Causal Perturbation Theory in Quantum Optics

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Outline

- Quantum optics vs relativistic QFT
- Causal perturbation theory
- Distribution splitting
- Spontaneous emission and Lamb shift
- Conclusion

Quantum Optics vs QFT

Quantum optics (QO) is the theory of (few, or many) particles consisting of photons and atoms

QO is a “dumbed down” version of full QED:

- Atoms = polarizable point dipoles with internal energy levels
- Radiation field is treated in Coulomb gauge
- Atom-light interaction via electric dipole coupling

$$H_{\text{int}} = -\vec{d} \cdot \vec{E}$$

Quantum Optics vs QFT

Features lost in QO:

- Not covariant
- Not causal since atoms are treated non-relativistically
- Gauge invariance can be tricky (electric dipole coupling vs minimal coupling)

QO has been incredibly helpful for the design of new experiments (lasers, Bell inequality violation, implementing quantum information, Bose-Einstein condensation and degenerate Fermi gases, ..)

Causal perturbation theory

Like other QFTs, QO is plagued by diverging results, but renormalization is not as developed as in QED

Consistent QED without infinities: causal perturbation theory (CPT)

In CPT, Feynman diagrams are replaced by a causal recursive construction of each order in perturbation theory

Key point: be careful when splitting distributions into retarded and advanced parts

Causal perturbation theory

CPT (Epstein and Glaser 1973), the general approach:

Standard expression for S matrix:

$$
\begin{aligned}
S &= \mathrm{Texp}\left(-\frac{i}{\hbar c}\int d^4x\, H_{\mathrm{int}}(x)\right) \\
&= 1 - \frac{i}{\hbar c}\int d^4x\, H_{\mathrm{int}}(x) \\
&\quad - \frac{1}{2\hbar^2 c^2}\int d^4x \int d^4y\, \mathrm{T}(H_{\mathrm{int}}(x) H_{\mathrm{int}}(y)) + \cdots
\end{aligned}
$$

Causal perturbation theory

Time ordering is usually done using step functions,

$$\mathrm{T}(H_{\mathrm{int}}(x)H_{\mathrm{int}}(y)) = \theta(x^0 - y^0)H_{\mathrm{int}}(x)H_{\mathrm{int}}(y) + \theta(y^0 - x^0)H_{\mathrm{int}}(y)H_{\mathrm{int}}(x)$$

It is these step functions that cause results to diverge

Reason: all S-matrix terms are based on distributions, and step functions are not test functions. One has to employ proper distribution splitting

Distribution splitting

Distributions are defined through linear functionals on well-defined function spaces.

Most famous example: Dirac distribution, defined through

$$\int_{-\infty}^{\infty} dx\, \delta(x) f(x) = f(0)$$

$\delta(x)$ is not a function, only a formal integral kernel
f(x) must be a test function, i.e., an element of the space of smooth integrable functions

Distribution splitting

What's wrong with step functions?
Consider the distribution $(x+ i\,0)^{-1}$, defined via

$$\int_{-\infty}^{\infty} dx\, \frac{f(x)}{x+i0} = -i\pi f(0) + \lim_{\epsilon\to 0} \int_{-\infty}^{-\epsilon} dx\, \frac{f(x)}{x} + \int_{\epsilon}^{\infty} dx\, \frac{f(x)}{x}$$

Each of the integrals on the r.h.s. diverges, but their sum remains finite

However, multiplying this distribution with a step function would produce diverging terms

$$\frac{\theta(x)}{x+i0}$$

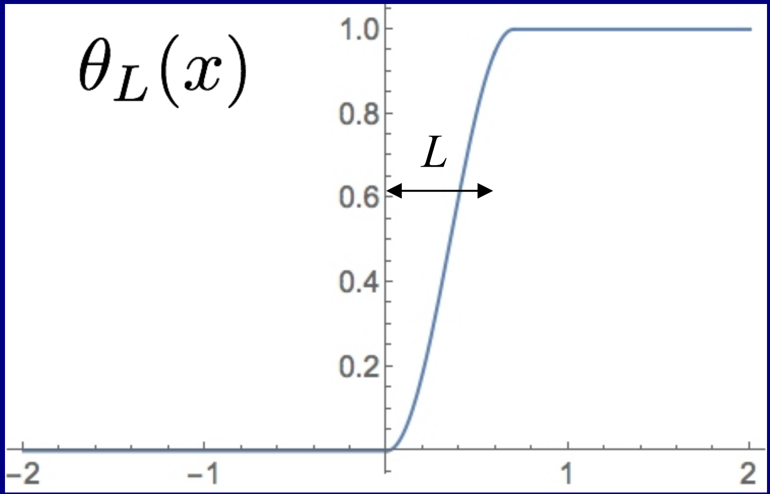
Distribution splitting

Solution to this problem: distribution splitting (Malgrange 1960)

Observation: $\dfrac{\theta(x)}{x+i0}$ remains finite as long as f(0)=0

Strategy for splitting a distribution *d*(x)

- Replace step function by a smooth step function $\theta_L(x)$ that varies over a width *L*. $\theta_L d(x)$ is then well defined



Distribution splitting

- In the limit $L \to 0$, distribution $\theta_L d(x)$ will diverge like $L^{-\omega}$. Distribution *d*(*x*) is the called singular of order ω

- Introduce a projector $\hat{P}_\omega$ that maps test functions on the subspace of functions where, at *x*=0, all derivatives up to order ω vanish. Then

$$\int_{-\infty}^{\infty} dx\, \theta_L(x) d(x) \hat{P}_\omega f(x)$$

is well defined

- The properly split distribution is $\lim_{L \to 0} \theta_L(x) d(x) \hat{P}_\omega$

Spontaneous emission

In CPT only causal distributions (light-like and/or time-like support) can be split (Scharf 2011)

In QO, the center-of-mass motion of atoms in electronic state $|E_n\rangle$ is described by a field operator $\hat{\Psi}_n(x)$

We take $\hat{\Psi}_n(x)$ to be a complex Klein-Gordon field of mass $m_0 + E_n/c^2$

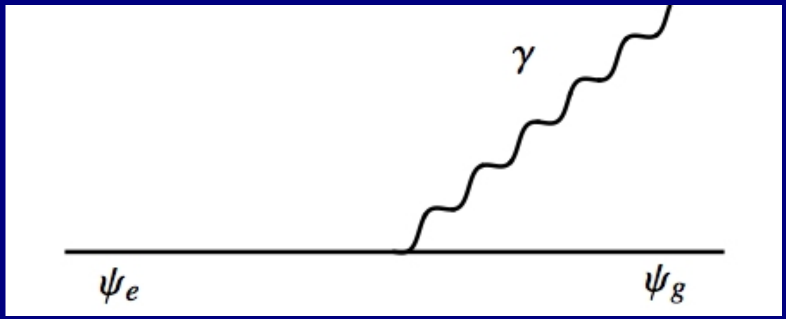
Hamiltonian for 2-level atom coupled to radiation field:

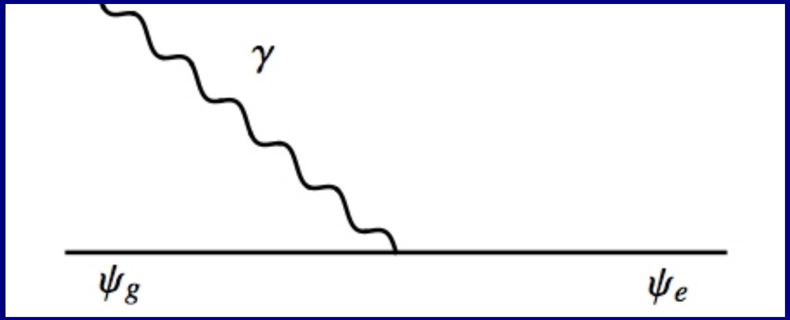
$$H_{\rm int}(x) = -\left(\hat{\Psi}_e^\dagger(x)\hat{\Psi}_g(x)\vec{d}_{eg} + \hat{\Psi}_g^\dagger(x)\hat{\Psi}_e(x)\vec{d}_{eg}^{\,*}\right)\cdot\vec{E}(x)$$

Spontaneous emission

Expansion of S-matrix: $S = \mathbb{1} + \hat{T}_1 + \hat{T}_2 + \cdots$

$\hat{T}_1 = -\frac{i}{\hbar c} \int d^4x\, H_{\rm int}(x)$ describes





time

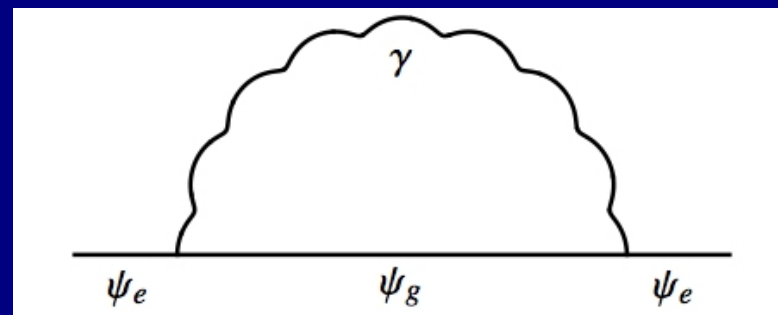
emission and absorption of photons

Spontaneous emission

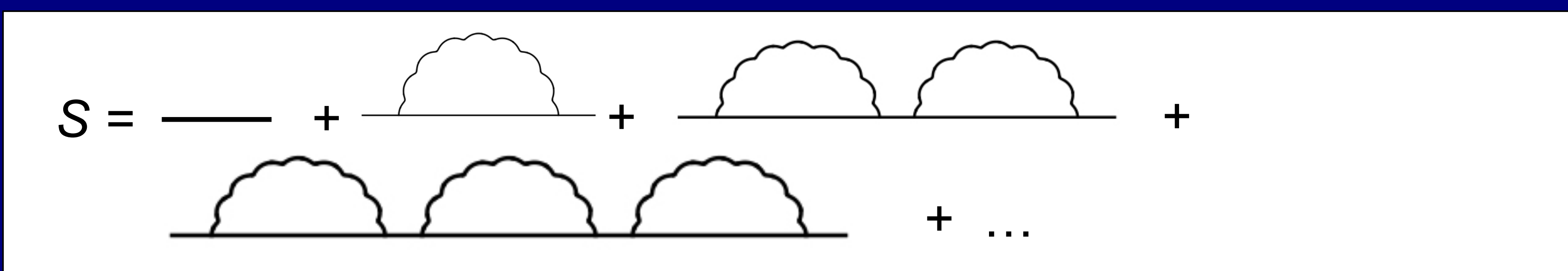
T_2 is related to T_1 via proper distribution splitting

$$\hat{T}_2 = \theta_L(x-y)\hat{T}_1(x)\hat{T}_1(y)$$

T_2 describes self-energy



Knight and Allan (1972): spontaneous emission can be described using the ladder approximation



Spontaneous emission

This means we only need to find T_2

Result before distribution splitting:

$$\hat{T}_2 = \int d^4x \int d^4y \, T_2(x-y) : \hat{\Psi}_e^\dagger(x) \hat{\Psi}_e(y) :$$

$$T_2(x-y) \approx [\hat{\Psi}_g(x), \hat{\Psi}_g^\dagger(y)] \, [\vec{d}_{eg} \cdot \vec{E}(x), \vec{d}^*_{eg} \cdot \vec{E}(y)]$$

The product of the two commutators has causal support and is singular of order 2.

Spontaneous emission

The projector $\hat{P}_\omega$ is best evaluated in momentum space (Aste, von Arx, Scharf 2010)

$$\hat{P}_\omega T_2(p) = \int d^4k \, \theta_L(k) \left(T_2(p-k) - \sum_{|b|=0}^{\omega} \frac{(p-q)^b}{b!} \partial_q^b T_2(q-k) \right)$$

Four-momentum *q* is called normalization point

q is not unique, its choice corresponds to renormalization parameters

We pick $q_\mu q^\mu = m_g^2 c^2 / \hbar^2$

Spontaneous emission

Result for T_2:

$$\theta_L T_2(p_\mu) = \frac{i(\sqrt{u}-1)^3}{192\pi^4 c\lambda^4 u^3 \epsilon_0 \hbar}\left(2\lambda^2|\vec{d}_{eg}\cdot\vec{p}|^2 + |d_{eg}|^2(u - 2\lambda^2 p_0^2)\right)$$
$$\times\left(4\pi(\sqrt{u}+1)^3\theta(-p_0)\theta(u-1) + i(3u^{3/2}+u)\right.$$
$$\left. - \pi(\sqrt{u}+1)^3 + i(\sqrt{u}+1)^3\log(u-1)\right)$$

with $u = \lambda^2 p_\mu p^\mu$ and $\lambda = \dfrac{\hbar}{m_g c}$

Spontaneous emission

Initial state: resting excited atom: $p_\mu = \left(\frac{m_e c}{\hbar}, \vec{0}\right)$

Resonance frequency fulfills $\hbar\omega_{eg} = (m_e - m_g)c^2 \ll m_g c^2$

Expanding T_2 to lowest order in ω_{eg} yields

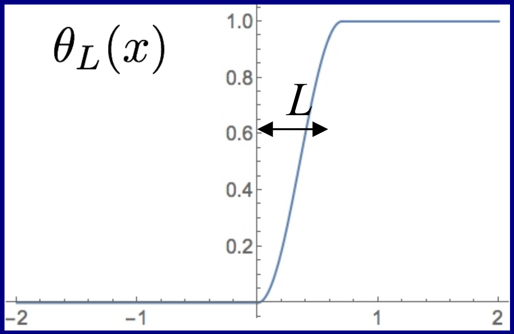
$$T_2(p) \approx \frac{\gamma \log\left(\frac{2\hbar\omega_{eg}}{m_g c^2}\right)}{\pi} + \frac{\gamma}{2\pi} + i\gamma$$

$$\gamma = \frac{|\vec{d}_{eg}|^2 \omega_{eg}^3}{3\pi\hbar\varepsilon_0 c^3}$$

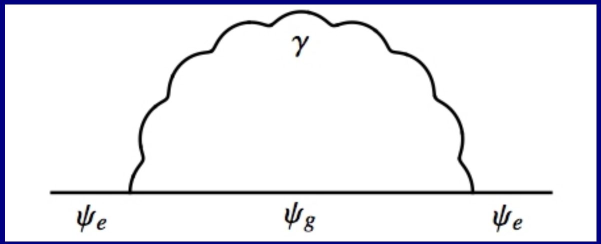
This is similar to the standard result for decay rate and Lamb shift

Conclusion

Causal perturbation theory is a way to avoid divergent terms in QFT



We used CPT for a non-covariant but causal model of atom-light interaction



Spontaneous decay rate and Lamb shift are similar to standard results

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Thanks!