The origin of nonextensivity and nonadditivity in thermostatistics

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Dedicated to Tamás Biró, on the occasion of his birthday.

What is thermal?

thermodynamics $\neq$ statistical physics

Thermal probability distributions - usually

Boltzmann-factor: $f \sim e^{-\frac{E}{T}}$

Maxwell-Boltzmann–distribution: $f_{MB}(p) = Z^{-1} e^{-\frac{p^2}{2mT}}$

Thermodynamical process

Fast internal equilibration:

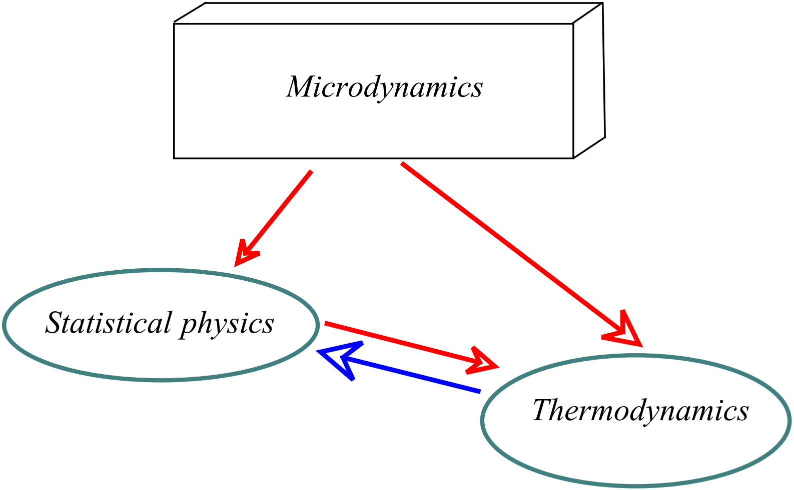
- homogenisation,
- changing thermal parameters of the momentum distribution.

$$f_1(p) = Z^{-1}(T(t))e^{-\frac{p^2}{2mT(t)}}$$

Central problem

thermodynamics AND statistical physics AND microdynamics

What is thermal?



Information entropy

History I

- Conditions leading to a definition of entropy for a probability distribution [Shannon, 1948]:

$$\exists! \quad S_S(P) := -\sum_{k=1}^{n} p_k \ln p_k .$$

  Additivity: $S(P * Q) = S(P) + S(Q)$

- Lemma of [Erdős, 1946] (and Fadeev and Rényi):
  If $f(nm) = f(n) + f(m)$, $n \in \mathbb{Z}$, *and* $\lim_{n \to +\infty}(f(n+1) - f(n)) = 0$, *then* $f(n) = c \ln(n)$.

- Complete conditions, incomplete distributions [Rényi 1961]:

$$\exists! \quad S_R(P) := \frac{1}{1-\alpha} \ln \left(\sum_{k=1}^{n} p_k^{\alpha} \right) .$$

  Generalized averages: $S = L^{-1} \sum_{k=1}^{n} p_k L(s)$

Information entropy and physical entropy

History II

- Separation of dynamical and static properties [Jaynes, 1957]. Additive entropy is unique and exists, therefore it is a primitive concept. There is no need of microdynamics. Extra conditions, canonical distribution.
- Entropy for power law distributions (Pareto, Zipf, ..) [Tsallis, 1988]. Fractals?

$$S_T(P) := \frac{1 - \sum_{k=1}^{n} p_k^q}{q-1} = \sum_{k=1}^{n} p_k \frac{1 - p_k^{q-1}}{q-1} = \frac{1 - e^{(1-q)S_R}}{q-1}.$$

  Additivity with Rényi.
  Subadditivity $S(P * Q) = S(P) + S(Q) + (1-q)S(P)S(Q)$
  Extensivity $\lim_{N\to\infty} S/N < \infty$. There is a classical density.

Tsallis is not additive but c-extensive. Rényi is additive but not c-extensive.

Composition rules for correlated systems?

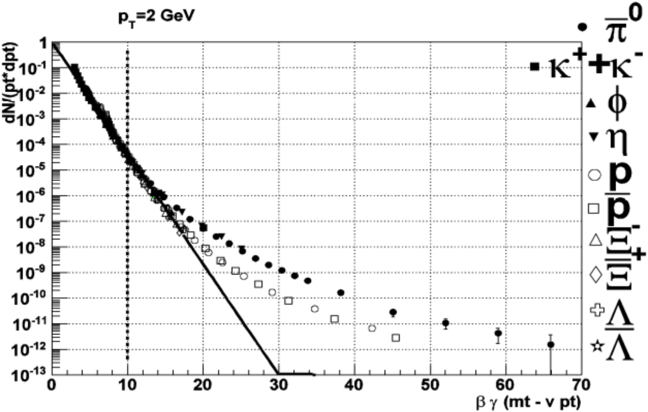
Requirements of a physical entropy.

History III. The entropy boom : 1988-2004

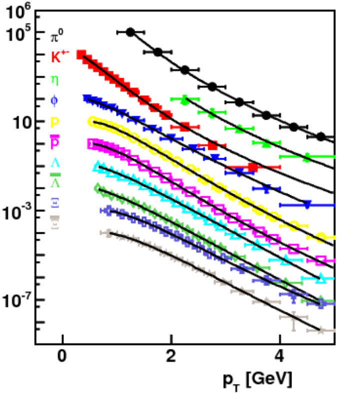
- Curve fitting : Tsallis everywhere.
- Many entropies (Landsberg-Vedral, Kaniadakis, etc..). Modifications in averages [Curado and Tsallis 1992, Tsallis et al. 1998].
- Further concepts : fluctuating temperature [Wilk and Wlodarczik 2000], that is superstatistics [Beck and Cohen 2003].
- Problems with generalized entropies [Nauenberg, 2003] :
  - Origin : static or dynamic ? Equilibrium or irreversible ? Fractals ? Long range forces ?
  - Which one is more physical ? Tsallis or Rényi ? Is there one above all ? Criteria : concavity, Lesche stability, etc..
  - Temperature : does your finger burned or not ?
  - Composition rules for other physical quantities ?
  - What is the physics of q ? How could it be derived ?

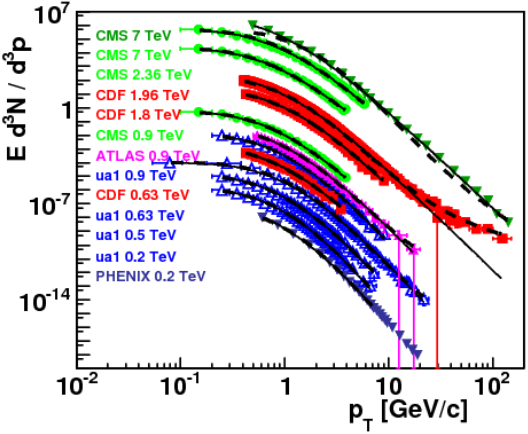
Existence. Physics of data evaluation.

TSB et al, J. Phys. G 37, 094027, (2010).



Domain. Where do you find?

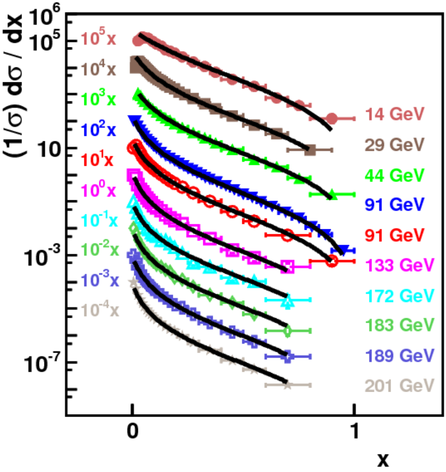




AuAu 200GeV,
KÜ and TSB, PLB 689: 14-17, (2010)

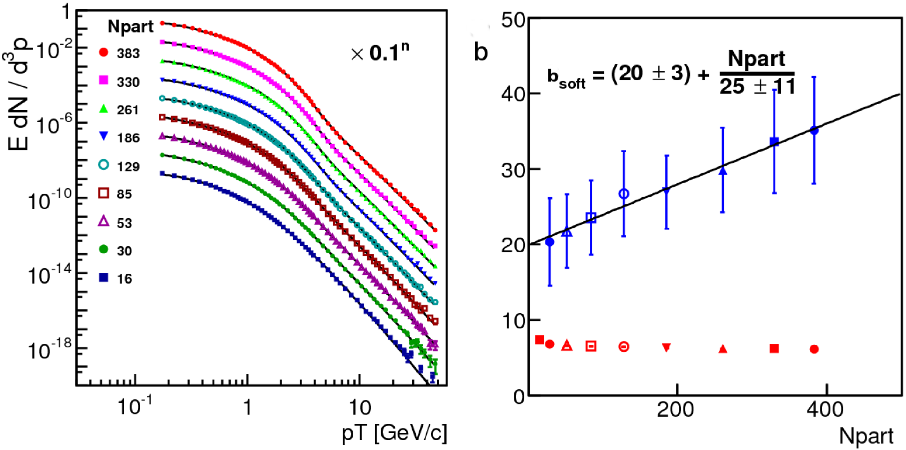
pp 0.2-t TeV
KÜ and TSB, JoP CS, 270 (2011) 012008

Domain. Microcanonical?



$e^- e^+$, 14-200 GeV
KÜ, GGB and TSB, PLB 701: 111-116 (2011)

Both together?



Pb-Pb, 2.76TeV

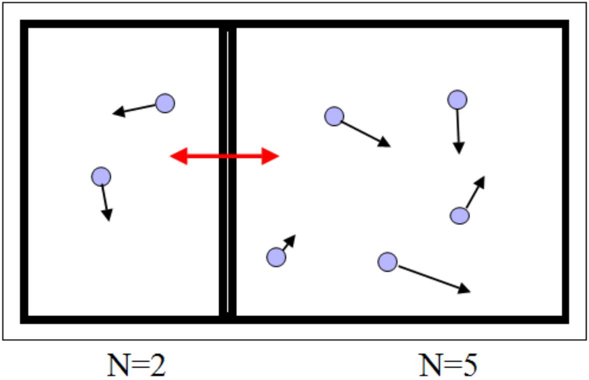
GGB, KÜ and TSB, JoP CS 612 (2015) 012048

Presentations of Takács, Ürmössy, etc...

Unique entropies? How strong is mathematics?

Microcanonical idea...

Insulated. Total energy is exactly constant.



Conditionally maximal entropy

$$\delta_f \left(S(f) - \beta(\langle E_1 \rangle - \bar{E}) - \alpha(\langle 1 \rangle - 1) \right) = 0$$

Tsallis:

$$S_T(f) = \frac{1}{1-q} \int (f^q - f) d\Omega$$

non additive, linear.

Rényi:

$$S_R(f) = \frac{1}{1-q} \log \int f^q d\Omega$$

additive, nonlinear

Formal logarithm:

$$S_R = L(S_T) = \tfrac{1}{1-q} \log \left(1 + (1-q) S_T\right)$$

Canonical equilibrium distributions:

$$f_*(E_1) = Z^{-1} \left(1 + (1-q)\beta_e E_1\right)^{\frac{1}{q-1}}$$

Classical ideal gas, microcanonical distribution.

Conditions:

N particles. Uniform distribution on the hypersphere, constant total energy.

One particle distribution function:

$$f_{mk}(E_1) = Z^{-1}\left(1 - a\beta_e E_1\right)^{\frac{1}{a}}$$

1. $E_1 = \frac{p^2}{2m}$
2. $a = \frac{2}{3N-4} = \frac{1}{C}$, C – heat capacity
3. $\beta_e = \frac{1}{T_e} = \frac{3N-4}{2E}$
4. $Z^{-1} = (1+a)\beta_e$

This is EXACTLY a Tsallis-Pareto distribution: $q = 1 + a$.
In particular

$$\lim_{a \to 0} f_{mk} = f_{MB}$$

E=const. correlations from constraints.

TSB, Physica A 392.15 (2013): 3132-3139.

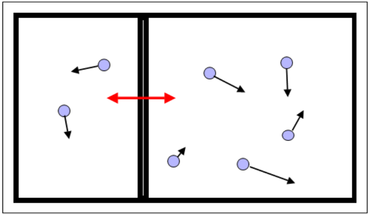
Thermal and microcanonical? Temperature concepts.

Lagrange multiplier (β) or equilibrium distribution or equipartition or entropy derivative.

Entropy	β_e	$1/\langle E_1 \rangle$	$\beta_t = \frac{\partial S}{\partial \langle E_1 \rangle}$
Boltzmann-Gibbs	β	β	β
Tsallis	$\frac{\beta^{1/q}}{q}$	$\frac{\beta^{1/q}(2q-1)}{q}$	$\beta = \left(\frac{3N-2}{2E}\right)^{\frac{3N-2}{3N-4}}$
Rényi	$\frac{\beta}{2q-1}$	β	$\beta = \frac{3N}{2E}$

$$f_*(E_1) = Z^{-1} \left(1 + (1-q)\beta_e E_1\right)^{\frac{1}{q-1}}$$

Internal equilibration



$$T_1 = \frac{2E_1}{3N_1} = T_2 = \frac{2E_2}{3N_2} = \frac{2(E - E_1)}{3N_2} \quad \rightarrow \quad E_1 = \frac{N_1}{N_1 + N_2} E$$

$$T_{microcanonical} = \frac{2E}{3(N_1 + N_2)}$$

Wall can be removed. Correlation is preserved. Equipartition.

0th law of thermodynamics

Transitivity and separability:
Separated but correlated bodies. Composition formula:

$$\exists S_{12}(S_1, S_2) \neq S_1 + S_2$$

$$E_1 + E_2 = const.$$

Maximal entropy:

$$dS_{12} = \frac{dS_{12}}{dE_1} dE_1 + \frac{dS_{12}}{dE_2} dE_2 = \left(\frac{dS_{12}}{dE_1} - \frac{dS_{12}}{dE_2} \right) dE_1 = 0$$

$$\frac{dS_{12}}{dE_1}(E_1, E_2) = \frac{dS_{12}}{dE_2}(E_1, E_2)$$

Formal logarithm:

$$\rightarrow \exists L, L_1, L_2 : \quad L(S_{12}) = L_1(S_1) + L_2(S_2)$$

Tsallis subadditivity $\rightarrow$ Rényi entropy.

TSB and PV, PRE, 83:061147 (2011); +et al, JoP CS, 394:012002 (2012)

Derivation by deviation

Thermometer - E_1 and heat reservoir - E_R. Insulated system: $E = E_1 + E_R$

Formal logarithm

$$L(S(E_1, E_R)) = L_1(S_1(E_1)) + L_R(S_R(E_R))$$

Zeroth law.

Maximal entropy, $L = L_1 = L_2$, $E_1 < E_R$

$$d_{E_1} L(S(E_1, E - E_1)) = L'(S(E_1))S'(E_1) - L'(S(E - E_1))S'(E - E_1) = 0$$

$$\beta_1 = L'(S(E_1))S'(E_1) = L'(S(E))S'(E) + \left[\underline{L''(S(E))S'^2(E) + L'(S(E))S''(E)}\right] E_1 + o({E_1}^2)$$

Reservoir condition

$$\boxed{\frac{L''}{L'}(S) = -\frac{S''}{S'^2}(E) = \frac{1}{C} = a = 1 - q}$$

UTI = Universal Thermostat Independence.

C = constant

$$L(S) = \frac{e^{aS} - 1}{a}, \qquad S(L) = \frac{1}{a}\log(1 + aL)$$

Generalized average: micro corrected density

$$S_T = L(S_R) = \sum (p_i \, L(-\log p_i))$$

- Ideal gas reservoir: Tsallis-Rényi entropy.
- EOS dependent entropy.

Reservoir EOS – entropy generator.

TSB, GGB and PV, *EPJA* 49:110 (2013).

Summary

- Additivity, extensivity and separability: more than thermodynamics.
- Unique entropies: Shannon and Rényi.
- Zeroth law: additivity is enforced by separability and thermodynamics.
- Thermostated temperature, reservoir driven averaging (Universal Thermostat Independence): EOS generated formal logarithm.
- Entropy generator principle. For ideal gas it leads to Tsallis-Rényi. Exactly.
- ARC = Additivity Restoration Condition.

TSB driven research line of thermostatistics

- Multiplicative noise: an origin of q [1],
- Generalized Boltzmann: q equilibration [2],
- Associative composition rules [3],
- p_T spectrum analysis [...],
- Zeroth law, formal logarithm [4],
- Ideal gas reservoir: Tsallis-Rényi entropies [5],
- Entropies for correlated systems (UTI principel) [6],
- Finite energy, $C > 0$ and temperature fluctuations [7] $\hat{q} = 1 - \frac{1}{C} + \frac{\Delta T^2}{T^2}$,
- Additivity Restoration Condition (ARC), $s_B = \ln(1 - \ln p_k)$: new entropy, eq. dist. Lambert-W function, e.g. large earthquakes [8],
- Nonadditive corrections to hydrodynamics [9],
- Black holes with Rényi entropy [10],
- Lattice QCD with generalized (q) thermodynamics,
- q-modified Fermi-Dirac [11],
- NBD, squeezed coherent states, network growth, financial markets, divergence measures,...

Thank you for the attention!

References

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History is the continuation of the expected future.

Microcanonical entropies and temperatures

The distribution is uniform and given. Entropies are discussed.

Boltzmann:

$$S_B = \log \omega(E)$$

state density:

$$\omega(E) = \int dx \delta(E - H(x))$$

$$\beta_B = \frac{\partial S_B}{\partial E} = \frac{3N-2}{2E}$$

Gibbs:

$$S_G = \log \Omega(E)$$

integrated state density:

$$\Omega(E) = \int \omega(E) dE$$

$$\beta_G = \frac{\partial S_G}{\partial E} = \frac{3N}{2E}$$

- There is no negative temperature [Dunkel and Hilbert 2013].
- Carnot efficiency cannot be improved.