Lattice computation of the nucleon sigma terms using the Feynman-Hellmann-theorem

Lukas Varnhorst

University of Wuppertal
Fak. 04 - Department of Physics

SFB TR55

BERGISCHE UNIVERSITÄT WUPPERTAL

BMW collaboration

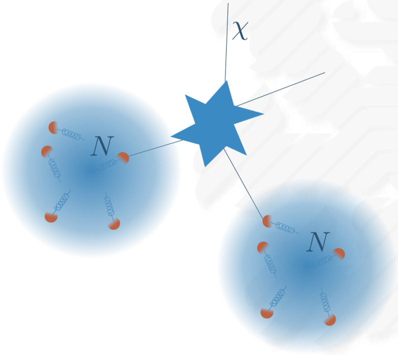
Introduction

The nucleon-sigma-terms are of significant interest for dark-matter searches, as they determine the coupling of several dark matter candidates to hadronic matter.

The nucleon sigma terms are defined as

$$\sigma_{\pi N} = f_{ud}^{N} M_N = m_{ud} \frac{\langle N \mid \bar{u}u + \bar{d}d \mid N \rangle}{2M_N}$$

$$\sigma_{sN} = f_{s}^{N} M_N = m_{s} \frac{\langle N \mid \bar{s}s \mid N \rangle}{2M_N}$$



Sigma terms and dark matter

Consider the spin independent interaction of dark matter WIMP particles χ with ordinary matter. We have an interaction between the WIMP and the quarks that is described by a Lagrangian of the following type:

$$\mathcal{L} = \lambda_q \bar{\chi}\chi\bar{q}q$$

The cross section of an interaction with a nucleon is given by

$$\sigma_{SI} \propto (Zf_p + (A - Z)f_n)^2$$

with

$$\frac{f_N}{M_N} = \sum_q f_q^N \frac{\lambda_q}{m_q}.$$

Hence the quark constens - or sigma terms - are of interest to dark matter detection experiments.

[1] J. R. Ellis, K. A. Olive and C. Savage, Phys. Rev. D **77** (2008) 065026 doi:10.1103/PhysRevD.77.065026 [arXiv:0801.3656 [hep-ph]].

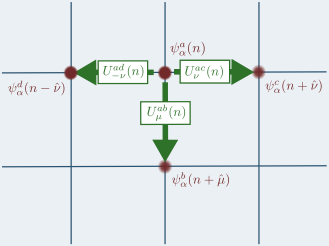
Lattice QCD

QCD Lagrangian in the continuum:

$$\mathcal{L} = \bar{\psi}(\mathrm{i}\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_{a,\mu\nu}F_a^{\mu\nu}$$

Lattice field theory is a method to non-perturbatively calculate quantities in a QFT.

Fields on the lattice



Define Fermion fields $\bar{\psi}$ and ψ on a regular space-time-lattice. Represent $SU(3)$ gauge fields by link variables U_μ.

Using this *regularization* the euclidean path integral becomes a very high but finite dimensional integral.

$\Rightarrow$ Use a computer to solve it!

Lattice QCD

Fermion fields in the path integral can be integrated out exactly

$$\langle O \rangle = \frac{1}{Z} \int \mathcal{D}U \mathcal{D}\bar{\psi} \mathcal{D}\psi \, O \, e^{-S_G[U] - S_F[U, \bar{\psi}, \psi]} = \frac{1}{Z} \int \mathcal{D}U \, O \, \det M[U] e^{-S_G[U]}$$

High dimensional integral over $SU(3)$ degrees of freedom at every lattice point. Use improtrance sampling.

Typical workflow:

simulation parameters

Methods of evaluation

There are two methods to evaluate the nucleon sigma terms on the lattice:

i. The direct method: Evaluate matrix elements directly. Needs three-point functions

ii. Use the Feynman-Hellmann theorem to calculate the sigma term:

$$f_{ud}^N = \frac{m_{ud}}{M_N}\frac{\partial M_N}{\partial m_{ud}} \quad \text{and} \quad f_s^N = \frac{m_s}{M_N}\frac{\partial M_N}{\partial m_s}$$

Only two-point functions necessary! This is the strategy pursued in this work.

General strategy

The general strategy of the calculation goes as follows:

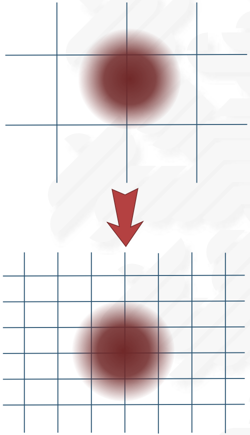
i. Generate a suitable set of QCD configuration
ii. Measure two-point functions needed for the extraction of masses.
iii. Fit two-point function to determine masses per ensemble.
iv. Using the masses from all ensembles to fit the nucleon mass as function of quark mass.
v. From the fit function determine the derivatives.

Setup

We use a tree-level improved Symmanzik gauge action and a tree-level improved clover Wilson fermion action with $N_f = 2 + 1$ and two levels of HEX smearing.

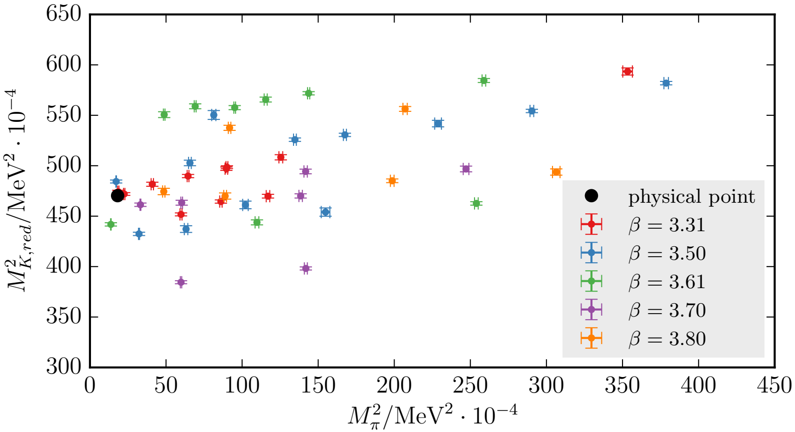
This lattice action is supposed to have cut-off effects of order $\alpha_s a$. Often a^2 effects are dominant.

We used simulations at 5 different lattice spacing (0.116 fm to 0.054 fm) and extrapolated results to the continuum. This extrapolation has been performed both with $\alpha_s a$ and a^2 to estimate the systematic uncertainty.



The data landscape

Landscape in the M_π^2 and $M_{K,red}^2 = 2M_K^2 - M_\pi^2$ plane:



Extracting hadron masses

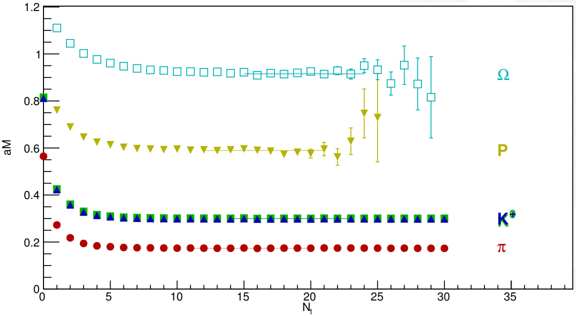
Suppose $\bar{O}$ and O are operators which create and annihilate a hadron H. The the correlation function between the two operators behaves as:

$$C(t) = \langle \bar{O}(t) O(0) \rangle = \sum_k \langle 0 \mid O \mid k \rangle \langle k \mid O^\dagger \mid 0 \rangle e^{-t \Delta E_k}$$

We can define a effective mass via

$$m_{\text{eff}}(t+\frac{1}{2}) = \ln \frac{C(t)}{C(t+1)}$$

and search for a plateau.



(Effective masses from one ensemble not used in this study)

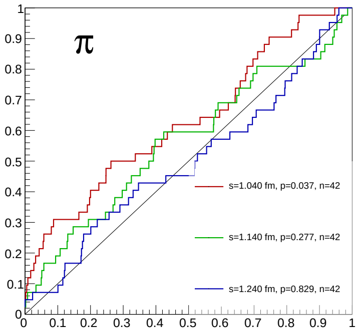
Extracting hadron masses

Even better: Fit correlation functions with

$$C(t) = \begin{cases} A\cosh(-m(t - N_t/2)) & \text{for mesons} \\ A\sinh(-m(t - N_t/2)) & \text{for baryons} \end{cases}$$

The question is: From which t_{min} on should the fit start?

We have many ensembles: We can make a Kolmogorov-Smirnov-Test to check wether the χ^2 values are properly distributed.



Parameterizing the nucleon mass

To leading order the nucleon mass can be fitted with the ansatz

$$M_N = (1 + f^{scale}(a))(1 + f^{fvol}(M_\pi, L)) \cdot \\ \cdot\, a_0^N(1 + a_1^N(m_{ud} - m_{ud}^{(\Phi)}) + a_2^N(m_s - m_s^{(\Phi)}) + \cdots)$$

to estimate systematic uncertainties we want to vary the fit function:

i. Continuum extrapolation may show an $f^{\text{scale}}(a) = a^2$ or an $f^{\text{scale}}(a) = \alpha_s(a)a$ behavior.
ii. The next-to-leading term in m_{ud} can be assumed to be m_{ud}^2 or $m_{ud}^{1.5}$.
iii. Finite volume effects can be parametrized via $f^{fvol}(M_\pi, L) = \sqrt{\frac{M_\pi}{L^3}}e^{-M_\pi L}$ or via $f^{fvol}(M_\pi, L) = e^{-M_\pi L}$
iv. There might be higher order contribution in m_s

The ratio difference method

Quark masses can be defined in several ways:

$$m_j^{VWI} = \frac{1}{Z_S} m_j^W \left(1 - \frac{1}{2} b_S a m_j^W - \bar{b}_S a \operatorname{tr} M + \mathcal{O}(a^2)\right)$$

$$m_j^{AWI} = \frac{Z_A}{Z_P} m_j^{PCAC} \left(1 + (b_A - b_P) a m_j^W + (\bar{b}_A - \bar{b}_P) a \operatorname{tr} M + \mathcal{O}(a^2)\right)$$

In practice it is advantageous to use the ratio-difference-method. Here one constructs quantities (and their improved counterparts [1])

$$d_{ij} = a(m_i^W - m_j^W) \quad \text{and} \quad r_{ij} = \frac{m_i^{PCAC}}{m_j^{PCAC}}$$

and from these one can extract the quark masses:

$$am_i^{rd,r} = \frac{1}{Z_S} am_i^{rd} = \frac{1}{Z_S} \frac{r_{ij} d_{ij}}{r_{ij} - 1} \quad \text{and} \quad am_j^{rd,r} = \frac{1}{Z_S} am_j^{rd} = \frac{1}{Z_S} \frac{d_{ij}}{r_{ij} - 1}.$$

[1] S. Durr *et al.*, "Lattice QCD at the physical point: Simulation and analysis details," JHEP **1108** (2011) 148 doi:10.1007/JHEP08(2011)148 [arXiv:1011.2711 [hep-lat]].

Renormalization

$$am_i^{rd,r} = \frac{1}{Z_S} am_i^{rd} = \frac{1}{Z_S}\frac{r_{ij} d_{ij}}{r_{ij}-1} \quad \text{and} \quad am_j^{rd,r} = \frac{1}{Z_S} am_j^{rd} = \frac{1}{Z_S}\frac{d_{ij}}{r_{ij}-1}.$$

We still need to know Z_S. For these ensembles they have already been determined [1]. From this we can define

$$m_i^{RGI} = \frac{am_i^{rd}}{aZ_s(1+f_{q,i}^{scale}(a))}$$

There quark masses can then be used in the parametrization of the nucleon mass:

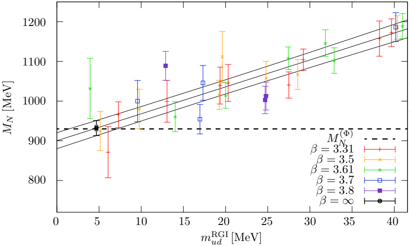
$$M_N = (1+f^{scale}(a))(1+f^{fvol}(M_\pi, L)) \cdot$$
$$\cdot\, a_0^N (1 + a_1^N (m_{ud}^{RGI} - m_{ud}^{(\Phi)RGI}) + a_2^N (m_s^{RGI} - m_s^{(\Phi)RGI}) + \cdots)$$

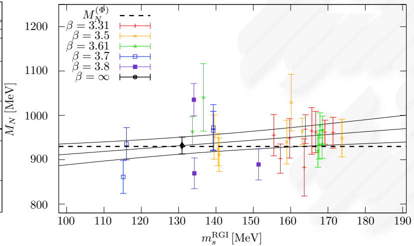
[1] S. Durr *et al.*, "Lattice QCD at the physical point: Simulation and analysis details," JHEP **1108** (2011) 148

doi:10.1007/JHEP08(2011)148 [arXiv:1011.2711 [hep-lat]].

The nucleon mass dependence

The following plots show a typical result of the nucleon mass dependence. The datapoints have been projected to the physical point in all but the shown direction.





Uncertainties

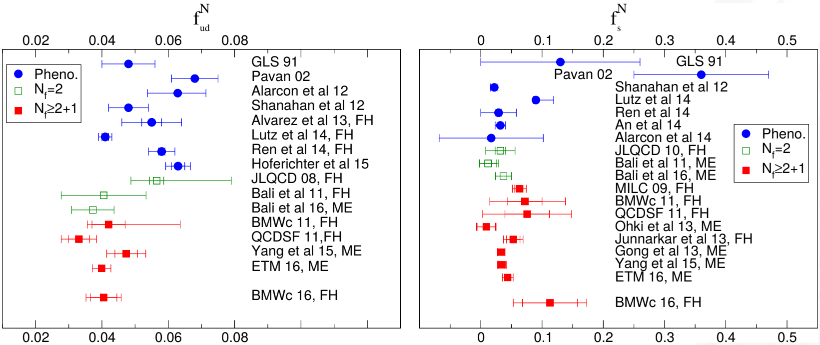
To estimate the systematic uncertainties we carry out a set of analyses, each of which is valid. We varied:

- We used the plateau-range as determined by the Kolmogorov-Smirnov-test and or an additional plateau-range starting 0.1 fm later.
- We applied two different pion mass cuts: 320 MeV and 480 MeV.
- We varied the higher order term in the fit functions
- We replaced Taylor expansion by Padé expansions of the same order.
- We performed continuum extrapolations with $\mathcal{O}(a^2)$ or $\mathcal{O}(\alpha_s a)$.

Altogether we have 192 different analyses. We make a histogram of thees analyses, weight them by the AIC weight, and determine the spread.

For the statistical error we have performed a bootstrap analysis.

Results



$$f^N_{ud} = 0.0405(40)(35)$$

$$f^N_{sN} = 0.113(45)(40)$$

$$\sigma_{\pi N} = 38(3)(3)\,\text{MeV}$$

$$\sigma_{sN} = 105(41)(37)\,\text{MeV}$$

Individual p- and n-quark-contents

One can rewrite the Individual quark contents as

$$f^{p}_{u/d} = \left(\frac{1}{2} \mp \frac{\delta m}{4 m_{ud}}\right) f_{ud,p} + \left(\frac{1}{4} \mp \frac{m_{ud}}{2\delta m}\right) \frac{\delta m}{2M_p^2} \langle p \mid \bar{d}d - \bar{u}u \mid p\rangle.$$

We use

$$H = H_{\text{iso}} + H_{\delta m} \qquad H_{\delta m} = \frac{\delta m}{2} \int \mathrm{d}^3x (\bar{d}d - \bar{u}u)$$

to derive

$$\Delta_{QCD} M_N = \frac{\delta m}{2M_p} \langle p \mid \bar{u}u - \bar{d}d \mid p\rangle$$

Using there relations one can derive ($r = m_u/m_d$)

$$f^{p/n}_{u} = \left(\frac{r}{1+r}\right) f^{N}_{ud} \pm \frac{1}{2}\left(\frac{r}{1-r}\right) \frac{\Delta_{QCD} M_N}{M_N} + \mathcal{O}(\delta m^2, m_{ud}\delta m)$$

$$f^{p/n}_{d} = \left(\frac{1}{1+r}\right) f^{N}_{ud} \mp \frac{1}{2}\left(\frac{1}{1-r}\right) \frac{\Delta_{QCD} M_N}{M_N} + \mathcal{O}(\delta m^2, m_{ud}\delta m)$$

Individual p- and n-quark-contents

$$f_u^{p/n} = \left(\frac{r}{1+r}\right) f_{ud}^N \pm \frac{1}{2}\left(\frac{r}{1-r}\right)\frac{\Delta_{QCD} M_N}{M_N} + \mathcal{O}(\delta m^2, m_{ud}\delta m)$$
$$f_d^{p/n} = \left(\frac{1}{1+r}\right) f_{ud}^N \mp \frac{1}{2}\left(\frac{1}{1-r}\right)\frac{\Delta_{QCD} M_N}{M_N} + \mathcal{O}(\delta m^2, m_{ud}\delta m)$$

Plugging in known values for r [1] and $\Delta_{QCD} M_N$ [2] one gets

$$f_u^p = 0.0139(13)(12) \qquad f_d^p = 0.0253(28)(24)$$
$$f_u^n = 0.0116(13)(11) \qquad f_d^n = 0.0302(28)(25)$$

[1] S. Aoki *et al.*, "Review of lattice results concerning low-energy particle physics," arXiv:1607.00299 [hep-lat].

[2] S. Borsanyi *et al.*, "Ab initio calculation of the neutron-proton mass difference," Science **347** (2015) 1452 doi:10.1126/science.1257050 [arXiv:1406.4088 [hep-lat]].

Summary

Quark content/Sigma terms:

$$f_{udN} = 0.0405(40)(35) \qquad \sigma_{\pi N} = 38(3)(3)\,\text{MeV}$$
$$f_{sN} = 0.113(45)(40) \qquad \sigma_{sN} = 105(41)(37)\,\text{MeV}$$

Individual quark contents:

$$f_u^p = 0.0139(13)(12) \qquad f_d^p = 0.0253(28)(24)$$
$$f_u^n = 0.0116(13)(11) \qquad f_d^n = 0.0302(28)(25)$$

Special thanks to all my collaborators: S. Durr, Z. Fodor, C. Hoelbling, S. D. Katz, S. Krieg, L. Lellouch, T. Lippert, T. Metivet, A. Portelli, K. K. Szabo, C. Torrero, B. C. Toth

More information: S. Durr *et al.*, Phys. Rev. Lett. **116** (2016) no.17, 172001 doi:10.1103/PhysRevLett.116.172001 [arXiv:1510.08013 [hep-lat]].