Thermalization by Rapid Repeated Interaction

Daniel Grimmer
Eduardo Martin-Martinez   Robert Mann

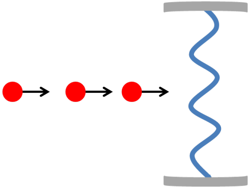
University of Waterloo
Institute for Quantum Computing

May 31, 2017

What dynamics emerge under bombardment by the environment?
Decoherence? Purification? Thermalization?

Atom bombarded by a series of atoms/light pulses:

Cavity bombarded by atoms (e.g. Entanglement Farming[1]):



[1] E. Martin-Martinez, E. Brown, W. Donnelly, A. Kempf; PRA 88, 052310 (2013)

Rapid Repeated Interaction Formalism (RRIF)

RRIF considers a systems, S, which has discrete-time evolution,

$$\text{For fixed } \delta t: \quad \rho_S(n\,\delta t) = \phi(\delta t)[\dots\phi(\delta t)[\rho_S(0)]\dots] = \phi(\delta t)^n\big[\rho_S(0)\big].$$

where $\phi(\delta t)$ is CPTP
has $\phi(0) = \mathbf{1}$ (nothing happens in no time),
and $\phi'(0)$ exists (things happen at a finite rate).

Rapid Repeated Interaction Formalism (RRIF)

RRIF considers a systems, S, which has discrete-time evolution,

$$\text{For fixed } \delta t: \quad \rho_S(n\,\delta t) = \phi(\delta t)\big[\dots\phi(\delta t)\big[\rho_S(0)\big]\dots\big] = \phi(\delta t)^n\big[\rho_S(0)\big].$$

where $\phi(\delta t)$ is CPTP
has $\phi(0) = \mathbf{1}$ (nothing happens in no time),
and $\phi'(0)$ exists (things happen at a finite rate).

For example, $\phi(\delta t)[\rho_\text{S}] = \text{Tr}_\text{A}\Big(\exp(-\mathrm{i}\,\delta t\,\hat{H}/\hbar)(\rho_\text{S}\otimes\rho_\text{A})\exp(\mathrm{i}\,\delta t\,\hat{H}/\hbar)\Big)$

Rapid Repeated Interaction Formalism (RRIF)

RRIF considers a systems, S, which has discrete-time evolution,

$$\text{For fixed } \delta t: \quad \rho_S(n\,\delta t) = \phi(\delta t)[\ldots\phi(\delta t)[\rho_S(0)]\ldots] = \phi(\delta t)^n\big[\rho_S(0)\big].$$

where $\phi(\delta t)$ is CPTP
has $\phi(0) = \mathbf{1}$ (nothing happens in no time),
and $\phi'(0)$ exists (things happen at a finite rate).

For example, $\phi(\delta t)[\rho_\text{S}] = \text{Tr}_\text{A}\Big(\exp(-\mathrm{i}\,\delta t\,\hat{H}/\hbar)(\rho_\text{S}\otimes\rho_\text{A})\exp(\mathrm{i}\,\delta t\,\hat{H}/\hbar)\Big)$

We find the **unique** continuous-time interpolation scheme which:
1) is (time independent) Markovian with effective Liouvillian $\mathcal{L}_{\delta t}$,
2) exactly fits the discrete time points, and
3) is well defined as $\delta t \to 0$.

$$\frac{d}{dt}\rho_\text{S}(t) = \mathcal{L}_{\delta t}[\rho_\text{S}(t)] \quad \text{with} \quad \mathcal{L}_{\delta t} = \frac{1}{\delta t}\text{Log}\big(\phi(\delta t)\big).$$

Series Expansions

Taking $\hat{H} = \hat{H}_\mathsf{S} \otimes \hat{\mathbf{1}} + \hat{\mathbf{1}} \otimes \hat{H}_\mathsf{A} + \hat{H}_\mathsf{SA}$ and

$$\phi(\delta t)[\rho_\mathsf{S}] = \mathrm{Tr}_\mathsf{A}\Big(\exp(-\mathrm{i}\,\delta t\,\hat{H}/\hbar)(\rho_\mathsf{S} \otimes \rho_\mathsf{A})\exp(\mathrm{i}\,\delta t\,\hat{H}/\hbar)\Big)$$

We can expand the update map as

$$\phi(\delta t) = \mathbf{1} + \delta t\,\phi_1 + \delta t^2\,\phi_2 + \delta t^3\,\phi_3 + \dots\,.$$

Using this we can expand the effective Liouvillian as,

$$\mathcal{L}_{\delta t} = \frac{1}{\delta t}\mathrm{Log}\big(\phi(\delta t)\big) = \mathcal{L}_0 + \delta t\,\mathcal{L}_1 + \delta t^2\,\mathcal{L}_2 + \delta t^3\,\mathcal{L}_3 + \dots\,.$$

ultimately in terms of in terms of $\hat{H}_{\mathrm{S}}$, $\hat{H}_{\mathrm{A}}$, $\hat{H}_{\mathrm{SA}}$, and ρ_{A}.

Series Expansions

Taking $\hat{H} = \hat{H}_\mathsf{S} \otimes \hat{\mathbf{1}} + \hat{\mathbf{1}} \otimes \hat{H}_\mathsf{A} + \hat{H}_\mathsf{SA}$ and

$$\phi(\delta t)[\rho_\mathsf{S}] = \mathrm{Tr}_\mathsf{A}\Big(\exp(-\mathrm{i}\,\delta t\,\hat{H}/\hbar)(\rho_\mathsf{S} \otimes \rho_\mathsf{A})\exp(\mathrm{i}\,\delta t\,\hat{H}/\hbar)\Big)$$

We can expand the update map as

$$\phi(\delta t) = \mathbf{1} + \delta t\,\phi_1 + \delta t^2\,\phi_2 + \delta t^3\,\phi_3 + \dots\,.$$

Using this we can expand the effective Liouvillian as,

$$\mathcal{L}_{\delta t} = \frac{1}{\delta t}\mathrm{Log}\big(\phi(\delta t)\big) = \mathcal{L}_0 + \delta t\,\mathcal{L}_1 + \delta t^2\,\mathcal{L}_2 + \delta t^3\,\mathcal{L}_3 + \dots\,.$$

ultimately in terms of in terms of $\hat{H}_\mathrm{S}$, $\hat{H}_\mathrm{A}$, $\hat{H}_\mathrm{SA}$, and ρ_A.

The master equation for the interpolation scheme is thus expanded as,

$$\frac{d}{dt}\rho_\mathsf{S}(t) = \mathcal{L}_0[\rho_\mathsf{S}(t)] + \delta t\,\mathcal{L}_1[\rho_\mathsf{S}(t)] + \delta t^2\mathcal{L}_2[\rho_\mathsf{S}(t)] + \delta t^3\mathcal{L}_3[\rho_\mathsf{S}(t)] + \dots\,.$$

When Does Rapid Bombardment Decohere?

Q: What happens as $\delta t \to 0$? (What is $\mathcal{L}_0$ like?)

[2] D. Layden, E. Martin-Martinez and A. Kempf; Phys. Rev. A 93, 040301(R) (2016)
[3] D. Grimmer, D. Layden, R. B. Mann, and E. Martin-Martinez; ArXiv:1605.04302;

When Does Rapid Bombardment Decohere?

Q: What happens as $\delta t \to 0$? (What is $\mathcal{L}_0$ like?)

A: The evolution is unitary, but nontrivial[2]!

$$\mathcal{L}_0[\,\rho\,] = \frac{-\mathrm{i}}{\hbar}[\hat{H}_{\text{eff}},\,\rho\,] \quad , \qquad \hat{H}_{\text{eff}} = \hat{H}_S + \text{Tr}_A\big(\hat{H}_{\text{SA}}\,\rho_{\text{A}}\big)$$

[2] D. Layden, E. Martin-Martinez and A. Kempf; Phys. Rev. A 93, 040301(R) (2016)
[3] D. Grimmer, D. Layden, R. B. Mann, and E. Martin-Martinez; ArXiv:1605.04302;

When Does Rapid Bombardment Decohere?

Q: What happens as $\delta t \to 0$? (What is $\mathcal{L}_0$ like?)

A: The evolution is unitary, but nontrivial[2]!

$$\mathcal{L}_0[\,\rho\,] = \frac{-\mathrm{i}}{\hbar}[\hat{H}_{\text{eff}},\,\rho\,] \quad , \qquad \hat{H}_{\text{eff}} = \hat{H}_S + \text{Tr}_A\big(\hat{H}_{\text{SA}}\,\rho_{\text{A}}\big)$$

The ancilla *push* the system dynamics.
But there is no quantum information flow.

[2] D. Layden, E. Martin-Martinez and A. Kempf; Phys. Rev. A 93, 040301(R) (2016)
[3] D. Grimmer, D. Layden, R. B. Mann, and E. Martin-Martinez; ArXiv:1605.04302;

When Does Rapid Bombardment Decohere?

Q: What happens as $\delta t \to 0$? (What is $\mathcal{L}_0$ like?)

A: The evolution is unitary, but nontrivial[2]!

$$\mathcal{L}_0[\,\rho\,] = \frac{-\mathrm{i}}{\hbar}[\hat{H}_{\text{eff}},\,\rho\,] \quad , \qquad \hat{H}_{\text{eff}} = \hat{H}_S + \text{Tr}_A\big(\hat{H}_{\text{SA}}\,\rho_{\text{A}}\big)$$

The ancilla *push* the system dynamics.
But there is no quantum information flow.

The system is insulated from its environment.
But still allows for universal unitary control.

[2] D. Layden, E. Martin-Martinez and A. Kempf; Phys. Rev. A 93, 040301(R) (2016)
[3] D. Grimmer, D. Layden, R. B. Mann, and E. Martin-Martinez; ArXiv:1605.04302;

When Does Rapid Bombardment Decohere?

Q: What happens as $\delta t \to 0$? (What is $\mathcal{L}_0$ like?)

A: The evolution is unitary, but nontrivial[2]!

$$\mathcal{L}_0[\,\rho\,] = \frac{-\mathrm{i}}{\hbar}[\hat{H}_{\text{eff}},\,\rho\,] \;\;, \qquad \hat{H}_{\text{eff}} = \hat{H}_S + \text{Tr}_A\big(\hat{H}_{\text{SA}}\,\rho_{\text{A}}\big)$$

The ancilla *push* the system dynamics.
But there is no quantum information flow.

The system is insulated from its environment.
But still allows for universal unitary control.

Outside of the limit $\delta t \to 0$, the dynamics generically has decoherence[3].

[2] D. Layden, E. Martin-Martinez and A. Kempf; Phys. Rev. A 93, 040301(R) (2016)
[3] D. Grimmer, D. Layden, R. B. Mann, and E. Martin-Martinez; ArXiv:1605.04302;

When Does Rapid Bombardment Purify?

Q: Can a system be purified by rapid bombardment?

[4] D. Grimmer, R. B. Mann, and E. Martin-Martinez; ArXiv:1611.07530; (PRA, 2017)
[5] D. A. Lidar, A. Shabani, and R. Alicki. Chem. Phys. 332, 82 (2006)

When Does Rapid Bombardment Purify?

Q: Can a system be purified by rapid bombardment?

A: Purification requires a sufficiently complex interaction[4].

[4] D. Grimmer, R. B. Mann, and E. Martin-Martinez; ArXiv:1611.07530; (PRA, 2017)

[5] D. A. Lidar, A. Shabani, and R. Alicki. Chem. Phys. 332, 82 (2006)

When Does Rapid Bombardment Purify?

Q: Can a system be purified by rapid bombardment?
A: Purification requires a sufficiently complex interaction[4].

Dynamics generated by some $\mathcal{L}$ can purify if and only if the dynamics moves the maximally mixed state, $\mathcal{L}[\hat{I}] \neq 0$[5].

[4] D. Grimmer, R. B. Mann, and E. Martin-Martinez; ArXiv:1611.07530; (PRA, 2017)
[5] D. A. Lidar, A. Shabani, and R. Alicki. Chem. Phys. 332, 82 (2006)

When Does Rapid Bombardment Purify?

Q: Can a system be purified by rapid bombardment?
A: Purification requires a sufficiently complex interaction[4].

Dynamics generated by some $\mathcal{L}$ can purify if and only if the dynamics moves the maximally mixed state, $\mathcal{L}[\hat{I}] \neq 0$[5].

Since $\mathcal{L}_0$ generates unitary evolution we have $\mathcal{L}_0[\hat{I}] = 0$.
When can $\mathcal{L}_1$ purify ($\mathcal{L}_1[\hat{I}] \neq 0$)?

[4] D. Grimmer, R. B. Mann, and E. Martin-Martinez; ArXiv:1611.07530; (PRA, 2017)
[5] D. A. Lidar, A. Shabani, and R. Alicki. Chem. Phys. 332, 82 (2006)

When Does Rapid Bombardment Purify?

Q: Can a system be purified by rapid bombardment?
A: Purification requires a sufficiently complex interaction[4].

Dynamics generated by some $\mathcal{L}$ can purify if and only if the dynamics moves the maximally mixed state, $\mathcal{L}[\,\hat{I}\,] \neq 0$[5].

Since $\mathcal{L}_0$ generates unitary evolution we have $\mathcal{L}_0[\,\hat{I}\,] = 0$.
When can $\mathcal{L}_1$ purify ($\mathcal{L}_1[\,\hat{I}\,] \neq 0$)?

Expanding a generic interaction Hamiltonian as $\hat{H}_{\text{SA}} = \sum_j \hat{Q}_j \otimes \hat{R}_j$ we find

$$\mathcal{L}_1[\,\hat{I}\,] = \frac{1}{2}\Big(\frac{-\mathrm{i}}{\hbar}\Big)^2 \sum_{i,j} [\hat{Q}_i, \hat{Q}_j] \; \text{Tr}_{\text{A}}\Big([\hat{R}_i, \hat{R}_j]\rho_{\text{A}}\Big).$$

[4] D. Grimmer, R. B. Mann, and E. Martin-Martinez; ArXiv:1611.07530; (PRA, 2017)
[5] D. A. Lidar, A. Shabani, and R. Alicki. Chem. Phys. 332, 82 (2006)

Purification Condition

In order for the interaction Hamiltonian $\hat{H}_{\mathsf{SA}} = \sum_j \hat{Q}_j \otimes \hat{R}_j$ to purify at first order in rapid bombardment one needs

$$\mathcal{L}_1[\hat{I}] = \frac{1}{2}\Big(\frac{-\mathrm{i}}{\hbar}\Big)^2 \sum_{i,j} [\hat{Q}_i, \hat{Q}_j] \, \mathrm{Tr}_{\mathsf{A}}\Big([\hat{R}_i, \hat{R}_j]\rho_{\mathsf{A}}\Big) \neq 0$$

Purification Condition

In order for the interaction Hamiltonian $\hat{H}_{\mathsf{SA}} = \sum_j \hat{Q}_j \otimes \hat{R}_j$ to purify at first order in rapid bombardment one needs

$$\mathcal{L}_1[\hat{I}] = \frac{1}{2}\Big(\frac{-\mathrm{i}}{\hbar}\Big)^2 \sum_{i,j} [\hat{Q}_i, \hat{Q}_j] \, \mathrm{Tr}_{\mathsf{A}}\Big([\hat{R}_i, \hat{R}_j]\rho_{\mathsf{A}}\Big) \neq 0$$

Thus the simple interaction Hamiltonian $\hat{H}_{\mathsf{SA}} = \hat{Q}_{\mathsf{S}} \otimes \hat{R}_{\mathsf{A}}$ **cannot** purify in rapid bombardment.

Purification Condition

In order for the interaction Hamiltonian $\hat{H}_{\mathrm{SA}} = \sum_j \hat{Q}_j \otimes \hat{R}_j$ to purify at first order in rapid bombardment one needs

$$\mathcal{L}_1[\hat{I}] = \frac{1}{2}\left(\frac{-\mathrm{i}}{\hbar}\right)^2 \sum_{i,j} [\hat{Q}_i, \hat{Q}_j]\ \mathrm{Tr}_{\mathrm{A}}\Big([\hat{R}_i, \hat{R}_j]\rho_{\mathrm{A}}\Big) \neq 0$$

Thus the simple interaction Hamiltonian $\hat{H}_{\mathrm{SA}} = \hat{Q}_{\mathrm{S}} \otimes \hat{R}_{\mathrm{A}}$ **cannot** purify in rapid bombardment.

The isotropic spin-spin interaction $\hat{H}_{SA} = \hat{\boldsymbol{\sigma}}_{\mathrm{S}} \cdot \hat{\boldsymbol{\sigma}}_{\mathrm{A}}$ in fact **can** purify in rapid bombardment.

Purification Condition

In order for the interaction Hamiltonian $\hat{H}_{\mathrm{SA}} = \sum_j \hat{Q}_j \otimes \hat{R}_j$ to purify at first order in rapid bombardment one needs

$$\mathcal{L}_1[\hat{I}] = \frac{1}{2}\left(\frac{-\mathrm{i}}{\hbar}\right)^2 \sum_{i,j} [\hat{Q}_i, \hat{Q}_j] \; \mathrm{Tr}_{\mathrm{A}}\Big([\hat{R}_i, \hat{R}_j]\rho_{\mathrm{A}}\Big) \neq 0$$

Thus the simple interaction Hamiltonian $\hat{H}_{\mathrm{SA}} = \hat{Q}_{\mathrm{S}} \otimes \hat{R}_{\mathrm{A}}$ **cannot** purify in rapid bombardment.

The isotropic spin-spin interaction $\hat{H}_{SA} = \hat{\boldsymbol{\sigma}}_{\mathrm{S}} \cdot \hat{\boldsymbol{\sigma}}_{\mathrm{A}}$ in fact **can** purify in rapid bombardment.

The light-matter interaction up to E4

$$\hat{H}_{SA} = \underbrace{\hat{\boldsymbol{x}}_S \cdot \hat{\boldsymbol{E}}_A}_{\text{Electric Dipole}} + \underbrace{\hat{\boldsymbol{L}}_S \cdot \hat{\boldsymbol{B}}_A}_{\text{Magnetic Dipole}} + \underbrace{\hat{\boldsymbol{x}}_S \, \nabla \hat{\boldsymbol{E}}_A \, \hat{\boldsymbol{x}}_S}_{\text{Electric Quadrupole}}$$

cannot purify in rapid bombardment[6].

When Does Rapid Bombardment Thermalize?

Q: Would a system rapidly bombarded by constituents of a thermal environment thermalize?

When Does Rapid Bombardment Thermalize?

Q: Would a system rapidly bombarded by constituents of a thermal environment thermalize?

A: Only a very specific family of interaction Hamiltonians thermalize in rapid bombardment.

When Does Rapid Bombardment Thermalize?

Q: Would a system rapidly bombarded by constituents of a thermal environment thermalize?
A: Only a very specific family of interaction Hamiltonians thermalize in rapid bombardment.

Consider a harmonic oscillator rapidly bombarded by thermal oscillators under a quadratic interaction Hamiltonian,

$$\begin{aligned}\hat{H}_{\text{SA}} &= \begin{pmatrix}\hat{x}_{\text{S}} & \hat{p}_{\text{S}}\end{pmatrix}\begin{pmatrix}g_{\text{xx}} & g_{\text{xp}}\\ g_{\text{px}} & g_{\text{pp}}\end{pmatrix}\begin{pmatrix}\hat{x}_{\text{A}}\\ \hat{p}_{\text{A}}\end{pmatrix}\\ &= \hat{\boldsymbol{X}}_{\text{S}}^{\intercal}\, G\, \hat{\boldsymbol{X}}_{\text{A}}\end{aligned}$$

When Does Rapid Bombardment Thermalize?

Q: Would a system rapidly bombarded by constituents of a thermal environment thermalize?
A: Only a very specific family of interaction Hamiltonians thermalize in rapid bombardment.

Consider a harmonic oscillator rapidly bombarded by thermal oscillators under a quadratic interaction Hamiltonian,

$$\begin{aligned}\hat{H}_{\text{SA}} &= \begin{pmatrix}\hat{x}_{\text{S}} & \hat{p}_{\text{S}}\end{pmatrix} \begin{pmatrix} g_{\text{xx}} & g_{\text{xp}} \\ g_{\text{px}} & g_{\text{pp}} \end{pmatrix} \begin{pmatrix}\hat{x}_{\text{A}} \\ \hat{p}_{\text{A}}\end{pmatrix} \\ &= \hat{\boldsymbol{X}}_{\text{S}}^{\intercal}\, G\, \hat{\boldsymbol{X}}_{\text{A}}\end{aligned}$$

Which G will take drive the system to a thermal state?

When Does Rapid Bombardment Thermalize?

Q: Would a system rapidly bombarded by constituents of a thermal environment thermalize?
A: Only a very specific family of interaction Hamiltonians thermalize in rapid bombardment.

Consider a harmonic oscillator rapidly bombarded by thermal oscillators under a quadratic interaction Hamiltonian,

$$\begin{aligned}\hat{H}_{\text{SA}} &= \begin{pmatrix}\hat{x}_{\text{S}} & \hat{p}_{\text{S}}\end{pmatrix}\begin{pmatrix}g_{\text{xx}} & g_{\text{xp}}\\ g_{\text{px}} & g_{\text{pp}}\end{pmatrix}\begin{pmatrix}\hat{x}_{\text{A}}\\ \hat{p}_{\text{A}}\end{pmatrix}\\ &= \hat{\boldsymbol{X}}_{\text{S}}^{\intercal}\, G\, \hat{\boldsymbol{X}}_{\text{A}}\end{aligned}$$

Which G will take drive the system to a thermal state?
When will this thermal state be at the same temperature as the environment, $T_{\text{S}} = T_{\text{E}}$?

Themalization Conditions

If $\det(G) \leq 0$, then the system will become "unboundedly noisy".

$$G = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} \longrightarrow \hat{H}_{\text{SA}} = \hat{x}_S \otimes \hat{x}_A - \hat{p}_S \otimes \hat{p}_A$$

Themalization Conditions

If $\det(G) \leq 0$, then the system will become "unboundedly noisy".

$$G = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} \longrightarrow \hat{H}_{\mathrm{SA}} = \hat{x}_S \otimes \hat{x}_A - \hat{p}_S \otimes \hat{p}_A$$

If $\det(G) > 0$, the system will thermalize to a temperature hotter than its environment,

$$T_{\mathrm{S}}(t = \infty) = \frac{\mathrm{Tr}(G^{\intercal} G)}{2 \det(G)} T_{\mathrm{E}} \geq T_{\mathrm{E}}.$$

Themalization Conditions

If $\det(G) \leq 0$, then the system will become "unboundedly noisy".

$$G = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} \longrightarrow \hat{H}_{\text{SA}} = \hat{x}_S \otimes \hat{x}_A - \hat{p}_S \otimes \hat{p}_A$$

If $\det(G) > 0$, the system will thermalize to a temperature hotter than its environment,

$$T_{\text{S}}(t = \infty) = \frac{\text{Tr}(G^{\intercal} G)}{2 \det(G)} T_{\text{E}} \geq T_{\text{E}}.$$

This inequality is saturated if and only if

$$G = g_1 \begin{pmatrix} +1 & 0 \\ 0 & +1 \end{pmatrix} + g_2 \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}$$

$$\longrightarrow \hat{H}_{\text{SA}} = g_1 \left(\hat{x}_S \otimes \hat{x}_A + \hat{p}_S \otimes \hat{p}_A\right) + g_2 \left(\hat{x}_S \otimes \hat{p}_A - \hat{p}_S \otimes \hat{x}_A\right)$$

Conclusion

- This formalism allows one to analytically deal with arbitrary quantum systems and couplings.
- In the limit of rapid bombardment, one finds non-trivial unitary dynamics, offering both isolation from the environment and control.
- Only sufficiently complex interactions purify through rapid bombardment.
- Interactions rarely thermalize through rapid bombardment.