Transport coefficients in QP-approximation and beyond

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Things to learn from Tamás

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simplest approach → complificate (just enough) → **still solvable!**

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random processes with multiplicative noise

PRL **94,** 132302 (2005)

finite-size thermo effects in a thermostat-independent manner

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Entropy **16**(12) 6497 (2014)
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generalized addition rules in thermodynamics

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mass-distributed ideal gas

PRC **75,** 034910 (2007)

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Outline

- **Motivation**
  thermo. & non-perturbative effects
  thermal QCD excitations
  fluent SIM

- **What is η?**
  kinetic theory point-of-view
  hydro. & linear response

- **A qualitative picture on fluidity**
  a possible extension of the QP-picture
  high- & low-T fluidity, liquid-gas crossover
  lower bound?

QCD near T_c

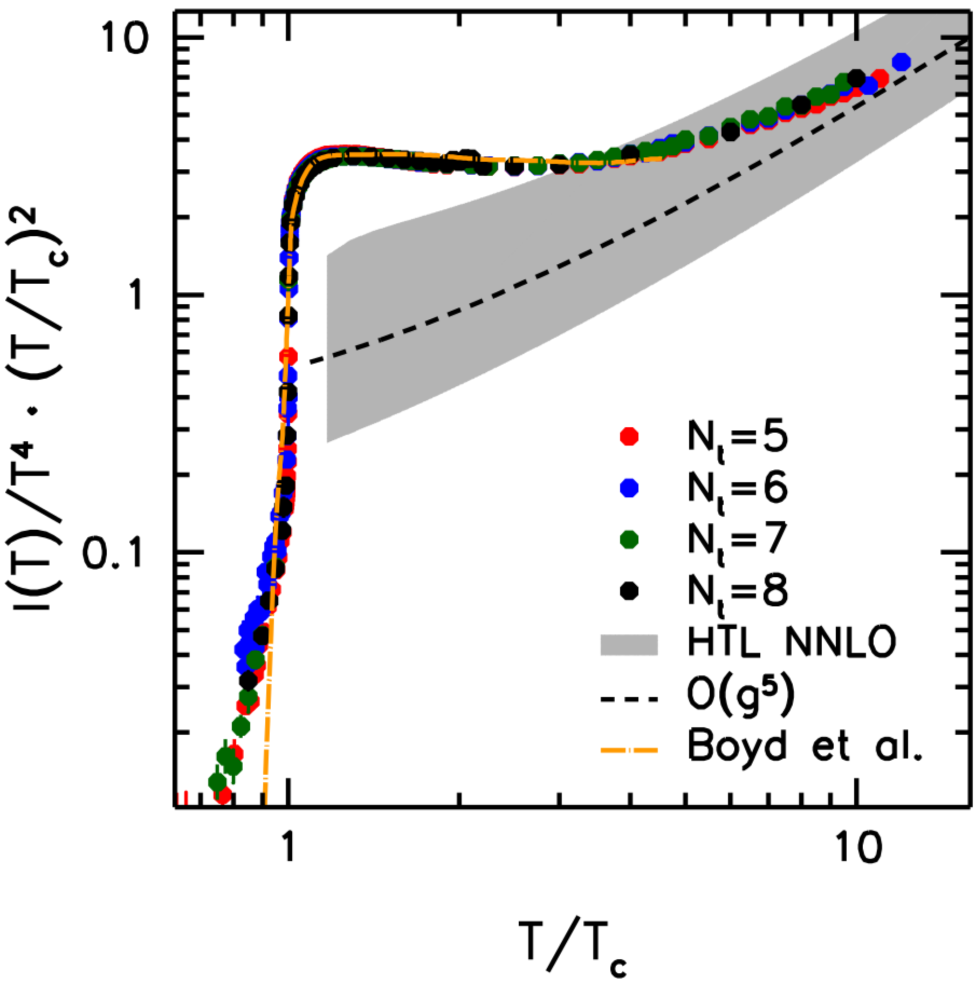
$$I = \varepsilon - 3P \sim \#_1 T^4 + \#_2 T_c^2 T^2$$
$(T_c \ll T)$

note:

$$P_{\text{id. gas}} \sim T^4 - \frac{m^2}{4} T^2 + \mathcal{O}(T^0)$$

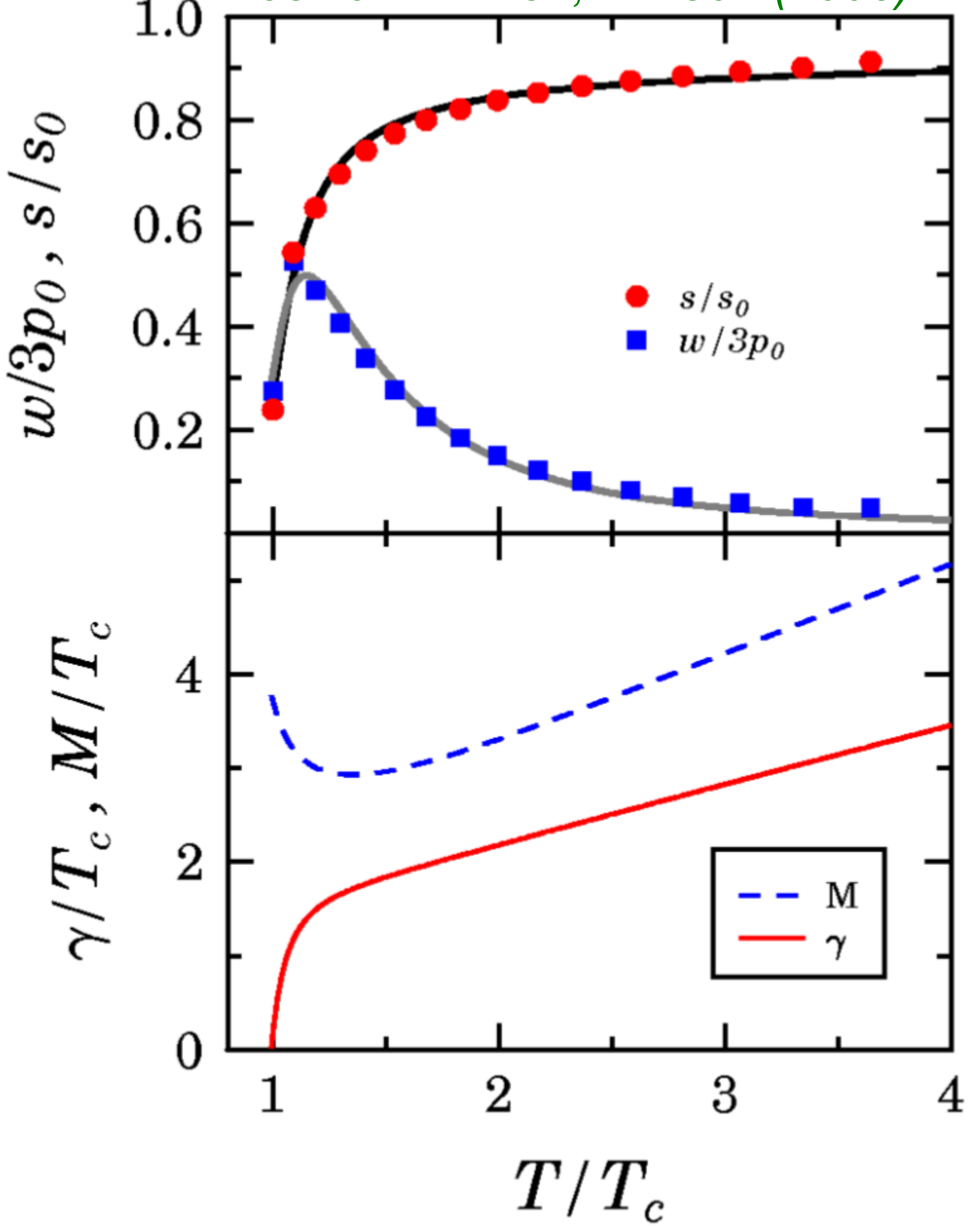
$$\frac{I_{\text{id. gas}}}{T^2} \sim m^2 + \mathcal{O}(T^{-2})$$

Borsányi etal. (2012): arXiv: 1204.6184



QCD near T_c

*Peshier: PRL **94**, 172301 (2005)*



quasiparticle like thermal excitations from 2PI treatment of field theory

$$M^2 = \frac{N_c}{6} g^2 T^2$$

$$\gamma = \frac{3}{4\pi} \frac{M^2}{T^2} T \log \frac{c}{(M/T)^2}$$

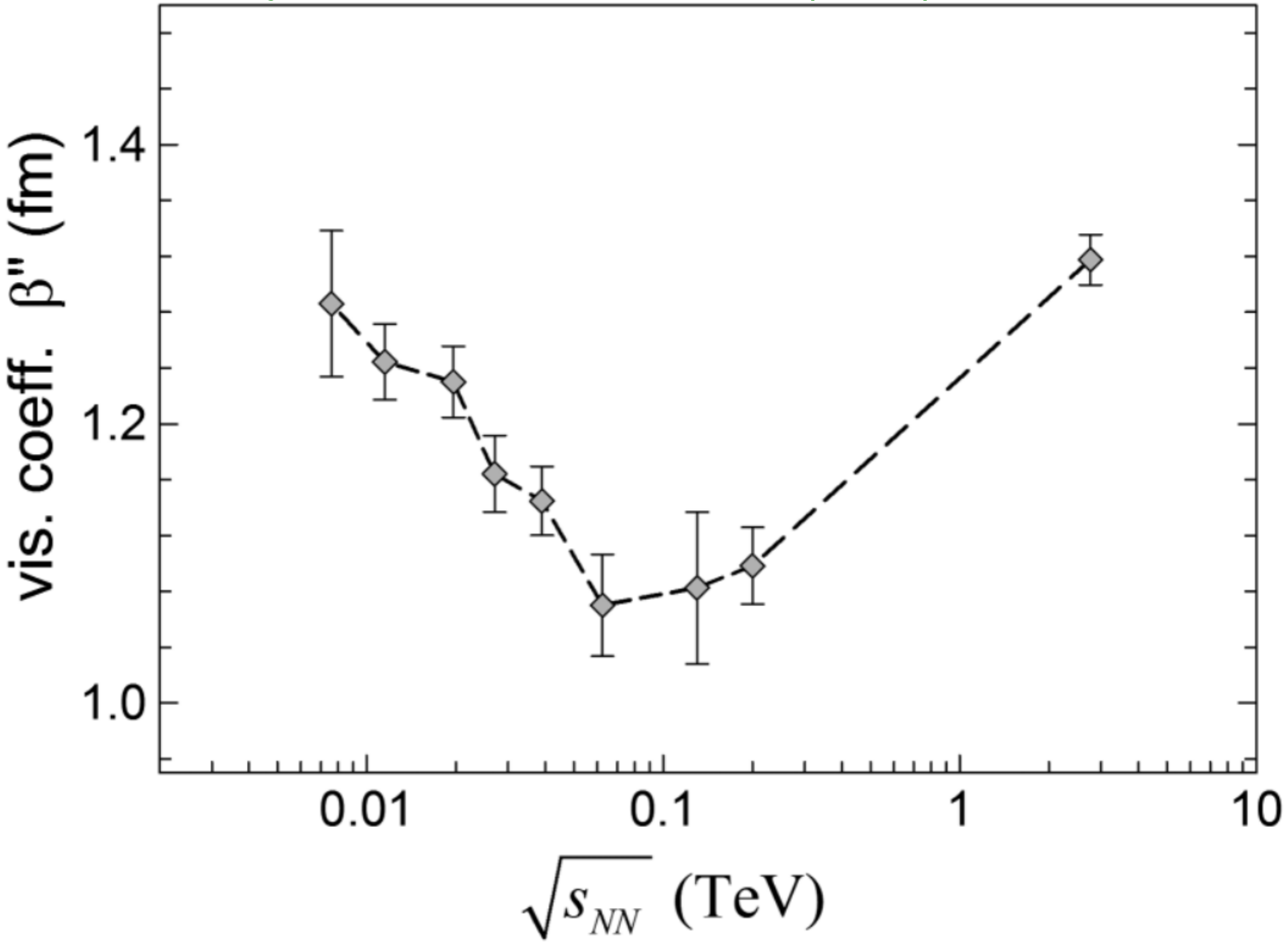
$$g^2(T) = \frac{48\pi^2}{11 N_c \log(\lambda(T - T_s)/T_c)^2}$$

Fluidity of SIM

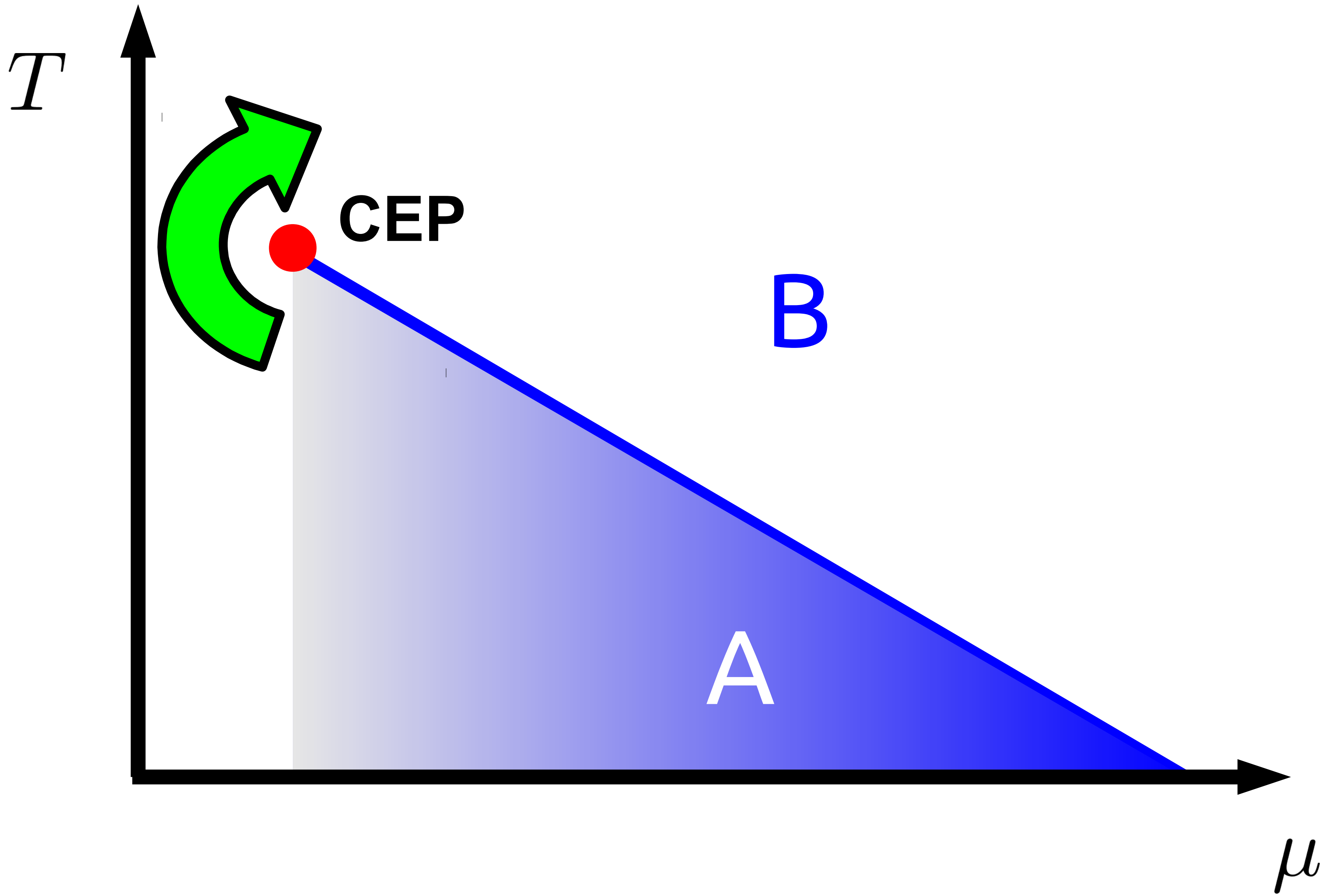
system size dependence of $v_{2,3}$ in HIC
translated into viscous damping

$$\beta'' \propto \frac{\eta}{s}$$

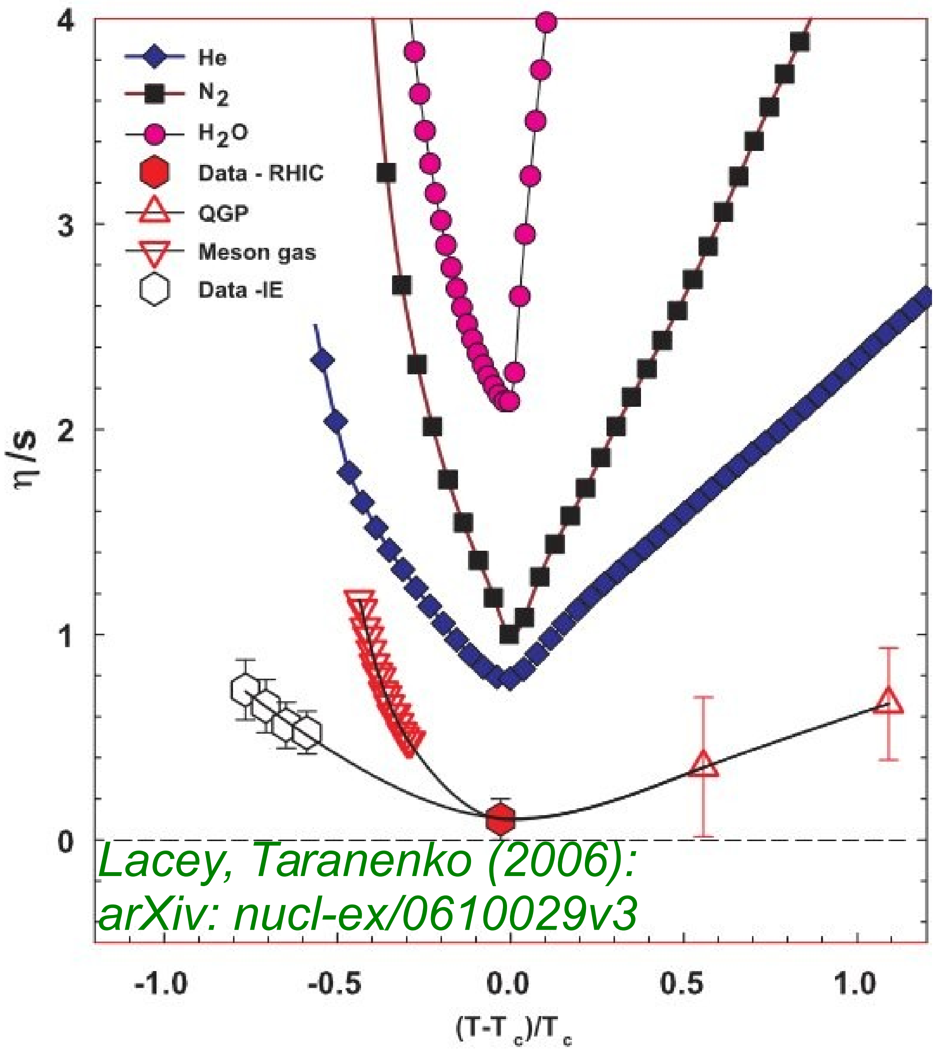
*Lacey etal.: PRL **112**, 082302 (2014)*



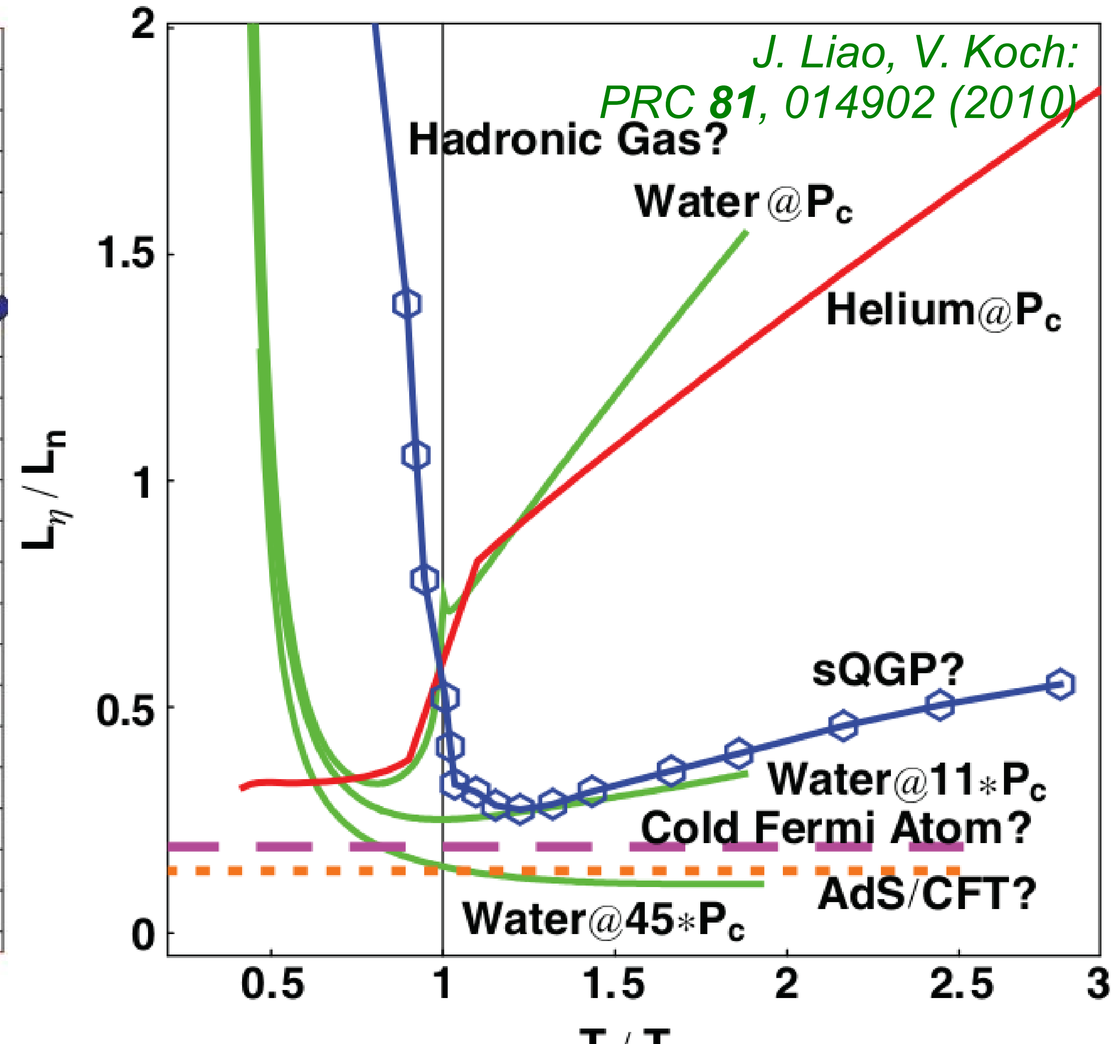
Near-critical behaviour



Measure of fluidity

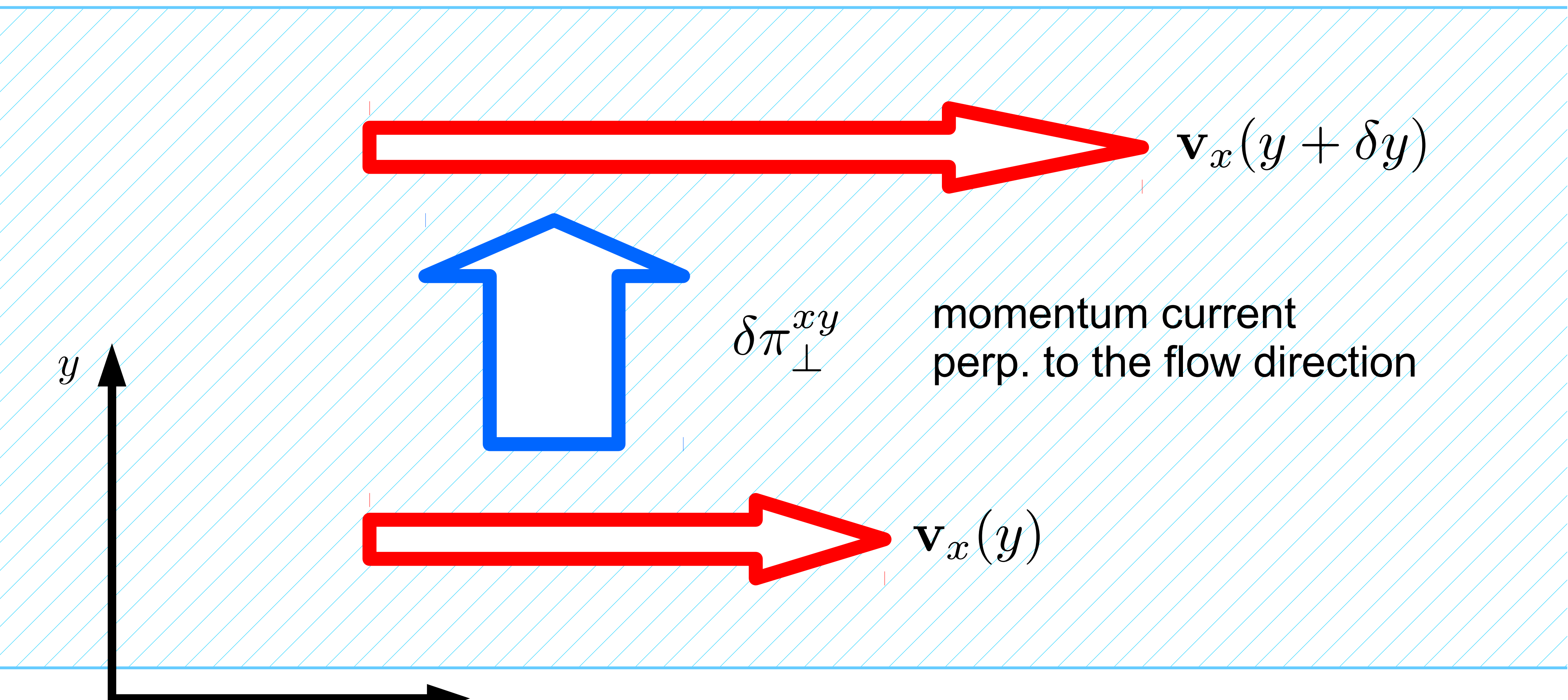


$$\delta\pi_{\perp}^{ij}(k,t) = \exp\left(-\frac{\eta}{s}\frac{k^2 t}{T}\right) \cdot \delta\pi_{\perp}^{ij}(k,0)$$



$$\frac{L_\eta}{L_n} = \frac{n^{\frac{1}{3}}\eta}{c_s\rho} \sim \frac{\text{min. wavelength to propagate}}{\text{inner scale, prop. to } n^{-1/3}}$$

Shear viscosity



$$\rho_m \partial_t \mathbf{v}_x \sim \eta \partial_y^2 \mathbf{v}_x \sim -\partial_y \pi_\perp^{xy}$$

$$\eta \sim \rho_m \frac{(\text{characteristic mom. transfer length scale})^2}{\text{char. mom. transfer time scale}} \sim \rho_m \lambda \langle p \rangle$$

Shear viscosity

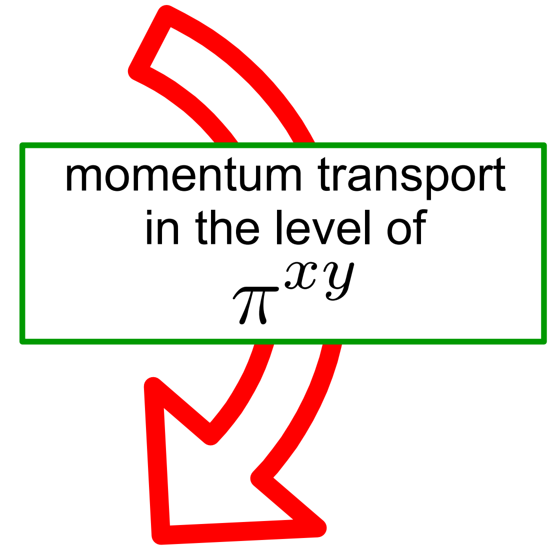
kinetic theory
(at simplest)

$$\eta \sim \rho_m \langle p \rangle \lambda \sim \frac{\langle p \rangle}{\sigma}$$

relaxation time approximation

$$\eta \sim \frac{1}{T} \int \mathrm{d}^3\mathbf{p}\,\tau(\mathbf{p}) \frac{\mathbf{p}^4}{E(\mathbf{p})} f_{\mathrm{eq.}}(E(\mathbf{p}/T))(1 + f_{\mathrm{eq.}}(E(\mathbf{p})/T))$$

possible order-by-order improvement (CE)



moving beyond quasiparticles:
linear response

$$\eta \sim \frac{1}{T} \int_0^\infty \mathrm{d}\omega \int \mathrm{d}^3\mathbf{p} \frac{\mathbf{p}^4}{\omega^2} \left(\frac{\partial K(\omega, \mathbf{p})}{\partial \omega} \rho(\omega, \mathbf{p}) \right) f_{\mathrm{eq.}}(\omega/T)(1 + f_{\mathrm{eq.}}(\omega/T))$$

η/s in hydrodynamics

See Kovtun: J. Phys. A **45,** 473001 (2012)
arXiv: 1205.5040

$$\partial_\nu T^{\mu\nu} = 0$$

$$(\partial_\mu J^\mu = 0, \ldots)$$

Hydro = evolution of energy-momentum (and other conserved charges)

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu + P\Delta^{\mu\nu} \underbrace{- \eta \Delta^{\mu\alpha}\Delta^{\nu\beta}\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{3} g_{\alpha\beta}\partial_\sigma u^\sigma\right) - \zeta\Delta^{\mu\nu}\partial_\sigma u^\sigma}_{\pi^{\mu\nu}} + \mathcal{O}(\partial^2)$$

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$$

linearization near the static equilibrium

$$\mathbf{k} || x$$

$$\partial_t \delta\varepsilon + i k_x T^{0x} = 0$$

$$\partial_t (T^{0x})^{||} + i k_x v_s^2 \delta\varepsilon + \frac{\frac{4}{3}\eta + \zeta}{s}\mathbf{k}^2 T^{0x} = 0$$

$$\partial_t (T^{0i})^\perp + \frac{\eta}{s}\mathbf{k}^2 (T^{0i})^\perp = 0$$

η in linear response

See Kovtun: J. Phys. A ***45,*** *473001 (2012)*
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perturbation in ***en-mom. current*** $\delta H = (u_\mu - \overline{u}_\mu) T^{0\mu}$

change of avr. ***transverse stress tensor*** to linear order in the strength of the perturbation $h^{ij} = \partial_i u_j + \partial_j u_i - \frac{1}{3}\delta_{ij}\partial_k u^k$

$$\delta\langle\pi^{ij}\rangle \sim h^{ij}(\omega, \mathbf{k}) \int_{-\infty}^{\infty} \mathrm{d}\widetilde{\omega} \frac{\rho_{\pi\pi}(\widetilde{\omega}, \mathbf{k})}{(\widetilde{\omega} - \omega + i\epsilon)^2}\Bigg|_{\epsilon\to 0} \to h^{ij}(k=0) \lim_{\omega\to 0} \frac{\rho_{\pi\pi}(\omega, \mathbf{k}=0)}{\omega}$$

$$\rho_{\pi\pi} = \left\langle \left[\pi^{kl}(\omega, \mathbf{k}), \pi_{kl}(x=0)\right] \right\rangle$$

η in linear response

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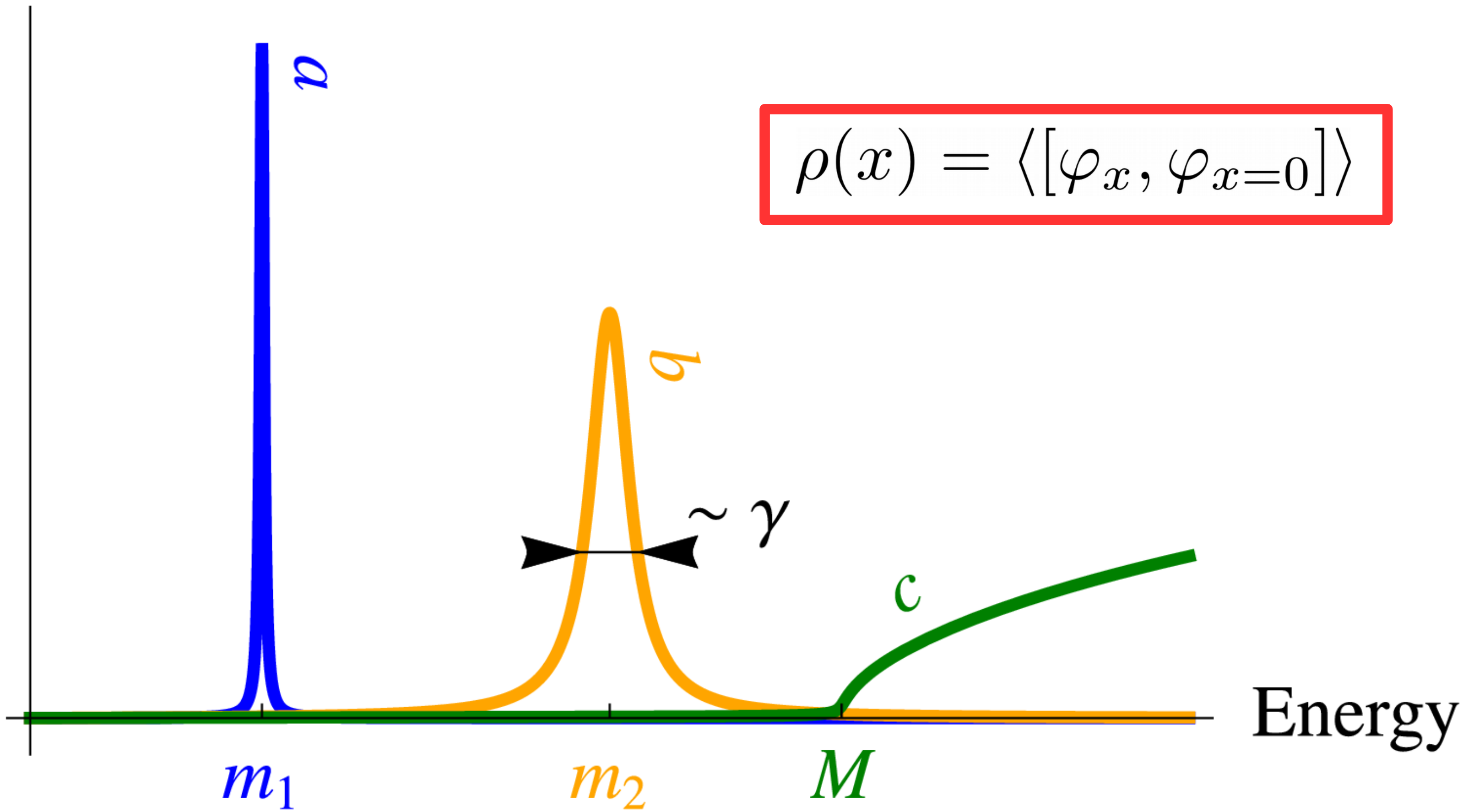
η

$$\rho_{\pi\pi} = \left\langle \left[\pi^{kl}(\omega, \mathbf{k}), \pi_{kl}(x=0) \right] \right\rangle$$

Possible extension of QPs

Parametrization: spectral function(s) =

density of quantum states as a function of the energy when other quantum numbers are fixed



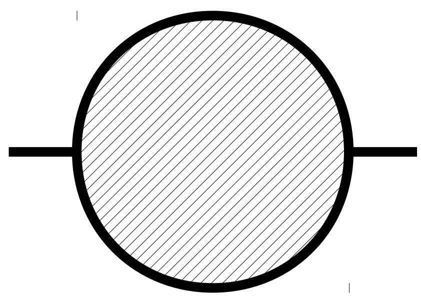
$$\rho(x) = \langle [\varphi_x, \varphi_{x=0}] \rangle$$

$$S_{\text{eff}}[\varphi] = \int_x \int_y \varphi_x \mathcal{K}_{x-y} \varphi_y + \int_x J_x \varphi_x$$

$$K^{-1}(p^0, \mathbf{p}) = P \int_{-\infty}^{\infty} \mathrm{d}\omega \frac{\rho(\omega, \mathbf{p})}{p^0 - \omega}$$

What is self-energy?

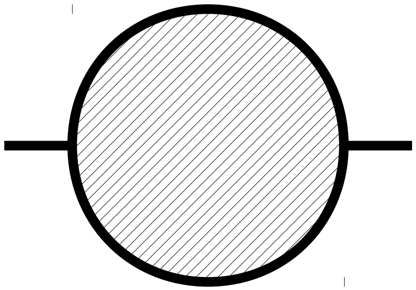
$$\rho(\omega, \mathbf{p}) = \frac{\mathrm{Im}\Sigma(\omega, \mathbf{p})}{(\omega^2 - \mathbf{p}^2 - \mathrm{Re}\Sigma(\omega, \mathbf{p}))^2 + \mathrm{Im}\Sigma^2(\omega, \mathbf{p})}$$

$\Sigma \sim$  = sum of ***all*** quantum corrections to the 2-point function (sum of 1-PI irreducible graphs)

What is self-energy?

$$\rho(\omega,\mathbf{p}) = \frac{\mathrm{Im}\Sigma(\omega,\mathbf{p})}{(\omega^2-\mathbf{p}^2-\mathrm{Re}\Sigma(\omega,\mathbf{p}))^2+\mathrm{Im}\Sigma^2(\omega,\mathbf{p})}$$

$\mathrm{Im}\Sigma \to 0$ $\quad$ $\mathrm{Re}\Sigma \to m^2$ $\quad$ $\rho(\omega,\mathbf{p}) \to \delta(\omega^2-\mathbf{p}^2-m^2)$

$\Sigma \sim$  $=$ sum of ***all*** quantum corrections to the 2-point function (sum of 1-PI irreducible graphs)

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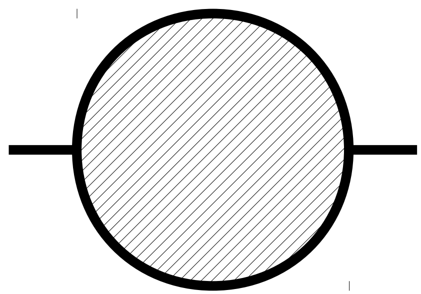
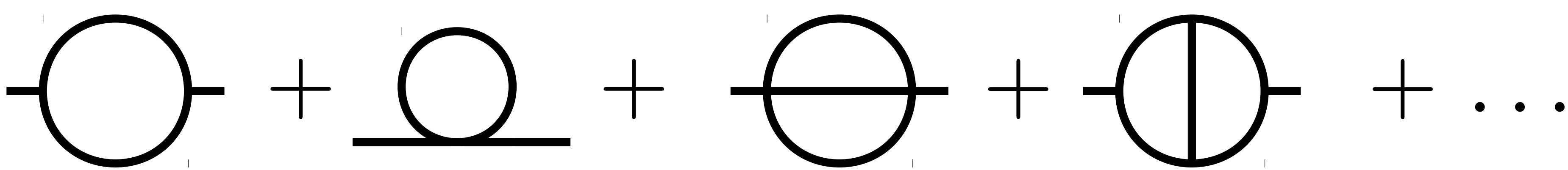
$\mathrm{Im}\Sigma \to 0 \quad \mathrm{Re}\Sigma \to m^2 \quad \rho(\omega, \mathbf{p}) \to \delta(\omega^2 - \mathbf{p}^2 - m^2)$

$\Sigma \sim$



= sum of ***all*** quantum corrections to the 2-point function (sum of 1-PI irreducible graphs)

think of:



EQP: thermo. and transport

thermodynamic quantity: entropy density

$$s = \int_0^\infty \mathrm{d}\omega \int \mathrm{d}^3\mathbf{p}\,\omega \boxed{\frac{\partial K(\omega,\mathbf{p})}{\partial\omega}\rho(\omega,\mathbf{p})} \frac{\chi_s(\omega/T)}{T}$$

transport coefficient: shear viscosity

$$\eta = \int_0^\infty \mathrm{d}\omega \int \mathrm{d}^3\mathbf{p}\,\frac{(p_x p_y)^2}{\omega^2} \boxed{\left(\frac{\partial K(\omega,\mathbf{p})}{\partial\omega}\rho(\omega,\mathbf{p})\right)^2} \frac{\chi_\eta(\omega/T)}{T}$$

$$\chi_s(y) = \frac{1}{e^y - 1} - \frac{\log(1-e^{-y})}{y} \qquad \chi_\eta(y) = n_{\mathrm{BE}}(y)(1+n_{\mathrm{BE}}(y)) = \frac{1}{2}\frac{1}{\cosh(y)-1}$$

Low/high-T asymptotics

$$s = \int_0^\infty \mathrm{d}\omega \int \mathrm{d}^3\mathbf{p}\,\omega \frac{\partial K(\omega,\mathbf{p})}{\partial \omega} \rho(\omega,\mathbf{p}) \frac{\chi_s(\omega/T)}{T} \sim T^3 \int_0^\infty \mathrm{d}y \int_0^\infty \mathrm{d}q\, q^2 y \frac{\partial K(Ty,Tq)}{\partial y} \rho(Ty,Tq) \chi_s(y)$$

high / low T ⇒ high / low ω asymptotics

Low/high-T asymptotics

$$s = \int_0^\infty \mathrm{d}\omega \int \mathrm{d}^3\mathbf{p}\,\omega \frac{\partial K(\omega,\mathbf{p})}{\partial\omega}\rho(\omega,\mathbf{p})\frac{\chi_s(\omega/T)}{T} \sim T^3 \int_0^\infty \mathrm{d}y \int_0^\infty \mathrm{d}q\, q^2 y \frac{\partial K(Ty,Tq)}{\partial y}\rho(Ty,Tq)\chi_s(y)$$

high / low T ⟹ high / low ω asymptotics

low T ($\omega \to 0$)

$$\mathrm{Re}\Sigma(\omega,\mathbf{p}) \approx m^2 + \mathcal{O}(\log(\omega/|\mathbf{p}|)\omega^2)$$

$$\mathrm{Im}\Sigma(\omega,\mathbf{p}) \approx \omega(\gamma + \mathcal{O}(\log(\omega/|\mathbf{p}|)\omega^2))$$

$$s \sim \begin{cases} \frac{\gamma}{m}T^3 \\ \sqrt{\frac{\gamma}{T}}T^3, \ (m=0) \end{cases}$$

$$\eta \sim \begin{cases} \frac{\gamma}{m}\frac{\gamma}{T}T^3 \\ \left(\frac{\gamma}{T}\right)^{3/2}T^3, \ (m=0) \end{cases}$$

$$\frac{\eta}{s} \sim \frac{\gamma}{T}$$

high T ($\omega \to \infty$)

$$\mathrm{Re}\Sigma(\omega,\mathbf{p}) \approx \mu^2\omega^2 + M^2 + \mathcal{O}(\log(\omega/|\mathbf{p}|)/\omega,\ 1/\omega)$$

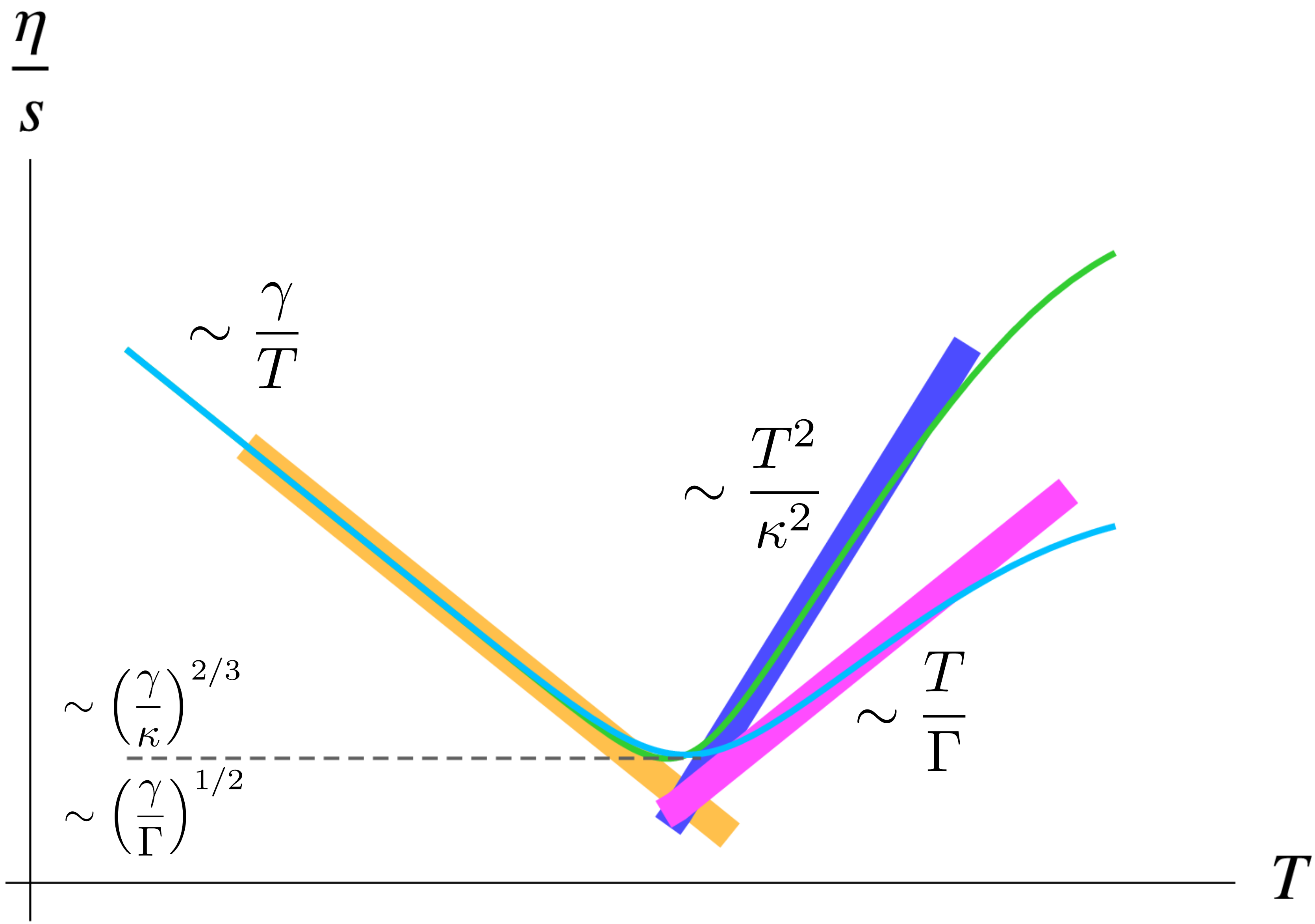
$$\mathrm{Im}\Sigma(\omega,\mathbf{p}) \approx \omega\Gamma + \kappa^2 + \mathcal{O}(\log(\omega/|\mathbf{p}|)/\omega,\ 1/\omega)$$

$$s \sim \begin{cases} (1-\mu^2)^{3/2}T^3 \\ T^3, \ (\mu=0,\ \Gamma=0) \end{cases}$$

$$\eta \sim \begin{cases} (1-\mu^2)^{9/2}\frac{T}{\Gamma}T^3 \\ \frac{T^2}{\kappa^2}T^3, \ (\mu=0,\ \Gamma=0) \end{cases}$$

$$\frac{\eta}{s} \sim \begin{cases} (1-\mu^2)^3\frac{T}{\Gamma} \\ \frac{T^2}{\kappa^2}, \ (\mu=0,\ \Gamma=0) \end{cases}$$

Qualitative understanding



Thank you for the attention!

Questions? Comments?

PRD *93*, 056010 (2016)
arXiv: 1605.08619

Back-up slides

EQP – energy-momentum conservation

$$\int_x \frac{\delta S[e^{\alpha\partial}\varphi]}{\delta\alpha_x^\mu}\alpha_x\Bigg|_{\alpha\equiv 0} = -\int_x \alpha_x(\partial_x\cdot T_x)^\mu$$

Noether's theorem

$$= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[\frac{1}{2}\int_x \varphi_{x^\mu+\varepsilon\alpha_x^\mu}\int_z \mathcal{K}_z e^{z\cdot\partial_x}\varphi_{x^\mu+\varepsilon\alpha_x^\mu}\right]_{\varepsilon=0}$$

The energy-momentum tensor in Fourier-space:

$$T_k^{\mu\nu} = \frac{1}{2}\int_p\int_q \varphi_{-p}\varphi_q\delta_{k+p-q}\frac{p^\mu(p+q)^\nu}{q^2-p^2}(K_q-K_p) \overset{k\to 0}{\rightarrow} \frac{1}{2}\int_p \varphi_{-p}\varphi_p\frac{p^\mu p^\nu}{|p|}\frac{\partial K_p}{\partial|p|}$$

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The energy-momentum tensor in thermal equilibrium

$$\langle T^{\mu\nu}_{x=0}\rangle = \frac{1}{2}\int_p \frac{p^\mu p^\nu}{|p|}\frac{\partial K_p}{\partial|p|}\rho_p\left(n(p^0/T)+\frac{1}{2}\right)$$

Transport coeff. derivation – backup

$$\rho_{(T^\dagger)^{ij}T^{ij}}(k) = iG^{21}_{(T^\dagger)^{ij}T^{ij}}(k) - iG^{12}_{(T^\dagger)^{ij}T^{ij}}(k) =$$

$$= \frac{1}{4}\int_{k'}\int_{p}\int_{q}\int_{r}\int_{s} \delta_{k+p-q}\delta_{k'+r-s} D^{ij}_{p,q} D^{ij}_{r,s} \left(\left\langle \varphi^{(2)}_p \varphi^{(2)}_{-q} \varphi^{(1)}_{-r} \varphi^{(1)}_s \right\rangle - \left\langle \varphi^{(1)}_p \varphi^{(1)}_{-q} \varphi^{(2)}_{-r} \varphi^{(2)}_s \right\rangle \right) =$$

$$= \frac{1}{4}\int_{k',p,q,r,s} \delta_{k+p-q}\delta_{k'+r-s} D^{ij}_{p,q} D^{ij}_{r,s} \left[\delta_{p-q}\delta_{r-s} \left(iG^{22}_p iG^{11}_r - iG^{11}_p iG^{22}_r \right) + \right.$$
$$\left. + \left(\delta_{p-r}\delta_{q-s} + \delta_{p+s}\delta_{q+r} \right) \left(iG^{12}_p iG^{21}_q - iG^{21}_p iG^{12}_q \right) \right] =$$

$$= \frac{1}{4}\int_p \left((D^{ij}_{p,p+k})^2 + D^{ij}_{p,p+k} D^{ij}_{p+k,p} \right) \rho_p \rho_{p+k} (n_p - n_{p+k})$$

$$(D^{ij}_{p,p+k})^2 \Big|_{\mathbf{k}=0} = D^{ij}_{p,p+k} D^{ij}_{p+k,p} \Big|_{\mathbf{k}=0} = \left[\frac{2p^i p^j}{\omega^2 - 2\omega\tilde{\omega}} (K_{\tilde{\omega}+\omega,\mathbf{p}} - K_{\tilde{\omega},\mathbf{p}}) \right]^2 \overset{\omega\to 0}{\approx} \left(\frac{p^i p^j}{\tilde{\omega}} \frac{\partial K_{\tilde{\omega},\mathbf{p}}}{\partial \tilde{\omega}} \right)^2 + o(\omega)$$

$$\eta = \lim_{\omega\to 0} \frac{\rho_{(T^\dagger)^{12}T^{12}}(\omega, \mathbf{k}=0)}{\omega} = \lim_{\omega\to 0} \frac{1}{2\omega} \int_p (D^{ij}_{p,p+k})^2 \Big|_{\mathbf{k}=0} \left[\rho^2_{\tilde{\omega},\mathbf{p}} + \omega \rho_{\tilde{\omega},\mathbf{p}} \frac{\partial \rho_{\tilde{\omega},\mathbf{p}}}{\partial \tilde{\omega}} + o(\omega^2) \right] \left(-\omega \frac{\partial n_{\tilde{\omega}}}{\partial \tilde{\omega}} + o(\omega^2) \right) =$$

$$= \frac{1}{2} \int_p \left(\frac{p^1 p^2}{\tilde{\omega}} \frac{\partial K_p}{\partial \tilde{\omega}} \right)^2 \rho^2_p (-n'_{\tilde{\omega}})$$

EQP: low-T power counting

$$\frac{\partial K}{\partial \omega}\rho = \frac{\partial}{\partial \omega}\left(2\omega^2 - \mathbf{p}^2 - \mathrm{Re}\Sigma(\omega,\mathbf{p})\right)\frac{\mathrm{Im}\Sigma(\omega,\mathbf{p})}{(\omega^2 - \mathbf{p}^2 - \mathrm{Re}\Sigma(\omega,\mathbf{p}))^2 + \mathrm{Im}\Sigma^2(\omega,\mathbf{p})}$$

$$\mathrm{Re}\Sigma(\omega,\mathbf{p}) \approx m^2 + \mathcal{O}(\log(\omega/|\mathbf{p}|)\omega^2) \qquad T \to 0$$

$$\mathrm{Im}\Sigma(\omega,\mathbf{p}) \approx \omega(\gamma + \mathcal{O}(\log(\omega/|\mathbf{p}|)\omega^2)) \qquad \omega \to 0$$

$$s = N_s T^4 \int_0^\infty \mathrm{d}y \int_0^\infty \mathrm{d}q q^2 \frac{\chi_s(y)}{T} T y \frac{2\gamma y^2}{T^2} \frac{1}{(y^2 - q^2 - \frac{m^2}{T^2})^2 + \frac{\gamma^2}{T^2} y^2} = T^3 f_s(m/T, \gamma/T)$$

$$\eta = N_\eta T^8 \int_0^\infty \mathrm{d}y \int_0^\infty \mathrm{d}q q^6 \frac{\chi_\eta(y)}{T} \frac{1}{T^2 y^2} \frac{4\gamma^2 y^2}{T^4} \frac{1}{\left[(y^2 - q^2 - \frac{m^2}{T^2})^2 + \frac{\gamma^2}{T^2} y^2\right]^2} = T^3 f_\eta(m/T, \gamma/T)$$

EQP: low-T power counting

$$\mathrm{Re}\Sigma(\omega,\mathbf{p}) \approx m^2 + \mathcal{O}(\log(\omega/|\mathbf{p}|)\omega^2) \qquad T \to 0$$

$$\mathrm{Im}\Sigma(\omega,\mathbf{p}) \approx \omega(\gamma + \mathcal{O}(\log(\omega/|\mathbf{p}|)\omega^2)) \qquad \omega \to 0$$

$$T\int_0^\infty \mathrm{d}q q^2 \frac{\frac{\gamma}{T}y}{(q^2 - y^2\frac{m^2}{T^2})^2 + \frac{\gamma^2}{T^2}y^2} \sim T\sqrt{\sqrt{(y^2-\frac{m^2}{T^2})^2+\frac{\gamma^2}{T^2}y^2}+y^2-\frac{m^2}{T^2}} \sim \begin{cases} \frac{\gamma}{m}Ty \\ \sqrt{\frac{\gamma}{T}y}T \ (m=0) \end{cases}$$

$$T^2\int_0^\infty \mathrm{d}q q^6 \frac{\frac{\gamma^2}{T^2}y^2}{\left[(q^2 - y^2\frac{m^2}{T^2})^2 + \frac{\gamma^2}{T^2}y^2\right]^2} \sim$$

$$T^2\left(y^2-\frac{m^2}{T^2}\right)\left[\left(2\frac{y^2-\frac{m^2}{T^2}}{\frac{\gamma}{T}y} + 3\frac{\frac{\gamma}{T}y}{y^2-\frac{m^2}{T^2}}\right)\sqrt{\sqrt{(y^2-\frac{m^2}{T^2})^2+\frac{\gamma^2}{T^2}y^2}+y^2-\frac{m^2}{T^2}}+\right.$$

$$\left.+\sqrt{\sqrt{(y^2-\frac{m^2}{T^2})^2+\frac{\gamma^2}{T^2}y^2}-y^2+\frac{m^2}{T^2}}\right] \sim \begin{cases} \frac{\gamma}{m}\frac{\gamma}{T}T^2 \\ \left(\frac{\gamma}{T}\right)^{3/2}T^2 \ (m=0) \end{cases}$$