Limiting temperature of pion gas with the van der Waals equation of state

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Outline

- Quantum VdW Equation of State
- Thermodynamic behavior of the pion gas with VdW EoS, emergence of limiting temperature
- Comparison with Hagedorn model

Van der Waals Equation of State

VdW canonical ensemble EoS

$$p(V,T,N) \;=\; \frac{NT}{V \;-\; b\,N} \;-\; a\,\frac{N^2}{V^2}$$

$b > 0$ - repulsive interaction e.g. $b = 16\pi r^3/3$ for hard-core particles with the hard-core r

$a > 0$ - attractive interaction

From Canonical Ensemble to Grand Canonical Ensemble

$$\left(\frac{\partial F}{\partial V}\right)_{T,N} ~=~ -~p(V,T,N)$$

$$\Rightarrow~ F(V,T,N) ~=~ F(V_0,T,N) ~-~ \int_{V_0}^{V} dV' p(V',T,N)$$

$F(V_0,T,N) ~\cong~ F_{\rm id}(V_0,T,N)$ becomes valid at $V_0 \to \infty$ (the role of particle interactions becomes negligible at very small density)
For relativistic ideal Boltzmann gas the free energy reads:

$$F_{\rm id}(V,T,N) ~=~ -~NT\left[1 ~+~ \ln\frac{V\,\phi(T;d,m)}{N}\right]~,$$

where

$$\phi(T;d,m) ~=~ \frac{d}{2\pi^2}\int_0^{\infty} k^2 dk~\exp(-\sqrt{k^2+m^2}/T) ~=~ \frac{d\,m^2\,T}{2\pi^2}\,K_2\left(\frac{m}{T}\right)$$

$$F(V,T,N) \; = \; F_{\rm id}(V_0,T,N) \; + \; \int_V^{V_0} dV' \left[\frac{NT}{V' - bN} \; - \; a\frac{N^2}{V'^2} \right]$$

$$\cong \; F_{\rm id}(V - bN, T, N) \; - \; a\,\frac{N^2}{V} \; .$$

$$\mu \; = \; \left(\frac{\partial F}{\partial N} \right)_{V,T} \; = \; -T \; \ln \frac{(V - bN)\,\phi(T;d,m)}{N} + b\,\frac{NT}{V - bN} - 2a\,\frac{N}{V}$$

VdW model in GCA

$$n(T,\mu) \; = \; \frac{n_{\rm id}(T,\mu^*)}{1 \; + \; b\,n_{\rm id}(T,\mu^*)} \; , \quad \mu^* \; = \; \mu \; - \; b\,\frac{n\,T}{1 - b\,n} \; + \; 2a\,n$$

$$p(T,n) \; = \; \frac{n\,T}{1 - b\,n} \; - a\,n^2 \; = \; p^{\rm id}(T,\mu^*) - a\,n^2.$$

Quantum VdW Equation of State

General requirements for the quantum version of VdW EoS:

1. It should be transformed to the ideal *quantum* gas at $a = 0$ and $b = 0$.
2. It should be equivalent to the classical VdW EoS in a region of thermodynamical parameters where quantum statistics can be neglected.
3. The entropy should be a non-negative quantity and go to zero at $T \to 0$.

Quantum formulation of the VdW EoS

$$\mu^* \;=\; \mu \;-\; b\,p(T,\mu) - a\,b\,n^2(T,\mu) + 2\,a\,n(T,\mu)\;,$$

$$p(T,\mu) \;=\; p^{\rm id}(T,\mu^*) - a\,n^2(T,\mu),\;\; n(T,\mu) \;=\; \frac{n^{\rm id}(T,\mu^*)}{1 + b\,n^{\rm id}(T,\mu^*)}$$

Two equations for two unknown functions: $p(T,\mu)$ and $n(T,\mu)$

Pion gas with the VdW EoS

The pion system with zero value of total electric charge is considered, thus, π^+, π^-, and π^0 all have zero chemical potentials, i.e., $\mu = 0$:

$$\mu^* = -b\,p(T) - a\,b\,n^2(T) + 2\,a\,n(T)\ ,$$

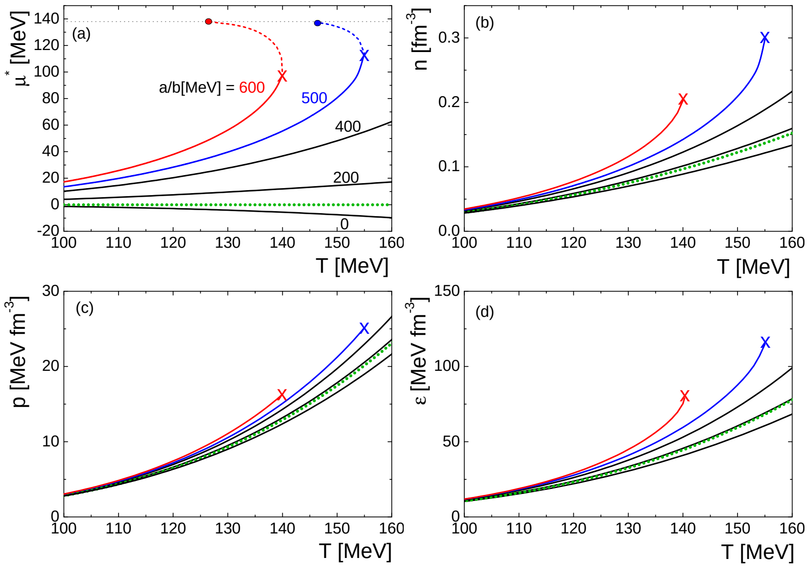
$$p^{\rm id}(T,\mu^*) = \frac{1}{2\pi^2}\int_0^\infty k^2 dk\, \frac{k^2}{\sqrt{m_\pi^2+k^2}}\left[\exp\left(\frac{\sqrt{m_\pi^2+k^2}-\mu^*}{T}\right)-1\right]^{-1}\ ,$$

$$n^{\rm id}(T,\mu^*) = \frac{3}{2\pi^2}\int_0^\infty k^2 dk\, \left[\exp\left(\frac{\sqrt{m_\pi^2+k^2}-\mu^*}{T}\right)-1\right]^{-1}\ .$$

- $g = 3$ for the pion degeneracy factor and $m_\pi \cong 138~\mathrm{MeV}$ for the pion mass.
- The value of b is fixed $b \cong 0.45$ fm^3 that corresponds to $r \cong 0.3~\mathrm{fm}^{-3}$ for the pion hard-core radius[1,2]
- The value of a is considered as a free model parameter.

[1]A. Andronic, P. Braun-Munzinger, J. Stachel, and M. Winn, Phys. Lett. B **718**, 80 (2012).
[2] V. Vovchenko, D. V. Anchishkin, and M. I. Gorenstein, Phys. Rev. C **91**, 024905 (2015).

Thermodynamic quantities of the pion gas (VdW EoS)



Thermodynamic quantities of the pion gas (VdW EoS)

$$s \;=\; \frac{dp}{dT}\;, \qquad \varepsilon(T) = T\frac{dp}{dT} \;-\; p \;=\; \frac{\varepsilon_{\rm id}(T,\mu^*)}{1 \;+\; b\, n_{\rm id}(T,\mu^*)} - a\, n^2$$

The scaled variance of particle number fluctuations:

$$\omega[N] \;=\; \frac{\langle N^2\rangle - \langle N\rangle^2}{\langle N\rangle}$$

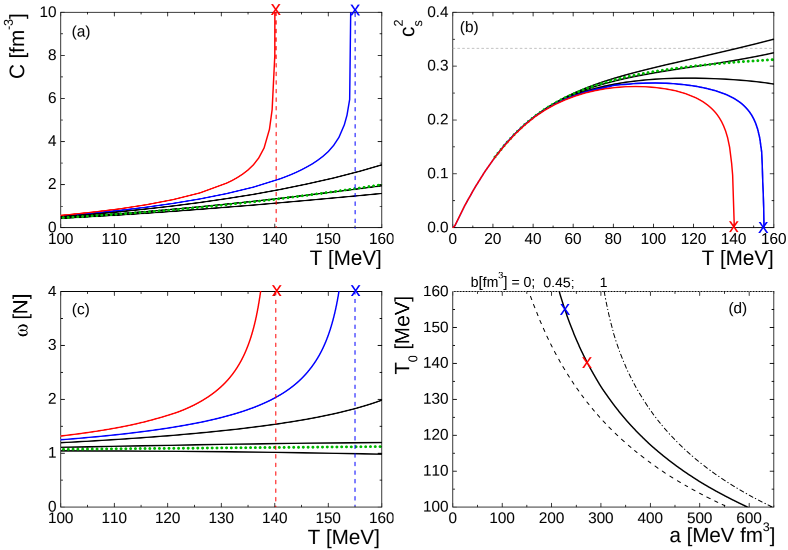
$$= \;\; \omega_{\rm id}(T,\mu^*)\left[\frac{1}{(1-bn)^2} - \frac{2an(T,\mu^*)}{T}\,\omega_{\rm id}(T,\mu^*)\right]^{-1}\;,$$

where

$$\omega_{\rm id}(T,\mu^*) \;=\; \frac{1}{n^{\rm id}(T,\mu^*)}\,\frac{3}{2\pi^2}\int_0^\infty k^2 dk\,\left[\exp\left(\frac{\sqrt{m_\pi^2+k^2}-\mu^*}{T}\right)\;-\;1\right]^{-2}\;,$$

Specific heat: $C = d\varepsilon/dT$
Speed of sound squared: $c_s^2 = dp/d\varepsilon$

Behaviour in the vicinity of the limiting temperature



Both C and $\omega[N]$ are proportional to $(T_0 - T)^{-1/2}$, and $c_s^2 \propto (T_0 - T)^{1/2}$

Hagedorn Model

Exponentially increasing mass spectrum $\rho(m)$ for the hadron excited states at large m:

$$\rho(m) \;\cong\; c\, m^{-\alpha} \exp\left(\frac{m}{T_0}\right) \theta(m - M_0)\;,$$

where c, M_0, T_0, and α are the model parameters.
In a simplest version of the model: only pions and the continuous mass spectrum $\rho(m)$ starting from $M_0 > 2m_\pi$.

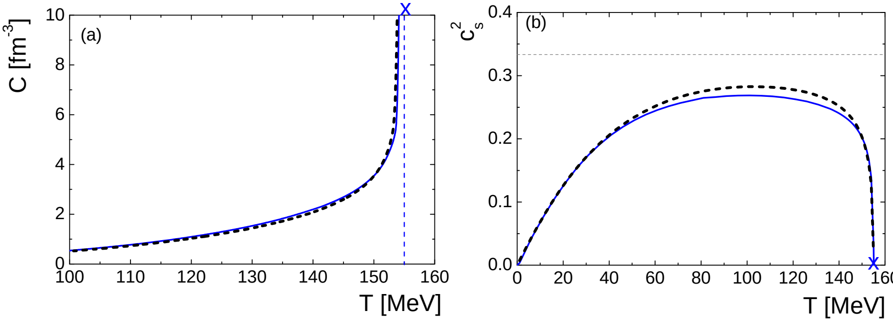
Pressure in the Hagedorn model

$$p_{\mathrm{H}}(T) \;=\; \int \frac{d^3k}{(2\pi)^3} \frac{k^2}{\sqrt{k^2+m_\pi^2}} \left[\exp\left(\frac{\sqrt{k^2+m_\pi^2}}{T}\right) \;-\; 1\right]^{-1} \;+\; \frac{T^2}{2\pi^2}\int_{M_0}^{\infty} dm\, \rho(m)\, m^2 K_2(m/T)\;.$$

where K_2 is the modified Bessel function.

Hagedorn Model in the vicinity of T_0

- At $T > T_0$ - divergence of the integral with respect to m .
  $p_H \propto (T_0 - T)^{\alpha - 5/2}, \qquad \varepsilon_H \propto (T_0 - T)^{\alpha - 7/2}$.
- An explicit solution of the statistical bootstrap equation $\Rightarrow \alpha = 3$. Thus, $\varepsilon_H \propto (T_0 - T)^{-1/2}$ and $\varepsilon_H \to \infty$ at $T \to T_0$.
- On the other hand, for $\alpha > 7/2$ all thermodynamical functions in the Hagedorn model remain finite in a vicinity of the limiting temperature $T = T_0$.
- In the vicinity of the limiting temperature T_0 the heat capacity and the speed of sound squared behave in Hagedorn model as $C \propto (T_0 - T)^{\alpha - 9/2}$ and $c_s^2 \propto (T_0 - T)^{9/2 - \alpha}$, respectively.
- Therefore, for $\alpha = 4$ the behavior of C and c_s^2 in the Hagedorn model at $T \to T_0$ is the same as in the pion gas with the VDW EoS, if the T_0 values in both models are set to be equal.



The solid lines correspond the VDW EoS with $b = 0.45$ fm^3 and $a/b = 500$ MeV. The dotted lines correspond to the Hagedorn model with $T_0 = 155$ MeV, $\alpha = 4$, $c = 3.57$ fm^{-3}, and $M_0 = 507$ MeV.

Summary and Conclusions

- Even though the realization of the attractive mechanism between particles in the Hagedorn model is different from that in the VDW model, we found that a very similar thermodynamical behavior emerges in both models.
- A presence of the limiting temperature T_0 in the VDW pion gas is definitely a signal of the restricted validity of this model. Similar to the Hagedorn model, the limiting temperature of the VDW pion system should be transformed to the temperature of deconfinement transition when the fundamental quark-gluon degrees of freedom are introduced.

Thank You for Attention!