Thermalization by Rapid Repeated Interaction

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Repeated Interaction with Bits of Environment

What dynamics emerge under bombardment by the environment? Decoherence? Purification? Thermalization?

Atom bombarded by a series of atoms/light pulses:

\[ \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \text{ or } \quad \text{\hline} \text{\hline} \text{\hline} \]

Cavity bombarded by atoms (e.g. Entanglement Farming\(^1\)):

\[ \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \]

\(^1\)E. Martin-Martinez, E. Brown, W. Donnelly, A. Kempf; PRA 88, 052310 (2013)
RRIF considers a system $S$, which has discrete-time evolution.

For fixed $\delta t$:

$\rho_S(n\delta t) = \phi(\delta t)[... \phi(\delta t)[\rho_S(0)]...] = \phi(\delta t)^n[\rho_S(0)]$.

where $\phi(\delta t)$ is CPTP

has $\phi(0) = 1$ (nothing happens in no time),

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For example, $\phi(\delta t)[\rho_S] = \text{Tr}_A \left( \exp(-i \delta t \hat{H}/\hbar)(\rho_S \otimes \rho_A) \exp(i \delta t \hat{H}/\hbar) \right)$
Rapid Repeated Interaction Formalism (RRIF)

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$$

We find the unique continuous-time interpolation scheme which:

1) is (time independent) Markovian with effective Liouvillian $\mathcal{L}_{\delta t}$,

2) exactly fits the discrete time points, and

3) is well defined as $\delta t \to 0$.

$$
\frac{d}{dt}\rho_S(t) = \mathcal{L}_{\delta t}[\rho_S(t)] \quad \text{with} \quad \mathcal{L}_{\delta t} = \frac{1}{\delta t} \text{Log}(\phi(\delta t)).
$$
Series Expansions

Taking $\hat{H} = \hat{H}_S \otimes \mathbf{1} + \mathbf{1} \otimes \hat{H}_A + \hat{H}_{SA}$ and

$$\phi(\delta t)[\rho_S] = \text{Tr}_A \left( \exp \left( -i \delta t \frac{\hat{H}}{\hbar} \right) (\rho_S \otimes \rho_A) \exp \left( i \delta t \frac{\hat{H}}{\hbar} \right) \right)$$

We can expand the update map as

$$\phi(\delta t) = 1 + \delta t \phi_1 + \delta t^2 \phi_2 + \delta t^3 \phi_3 + \ldots.$$ 

Using this we can expand the effective Liouvillian as,

$$\mathcal{L}_{\delta t} = \frac{1}{\delta t} \log(\phi(\delta t)) = \mathcal{L}_0 + \delta t \mathcal{L}_1 + \delta t^2 \mathcal{L}_2 + \delta t^3 \mathcal{L}_3 + \ldots.$$ 

ultimately in terms of in terms of $\hat{H}_S$, $\hat{H}_A$, $\hat{H}_{SA}$, and $\rho_A$. 
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Using this we can expand the effective Liouvillian as,

$$L_{\delta t} = \frac{1}{\delta t} \log(\phi(\delta t)) = L_0 + \delta t L_1 + \delta t^2 L_2 + \delta t^3 L_3 + \ldots.$$ 

ultimately in terms of $\hat{H}_S$, $\hat{H}_A$, $\hat{H}_{SA}$, and $\rho_A$.

The master equation for the interpolation scheme is thus expanded as,

$$\frac{d}{dt}\rho_S(t) = L_0[\rho_S(t)] + \delta t L_1[\rho_S(t)] + \delta t^2 L_2[\rho_S(t)] + \delta t^3 L_3[\rho_S(t)] + \ldots.$$
When Does Rapid Bombardment Decohere?

**Q:** What happens as $\delta t \to 0$? (What is $L_0$ like?)

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$^{3}$D. Grimmer, D. Layden, R. B. Mann, and E. Martin-Martinez; ArXiv:1605.04302;
**Q:** What happens as $\delta t \to 0$? (What is $L_0$ like?)

**A:** The evolution is unitary, but nontrivial$^2$!

$$L_0[\rho] = -\frac{i}{\hbar} [\hat{H}_{\text{eff}}, \rho], \quad \hat{H}_{\text{eff}} = \hat{H}_S + \text{Tr}_A (\hat{H}_{SA} \rho_A)$$

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The system is insulated from its environment.
But still allows for universal unitary control.

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Outside of the limit $\delta t \to 0$, the dynamics generically has decoherence$^3$.

$^3$D. Grimmer, D. Layden, R. B. Mann, and E. Martin-Martinez; ArXiv:1605.04302;
When Does Rapid Bombardment Purify?

Q: Can a system be purified by rapid bombardment?

\[ \text{D. Grimmer, R. B. Mann, and E. Martin-Martinez; ArXiv:1611.07530; (PRA, 2017)} \]

\[ \text{D. A. Lidar, A. Shabani, and R. Alicki. Chem. Phys. 332, 82 (2006)} \]
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Dynamics generated by some $\mathcal{L}$ can purify if and only if the dynamics moves the maximally mixed state, $\mathcal{L}[\hat{I}] \neq 0$\textsuperscript{5}.

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Since \( \mathcal{L}_0 \) generates unitary evolution we have \( \mathcal{L}_0[\hat{I}] = 0 \). When can \( \mathcal{L}_1 \) purify (\( \mathcal{L}_1[\hat{I}] \neq 0 \))?

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When can $\mathcal{L}_1$ purify ($\mathcal{L}_1[\hat{I}] \neq 0$)?

Expanding a generic interaction Hamiltonian as $\hat{H}_{SA} = \sum_j \hat{Q}_j \otimes \hat{R}_j$ we find

$$\mathcal{L}_1[\hat{I}] = \frac{1}{2} \left( -\frac{i}{\hbar} \right)^2 \sum_{i,j} [\hat{Q}_i, \hat{Q}_j] \text{Tr}_A \left( [\hat{R}_i, \hat{R}_j] \rho_A \right).$$

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Purification Condition

In order for the interaction Hamiltonian $\hat{H}_{SA} = \sum_j \hat{Q}_j \otimes \hat{R}_j$ to purify at first order in rapid bombardment one needs

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The isotropic spin-spin interaction $\hat{H}_{SA} = \hat{\sigma}_S \cdot \hat{\sigma}_A$ in fact can purify in rapid bombardment.
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The light-matter interaction up to E4

\[
\hat{H}_{SA} = \hat{x}_S \cdot \hat{E}_A + \hat{L}_S \cdot \hat{B}_A + \hat{x}_S \nabla \hat{E}_A \hat{x}_S
\]

cannot purify in rapid bombardment\(^6\).

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Consider a harmonic oscillator rapidly bombarded by thermal oscillators under a quadratic interaction Hamiltonian,

\[ \hat{H}_{SA} = (\hat{x}_S \quad \hat{p}_S) \begin{pmatrix} g_{xx} & g_{xp} \\ g_{px} & g_{pp} \end{pmatrix} (\hat{x}_A \quad \hat{p}_A) = \hat{X}^T_S \ G \ \hat{X}_A \]
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Which $G$ will take drive the system to a thermal state?
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Which \( G \) will take drive the system to a thermal state?
When will this thermal state be at the same temperature as the environment, \( T_S = T_E \)?
Thermalization Conditions

If \( \det(G) \leq 0 \), then the system will become “unboundedly noisy”.

\[
G = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} \quad \rightarrow \quad \hat{H}_{SA} = \hat{x}_S \otimes \hat{x}_A - \hat{p}_S \otimes \hat{p}_A
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If $\det(G) > 0$, the system will thermalize to a temperature hotter than its environment,

$$T_S(t = \infty) = \frac{\text{Tr}(G^T G)}{2 \det(G)} T_E \geq T_E.$$
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\]

This inequality is saturated if and only if

\[
G = g_1 \begin{pmatrix} +1 & 0 \\ 0 & +1 \end{pmatrix} + g_2 \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}
\]

\[
\rightarrow \hat{H}_{SA} = g_1 (\hat{x}_S \otimes \hat{x}_A + \hat{p}_S \otimes \hat{p}_A) + g_2 (\hat{x}_S \otimes \hat{p}_A - \hat{p}_S \otimes \hat{x}_A)
\]
This formalism allows one to analytically deal with arbitrary quantum systems and couplings. In the limit of rapid bombardment, one finds non-trivial unitary dynamics, offering both isolation from the environment and control. Only sufficiently complex interactions purify through rapid bombardment. Interactions rarely thermalize through rapid bombardment.