Universality of low-energy Rashba scattering

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Spin degeneracy is a consequence of time-reversal + inversion symmetry

\[ E(k \uparrow) = E(-k \downarrow) \quad \text{Time-reversal} \]
\[ E(k \uparrow) = E(-k \uparrow) \quad \text{Inversion} \]

\[ \Rightarrow E(k \uparrow) = E(k \downarrow) \]

Rashba spin-orbit coupling
Rashba spin-orbit coupling

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\[ \Rightarrow E(k \uparrow) \neq E(k \downarrow) \]

- Inversion asymmetry causes “spin-split” dispersion

E.g. surfaces, interfaces, quantum well with confining potential
**Rashba spin-orbit coupling**

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- Inversion asymmetry causes “spin-split” dispersion

E.g. surfaces, interfaces, quantum well with confining potential
Spin and momentum are locked. Lots of potential applications!

$E_0 = -\frac{1}{2}m\lambda^2$

Dirac point

Spin and momentum are locked. Lots of potential applications!
Low-energy Rashba

- **2D Hamiltonian:** \( H(k) = \frac{k^2}{2m} + \lambda \hat{z} \cdot (\sigma \times k) \)

There are two different scattering states at each angle.

\( k_{\geq} \equiv k_0(1 \pm \delta) \quad \delta \equiv \sqrt{1 - |E|/E_0} \quad E_0 = \frac{1}{2}m\lambda^2 \)
Low-energy Rashba
Basic question: Is there anything fundamentally different about Rashba scattering in this regime, independent of interactions and many-body physics?
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\[ V = \begin{cases} \infty & r < R \\ 0 & r > R \end{cases} \]

- Wavefunction computed analytically from matching conditions.

\[ \Psi(r, \theta) = \sum_{l=-\infty}^{\infty} e^{il\theta} \left[ a_l \left( \frac{H_l^+(k<r)}{-H_{l+1}^+(k<r)e^{i\theta}} \right) + b_l \left( \frac{H_l^-(k<r)}{-H_{l+1}^-(k<r)e^{i\theta}} \right) + c_l \left( \frac{H_l^+(k>r)}{-H_{l+1}^+(k>r)e^{i\theta}} \right) + d_l \left( \frac{H_l^-(k>r)}{-H_{l+1}^-(k>r)e^{i\theta}} \right) \right] \]

- Cross-sections and S-matrix extracted.
Differential cross-section in conventional 2D system (no Rashba):

Differential cross-section in Rashba system:

\[ kR = 0.01 \]

\[ kR = 0.1 \]

\[ kR = 0.25 \]

\[ kR = 0.5 \]

\[ kR = 0.75 \]

\[ kR = 1.0 \]
Example: Hard Disk

- Differential cross-section in conventional 2D system (no Rashba):

- Differential cross-section in Rashba system:

- In the low energy limit, scattering looks 1D!

\[
\left( \frac{d\sigma}{d\theta} \right) \bigg|_{E=-E_0} = \frac{2\pi}{k_0} \left[ \delta^2(\theta) + \delta^2(\theta - \pi) \right]
\]

Example: Hard Disk

\[
V = \begin{cases} 
\infty & r < R \\
0 & r > R 
\end{cases}
\]

- S-matrix decomposed in partial waves.

---

Example: Hard Disk

\[ V = \begin{cases} \infty & r < R \\ 0 & r > R \end{cases} \]

\[ S^l = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \]

- S-matrix decomposed in partial waves.
- Low energy limit:
- Independent of \( l \)
- Off diagonal
- Universal?
Scattering Formalism

- Relate T and S matrices through Lippmann-Schwinger equation:

\[
\psi_{k\sigma}(\mathbf{r}; E) = \psi^\text{in}_{k\sigma}(\mathbf{r}; E) + \sum_{\sigma',\sigma''} \int \frac{d^2 \mathbf{r}'}{(2\pi)^2} G^+_{\sigma\sigma'}(\mathbf{r}, \mathbf{r}'; E) T^k_{\sigma'\sigma''} e^{i \mathbf{k} \cdot \mathbf{r}'} \eta^-_{\sigma''}(\theta_k)
\]

- For negative energies:

\[
S^l_{\mu\nu} = \mathbb{I}_{\mu\nu} - \frac{im}{k_0 \delta} \sqrt{k_\mu k_\nu} T^l(k_\nu, k_\mu)
\]

Partial wave expansion

Lower helicity S matrix

Lower helicity T-matrix

Indices
Scattering Formalism

- Cross-sections from Fermi’s golden rule:
  \[
  \frac{d\sigma}{d\theta} \bigg|_{\mu\nu} = \frac{w_{\mu\rightarrow\nu}}{|\mathbf{j}_\mu|}
  \]
  \[
  \sigma_\mu = \frac{2}{k_\mu} \sum_{l=-\infty}^{\infty} (1 - \text{Re}(S^{l}_{\mu\mu}))
  \]

- Optical theorem:
  \[
  \text{Im}(T_\mu^{-k_\mu}k_\mu (\theta = 0)) = -\frac{k_0\delta}{2m} \sigma_\mu
  \]
  Is there a generic form for the T-matrix?
Generic Rashba T-matrix

Claim: The low-energy T-matrix takes a universal form for any circular-symmetric, finite range, spin-independent potential.

\[ \delta \ll \Lambda \ll 1 \]

Allowed virtual transitions within cutoff
Generic Rashba T-matrix

- **Claim:** The low-energy T-matrix takes a universal form for any circular-symmetric, finite range, spin-independent potential.

Born series:

\[
T_{ji}^{k_{\nu}, k_{\mu}} = V_{ji}(k_{\nu}, k_{\mu}) + \sum_{n=\pm} \int \frac{d^2q}{(2\pi)^2} V_{jn}(k_{\nu}, q) G_{nn}^+(q) T_{ni}^q k_{\mu}
\]

"On-shell"

"Off-shell"
Generic Rashba T-matrix

\[ V_{ji}(k_\nu, k_\mu) \approx V_{ji}(k'_0 \hat{k}_\nu, k'_0 \hat{k}_\mu) + O(\delta) \]

\[ T_{--}^l \approx \frac{1}{m} \frac{\delta^*_l}{1 + i\delta^*_l/\delta} = -\frac{i\delta}{m} + O(\delta^2) \]

With \[ \delta^*_l \equiv \frac{m}{2} (V^l(k_0, k_0) + V^{l+1}(k_0, k_0)) \]

\[ V^l(k, k') = \int_0^{2\pi} \frac{d\theta}{2\pi} \int_0^\infty dr r V(r) J_0(|k - k'|r) e^{-i\ell \theta} \]
Remarks:

1) To lowest order, T-matrix is independent of potential and partial wave!

\[ T_{-\ell} = -\frac{i\delta}{m} + O(\delta^2) \]

2) We obtain previous S-matrix limit.

\[ S^l = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \]

3) The energy dependence is fundamentally different than in conventional 2D scattering.

\[ T^{kk'} \approx T^0(E) \approx \frac{1/m}{i - \frac{1}{\pi} \ln(E/E_a)} \]

4) The energy dependence is that of a 1D T-matrix!

\[ T_{1D} \approx \frac{i}{m} \sqrt{2mE} \]
Example: Delta-shell

\[ V(r) = V_0 \delta(r - R) \]
Example: Delta-shell

\[ V(r) = V_0 \delta(r - R) \]
$$V(r) = V_0 \delta(r - R)$$
Example: Circular Barrier

\[ V(r) = \begin{cases} V_0 & r < R \\ 0 & r > R \end{cases} \]

1st Born approximation
Low-energy approximation

1st Born approximation
Low-energy approximation

\[ \delta \]

\[ m|T^l| \]
Conductivity

- Optical theorem gives low-energy cross section:
  \[
  \sigma \approx \frac{2}{k_0} \sum_{l=-\infty}^{\infty} \frac{\delta_l^* \delta}{1 + \delta_l^* \delta} \delta_l^2 / \delta^2
  \]

- Semi-classical Boltzmann:
  \[
  0 = \partial_t n_k + \hat{k} \cdot \nabla_k n_k + v \cdot \nabla_r n_k - \left( \frac{\partial n_k}{\partial t} \right)_{\text{collisions}}
  \]
Conductivity

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  \[ \sigma \approx \frac{2}{k_0} \sum_{l=-\infty}^{\infty} \frac{\delta_l^*}{1 + \frac{\delta_l^*}{\delta_l}} \]

- Semi-classical Boltzmann:
  \[ 0 = \partial_t n_k + \dot{k} \cdot \nabla_k n_k + v \cdot \nabla_r n_k - \left( \frac{\partial n_k}{\partial t} \right)_{\text{collisions}} \]

- Current density:
  \[ j = -e \sum_{\nu} \int d\phi \int dE \rho_{\nu}(E)n_{k_{\nu}}(E)v_{\nu}(E, \phi) \]

- Conductivity:
  \[ \sigma_e = \frac{e^2 k_0}{2\pi n_i \sigma} \]
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The low energy limit of a Rashba system contains interesting physics not seen at energies above the Dirac point:

- Change in the topology of the Fermi surface (Lifschitz transition).
- Low energy scattering quantities have a 1D character:
  - Differential cross sections become confined to a line (incident wave axis).
  - T matrix has an energy dependence inherent to 1D systems.
- Low energy T matrix is universal - independent of potential features
- Low energy ≠ s-wave!
- Conductivity displays quantized plateaus.
Thank you!
\[
\psi_\mu (\mathbf{r}; E) \approx \psi^\text{in}_\mu (\mathbf{r}; E) - \frac{m}{(k^>_2 - k^<_2)} \left( \sqrt{\frac{2i}{\pi r}} (\sqrt{k^>_2} e^{ik^>_2 r} \eta^- (\theta_r) \eta^- (\theta_r) + T^{k^>_2} \eta^- (0))
\right.
+ i \sqrt{k^<_2} e^{-ik^<_2 r} \eta^+ (\theta_r) \eta^+ (\theta_r) + T^{-k^<_2} \eta^- (0))
\]

\[
\Psi (r, \theta) = \sum_{l=-\infty}^{\infty} e^{i l \theta} \left[ a_l \left( \begin{array}{c} H^+_l (k^<_r) \\
-H^+_l (k^<_r) e^{i \theta} \end{array} \right) + b_l \left( \begin{array}{c} H^-_l (k^<_r) \\
-H^-_l (k^<_r) e^{i \theta} \end{array} \right) + c_l \left( \begin{array}{c} H^+_l (k^>_r) \\
-H^+_l (k^>_r) e^{i \theta} \end{array} \right) + d_l \left( \begin{array}{c} H^-_l (k^>_r) \\
-H^-_l (k^>_r) e^{i \theta} \end{array} \right) \right]
\]

\[
|k^\mu - k^\nu| r = r \sqrt{k^2_\mu + k^2_\nu - 2k^\mu k^\nu \cos \theta_{k^\nu - k}}
\]

\[
= \sqrt{2k_0 r} \sqrt{1 - \cos \theta_{k^\nu - k}} + O(\delta)
\]
Consider an incident helicity band at positive energies (negative-helicity state). The differential cross section for scattering has a universal feature of Rashba scattering in the low-energy limit, at least for spin-independent potentials matching conditions, which is precisely the result from the Neumann function $(37)$ and $(36)$.

In summary, we have studied the scattering of electrons near the band bottom (between: (a) helicity bands at positive energies $(37)$, (b) differential cross section $(38)$), and (c) $(39)$.

We conjecture that this extends to any hard-disk and delta-shell potentials of any radius. In the region $(37)$, we found the Neumann function $(38)$ and integrating the Schrödinger equation along the radial direction from $(39)$ gives two more equations, $(40)$.

The wave function has two tunable parameters, which is precisely the result from the Neumann function $(37)$ and $(36)$.

For an incident positive-helicity (negative-helicity) state as a function of energy, (c) total cross section $(41)$, this shows scattering from an arbitrary $(42)$.

The Neumann function $(37)$ has two tunable parameters, which is precisely the result from the Neumann function $(36)$.

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These determine the poles of the Green's function, which lower helicity wave vectors by a (dashed). The dimensionless parameters used are $G_{\mathbf{r}} \cdot G_{\mathbf{r}} = 0$, $\Gamma_{\mathbf{r}} = \mathbf{r}$, and $\omega_{\mathbf{r}} = \mathbf{r}$. Curves are obtained from an exact calculation of $r$, $r \cdot \mathbf{r}$, and $\mathbf{r}$.

\[ G_{\mathbf{r}} \cdot G_{\mathbf{r}} = 0, \quad \Gamma_{\mathbf{r}} = \mathbf{r}, \quad \omega_{\mathbf{r}} = \mathbf{r}. \]

\[ \mathbf{r} \cdot \mathbf{r} = 0, \quad \mathbf{r} \cdot \mathbf{r} = \mathbf{r}. \]

So upon combining the first and last terms as well as the identity:

\[ G_{\mathbf{r}} \cdot G_{\mathbf{r}} = 0, \quad \Gamma_{\mathbf{r}} = \mathbf{r}, \quad \omega_{\mathbf{r}} = \mathbf{r}. \]

\[ (A4) (A1) \]

\[ \phi = \frac{1}{2} \left( \mathbf{r} \cdot \mathbf{r} + \mathbf{r} \cdot \mathbf{r} \right). \]

\[ (b) \]

\[ k \cdot k + 2 \]

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