duality. We deflotes by
*g*_s the coupling constant of string theory and by *n* ∼ *g* — the Planck constant appearing in the
Savakious,exactivestils in tomologicalist chargeand u spectral theory side of the correspondence. **The TS/ST** duality **T₂** has led to ve ange of the couplings in both side of the quality. We denotes by
 q_s the coupling constant of string theory and by $h \sim q_s$. the Planck constant appearing in the who appoint came us to explore all range of the duality of the duality of the duality of the duality of the du
Bectral theory side of the correspondence. **gave algebra string the couplings in both side of the duality. We denotes by** $\frac{1}{2}$ We test that the TS/ST duality $\prod_{i=1}^{\infty}$ has led to various exact results in topological string and in spectral theory **Klumes** in both side of the quality. We denotes by $\hbar \sim a$, the Planck constant appearing in $\frac{1}{2}$ the $\frac{1}{2}$ determining $\frac{1}{2}$ has led to various exact results in topological string and in spectral theory to explore all range of the couplings in both side of the duality. We denotes by **g**
gradital decoupling the computer of the constant of the string and appearing in the property is a search of the constant and appearing in the property side of the correspondence. Ω of α and α and α the correction of the correc ittyhl**e** 5,544620 ctaste tanh as 186 to $\frac{3}{2}$ and $\frac{1}{2}$ cannot results in topological strong and in spectral theory ω algorithm algorithm and ω in both side of the duality. We denotes by **We test in the 15/51 quality 19 has led to various exact results in topological stri**
Let the SW as 20 degs existing and the Symbol of the Symbol and in spectral theory **EQUILES IN DOULE SIDE OF THE GRAILLY. WE DENOTES DY** Pu**lts ar toppel Pelgad Steemstand— upsjeertna**l tii

Perturbing the massless monopole point If we can entry the massive *i*=1 *i,j* 2 cosh ⇣ *^xix^j* ² + i⇡*,*

 $a_D^{\iota} = 0$ massless monopole point *D* 6.40, (4.30) *D* 6.40, (4.30) *A* 6.40, ($\tilde{\mathbf{r}}$ matrix model computes the following kernel traces of t

¹

gs cosh(*xi*)

One cut matrix model, with e.g. $a_D^+ \neq 0$ K(*x, y*)=e cosh(*x*)*/*2*g^s* $e \cdot g \quad a_D^1 \neq 0$

Y

$$
Z_{4d}^{(1)}(M) = \frac{1}{M!} \int \frac{d^M x}{(2\pi)^M} \prod_{i=1}^M e^{-\frac{1}{g_s} \cosh(x_i)} \frac{\prod_{i < j} 4 \sinh\left(\frac{x_i - x_j}{2}\right)^2}{\prod_{i,j} 2 \cosh\left(\frac{x_i - x_j}{2} + i\pi\beta\right)}, \quad \beta = \frac{2 - N}{2N}.
$$

² + i⇡*,*

i=1

i<j 4 sinh ⇣ *^xix^j*

2

2 *N*

⌘e cosh(*y*)*/*2*g^s , x, y* ² ^R*,* (4.32)

²*^N .* (4.31)

1*.* (4.36)

This matrix model computes the fermionic spectral traces of the following kernel σ deposition models of the fermionic spectral traces of the following kernel of the following kernel of the following kernel fermionic spectral traces of 2*S^M* \sim 101111101110 opoutier trace

Z d*Mx*

Z(1)

namely

where

4d (*M*) = ¹

$$
K(x, y) = e^{-\cosh(x)/2g_s} \frac{1}{4\pi \cosh\left(\frac{x-y}{2} - i\pi \frac{(N-2)}{2N}\right)} e^{-\cosh(y)/2g_s}, \quad x, y \in \mathbb{R},
$$

spec tisfies the equation \overline{y} spectral determinant satisfies the equations of A_{N-1} Toda chain: *K*(*xi, x*(*i*))*.* (4.33)

$$
q_\ell^{''}+\frac{1}{t}q_\ell^{'}= {\rm e}^{q_\ell-q_{\ell-1}}-{\rm e}^{q_{\ell+1}-q_\ell},
$$

q is the connection of the team in the team of the subset of the connection

ar with regular singularities at 0 and ∞ [Cecotti-Vafa, Guest-Its-Lin]. i onds to **isomonodromic deformation** \overline{C} *t* rities at 0 and ∞ [Cecotti-Vafa, Guest-I $\frac{1}{2}$ $\$ $1.5L(11, 8)$ ¹¹. (4.436)¹¹. (4.436)¹ corresponds to **isomonodromic deformations** of $SL(N,\mathbb{C})$ flat connection on the cylinder with regular singularities at 0 and ∞ [Cecotti-Vafa, Guest-Its-Lin].

q-Painleve', five dimensional gauge theories and topological strings

The 4d Renormalization Group diagram is part of a bigger picture which comes from the embedding of gauge theories in string theory.

This also has a counterpart in Painleve' theory by going to the **multiplicative** case.

$SU(2)$ five dimensional gauge theories with N_f fundamental hypers

are relevant deformations in the IR of strongly coupled SCFTs with exceptional global symmetries

Seiberg's classification:

$$
E_8 \to E_7 \to E_6 \to E_5 \to E_4 \to E_3 \to E_2 \to E_1
$$

 $\mathbf{N}_{\mathbf{f}}$ from 7 to 0 via holomorphic decoupling of masses

$$
\qquad \qquad \text{reduction to four dimensions:} \qquad \qquad \mathbb{R}
$$

$$
\mathbb{R}^4\times S^1
$$

$$
S^1
$$
 - radius going to zero

Sakai's classification of q-Painleve' :

• higher rank/genus isomonodromy ?

$$
E_8^{(1)} \rightarrow E_7^{(1)} \rightarrow E_6^{(1)} \rightarrow D_5^{(1)} \rightarrow A_4^{(1)} \rightarrow (A_1 + A_2)^{(1)} \rightarrow (A_1 + A_1)^{(1)} \rightarrow A_1^{(1)} \rightarrow A_1^{(1)} \rightarrow A_2^{(1)} \rightarrow A_2^{(1)} \rightarrow A_3^{(1)} \rightarrow A_4^{(1)} \rightarrow A_2^{(1)} \rightarrow A_1^{(1)} \rightarrow A_2^{(1)} \rightarrow A_
$$

of these theories are expected to be related to the solutions of the discrete Painlevé **symmetry group** of the corresponding 5d/4d gauge theory. Backlund symmetries table: Weyl group of the **affinization** of **flavour**

 $\mathbb{Z}L^2(\Omega)$ function \mathbb{Z}^4 theory. It would be interesting be interesting be interesting be interesting by interesting \mathbb{Z}^4 σ (2) yaago thoones of **by topological strings** on local delli ezzo sunaces. $SU(2)$ gauge theories on $\mathbb{R}^+ \times S^+$ are geometrically engineered by **topological strings** on local del Pezzo surfaces.

e.g. up to $N_f = 4$:

topological strings on local Hirzebruch and their blow-ups provide conjectural tau-functions for q-Paihleve' $\rm III_2$

for toric geometries non-perturbative topological string would provide a candidate *spectral determinant* for those tau-functions.

IV IIFN **it works for q-PIII3 tau function** [Bonelli-Grassi-A.T., to appear]

Figure 1. In the upper line we have a list of polyhedra representing del Pezzo surfaces *S* [53, 54]

connected to the coalescence diagram of Painlev´e equations through an arrow. The total space of the

Concluding remarks

gauge theory partition functions/BPS correlators and isomonodromy: window on strongly coupled sectors of gauge and string theories quantum statistical systems capturing self-dual Omega background a new class of matrix models for magnetic phase

Some open problems

direct derivation of matrix model from gauge theory

including matter fields

breaking to N=1, confinement and condensates

non-perturbative string: frame dependence, relation to isomonodromy

THANKS!!