Superfluid black holes

Robie Hennigar

Work with Erickson Tjoa, Hannah Dykaar and Robert Mann

University of Waterloo

May 30, 2017

The thermodynamics of black holes

In the mid 1970s, work to combine quantum mechanics and relativity revealed that black holes have temperature and entropy: black holes obey the laws of thermodynamics:

$$dM = TdS \quad \text{with} \quad S = \frac{k_B c^3}{\hbar}\frac{A}{4G}\,, \quad T = \frac{\hbar\kappa}{2\pi c k_B}\,.$$

Bekenstein, J. *Phys. Rev. D.* **7** (8) (1973); Hawking, S. *Communications in Mathematical Physics*. **43** (1975)

Padmanabhan, T. *Entropy* **17** (2015), 7420-7452

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"*If you can heat it, it has microscopic structure*"

Provides a window into the quantum properties of the gravitational field, motivated the holographic principle, and more.

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The chemistry of black holes

- When the cosmological constant (Λ) is non-vanishing, it should be included as a thermodynamic parameter appearing in the first law of black hole thermodynamics $\Rightarrow$ mass is enthalpy.
- For $\Lambda < 0$ (asymptotically AdS), the cosmological constant corresponds to a pressure, and its thermodynamic conjugate is the thermodynamic volume:

$$\text{Pressure} + \text{Volume} \Rightarrow \text{black hole equations of state } P(T, v, Q_i)$$

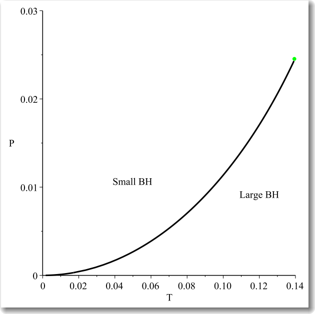
- Has come to be known as black hole chemistry as connections between black holes and everyday thermal systems have emerged: triple points, re-entrant phase transitions, etc.

$$dM = TdS + VdP + \text{ work terms}\,, \quad P = -\frac{\Lambda}{8\pi} = \frac{(d-1)(d-2)}{16\pi\ell^2}$$

Kastor, D. *et al.* *Class. Quant. Grav.* **26** (2009) [arXiv:0904.2765].
Kubiznak, D. and Mann, R. *JHEP* **1207** (2012) [arXiv:1205.0559].
Kubiznak, D. *et al.* *Class. Quant. Grav.* **34** (2017) [arXiv:1608.06147].

Phase transitions for black holes: charged AdS black holes

- The seminal work on phase transitions for AdS black holes was carried out by Hawking and Page, but here we focus on the result of Kubiznak and Mann.



Charged AdS BH / van der Waals fluid:

- First order phase transition, terminating at a critical point where the EOS has an inflection point.
- The critical point is characterized by mean field theory critical exponents.
- Universal ratio of critical values,

$$\frac{P_c v_c}{T_c} = \frac{3}{8}$$

[here, for BH, $v = 2r_+$]

Hawking, S. and Page, D. *Commun. Math. Phys.* **87** (1983) 577.
Kubiznak, D. and Mann, R. *JHEP* **1207** (2012) [arXiv:1205.0559].

Black holes with hair

- BH has "hair" if it has more than mass, charge and angular momentum.
- In $4D$ Einstein gravity with $\Lambda = 0$, black holes have no hair. But hair can exist for $\Lambda \neq 0$. Our goal was to explore the consequences of hair for black hole chemistry.
- We considered a model that was first described by Oliva and Ray[1]: a (real) scalar field conformally coupled to Lovelock gravity.

[1] Oliva, J. and Ray, S. *Class. Quant. Grav.* **29** (2012) 205008 [arXiv:1112.4112].

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- We considered a model that was first described by Oliva and Ray[1]: a (real) scalar field conformally coupled to Lovelock gravity.

  Aside: Lovelock gravity modifies Einstein gravity in $D > 4$ through the addition of higher curvature corrections (the dimensionally continued Euler densities),

$$\mathcal{I} = \int d^d x \sqrt{-g} \left[-2\Lambda + R + c_2 \left(R_{abcd} R^{abcd} - 4 R_{ab} R^{ab} + R^2 \right) + \cdots \right]$$

  and serve as useful toy models for the types of corrections expected in a UV complete theory of quantum gravity.

[1] Oliva, J. and Ray, S. *Class. Quant. Grav.* **29** (2012) 205008 [arXiv:1112.4112].

More details on the model

Cosmological constant + Ricci scalar + Lovelock terms + Maxwell field with scalar field ϕ conformally coupled to Lovelock terms.

$$\mathcal{I} = \frac{1}{16\pi G} \int d^d x \sqrt{-g} \left(\sum_{k=0}^{k_{\max}} \left[a_k \mathcal{L}^{(k)} + b_k \mathcal{S}^{(k)} \right] - 4\pi G F_{\mu\nu} F^{\mu\nu} \right)$$

$\mathcal{L}^{(k)}$: k^{th}-order Lovelock term (Λ, Ricci scalar, Gauss-Bonnet)

$\mathcal{S}^{(k)}$: scalar field, ϕ, coupled conformally to k^{th} order Lovelock term

$F_{\mu\nu}$: Electromagnetic field

- Giribet *et al.* found that the theory admits analytic hairy black hole solutions in any dimension[2] $D > 4$, evading No-Go results that had been reported for (conformal) scalar hair in $D > 4$,[3]

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Sigma_k^2$$

with $f(r)$ determined by a polynomial equation.

[2] Giribet, G. *et al.* *Phys. Rev. D* **91**, (2015) [arXiv: 1401.4987]

[3] C. Martinez in *Quantum mechanics of fundamental systems*.

Thermodynamics of hairy black holes

The general thermodynamic behaviour of the hairy black holes was considered in a recent paper,[4] here we will focus on a new phase transition that was found for these black holes.[5]

The equation of state has the form, when we consider cubic Lovelock gravity (i.e. up to $\mathcal{L}^{(3)}$),

$$p = \frac{t}{v} + \frac{(d-3)(d-2)}{4\pi v^2} - \frac{2\alpha t}{v^3} - \frac{\alpha(d-2)(d-5)}{4\pi v^4} + \frac{3t}{v^5} + \frac{(d-7)(d-2)}{4\pi v^6} + \frac{q^2}{v^{2(d-2)}} - \frac{h}{v^d}.$$

h is the hair, while p, t, v and q are the rescaled, dimensionless pressure, temperature, volume and charge; α is a dimensionless combination of the Lovelock couplings.

This equation of state has a remarkable property: under certain conditions, it admits **infinitely many critical points**!

[4] Hennigar, R. *et al*, *JHEP* 1702 (2017) 070, [arXiv: 1612.06852].

[5] Hennigar, R. *et al*, Phys.Rev.Lett. 118 (2017) no.2, 021301, [arXiv: 1609.02564]

A new type of phase transition

For certain choices of parameters (α, q, h), the necessary condition for a critical point,

$$\frac{\partial p}{\partial v}\bigg|_{\alpha,h,q} = \frac{\partial^2 p}{\partial v^2}\bigg|_{\alpha,h,q} = 0\,.$$

has **infinitely many solutions**! The critical temperature, t_c is a free parameter!

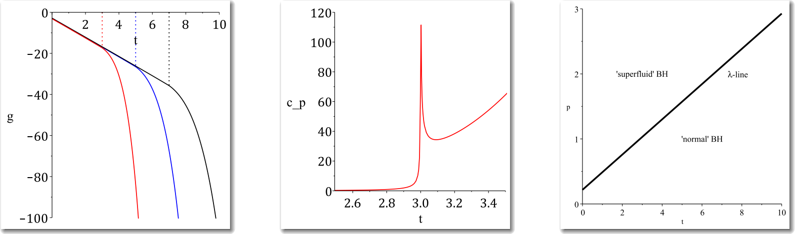


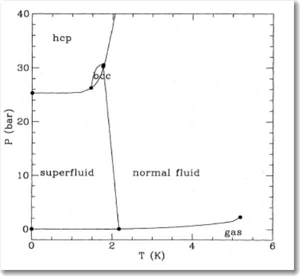
Figure: Example plots for $d = 7$. Right: Solid black line is locus of critical points.

To our knowledge, the first example of a "λ"-line in black hole thermodynamics.

What does it mean?

A 'line of critical points' is also found in the context of

- Superconductivity
- Liquid crystals
- Ferromagnetism
- Superfluidity



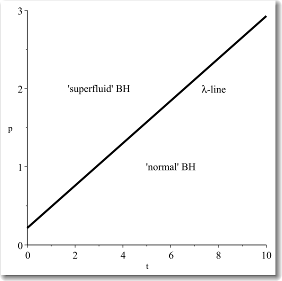


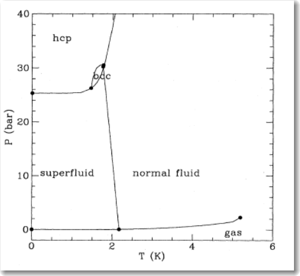
Figure: Pressure vs. Temperature plots. Left: Helium-4, Right: superfluid BH

D.M. Ceperley, Rev.Mod.Phys. 67 (1995) 279-355)

What does it mean?

A 'line of critical points' is also found in the context of

- Superconductivity
- Liquid crystals
- Ferromagnetism
- Superfluidity $\Rightarrow$ Seems natural in BH chemistry



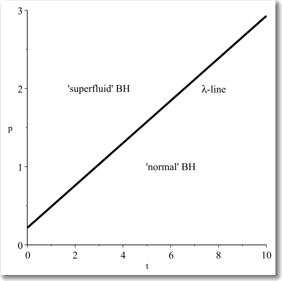


Figure: Pressure vs. Temperature plots. Left: Helium-4, Right: superfluid BH.

D.M. Ceperley, Rev.Mod.Phys. 67 (1995) 279-355

Properties

- Using standard method, the critical points along λ-line is found to have mean field theory critical exponents,

$$\alpha = 0,\ \beta = \frac{1}{2},\ \gamma = 1,\ \delta = 3$$

  The ordering field can be q, α, h.
- No spacetime pathology, except the usual curvature singularity inside the horizon.
- Positive entropy and specific heat,
- Vacuum stability: the theory is free from ghost and tachyon instabilities.

A necessary condition

The equation of state takes the form

$$p = a_1(v, h, q, \alpha)t + a_2(v, h, q, \alpha)\,.$$

The key to having infinitely many critical points was that t_c was a free parameter. For this to occur we must have,

$$\frac{\partial a_1}{\partial v} = \frac{\partial^2 a_1}{\partial v^2} = 0\,,$$
$$\frac{\partial a_2}{\partial v} = \frac{\partial^2 a_2}{\partial v^2} = 0$$

Necessary condition: The black hole must possess at least three 'charges' or couplings (here α, q and h).[6]

NOT sufficient: for example, black holes in pure Lovelock gravity (without hair) cannot meet these conditions.

[6] Unless there is some degeneracy in the equations.

Conclusions

- First example of a black hole λ-line: a line of second order (continuous) phase transitions in black hole thermodynamics.
- Presented simple, necessary conditions for such a phase transition to be possible (three external charges or couplings).
- Future study: Additional examples of these black holes; explore underlying degrees of freedom (via, e.g. AdS/CFT) to further elucidate the features of the phase transition.

Conclusions

- First example of a black hole λ-line: a line of second order (continuous) phase transitions in black hole thermodynamics.
- Presented simple, necessary conditions for such a phase transition to be possible (three external charges or couplings).
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Relevant papers:

- RAH, Erickson Tjoa, and Robert B. Mann, Phys.Rev.Lett. 118 (2017) no.2, 021301, [arXiv: 1609.02564],
- RAH, Erickson Tjoa, and Robert B. Mann, *JHEP* 1702 (2017) 070, [arXiv: 1612.06852],
- Hannah Dykaar, RAH and Robert B. Mann, *JHEP* 1705 (2017) 045, [arXiv: 1703.01633].

Acknowledgements: NSERC for funding, Erickson Tjoa, Hannah Dykaar, Robert Mann for fruitful collaborations.

Explicit details on the model

In 2011, Oliva and Ray demonstrated[7] that a scalar field can be conformally coupled to higher curvature terms making use of the tensor

$$S_{\mu\nu}{}^{\gamma\delta} = \phi^2 R_{\mu\nu}{}^{\gamma\delta} - 2\delta^{[\gamma}_{[\mu}\delta^{\delta]}_{\nu]}\nabla_\rho\phi\nabla^\rho\phi - 4\phi\delta^{[\gamma}_{[\mu}\nabla_{\nu]}\nabla^{\delta]}\phi + 8\delta^{[\gamma}_{[\mu}\nabla_{\nu]}\phi\nabla^{\delta]}\phi$$

Note: $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$ and $\phi \to \Omega^{-1}\phi \Rightarrow S_{\mu\nu}{}^{\gamma\delta} \to \Omega^{-4} S_{\mu\nu}{}^{\gamma\delta}$
This leads to the following theory:

$$\mathcal{I} = \frac{1}{16\pi G}\int d^d x\sqrt{-g}\left(\sum_{k=0}^{k_{\max}}\mathcal{L}^{(k)} - 4\pi G F_{\mu\nu}F^{\mu\nu}\right)$$

$$\mathcal{L}^{(k)} = \frac{1}{2^k}\delta^{\mu_1\nu_1\cdots\mu_k\nu_k}_{\alpha_1\beta_1\cdots\alpha_k\beta_k}\left(a_k\prod_r^k R^{\alpha_r\beta_r}_{\mu_r\nu_r} + b_k\phi^{d-4k}\prod_r^k S^{\alpha_r\beta_r}_{\mu_r\nu_r}\right)$$

$\Rightarrow$ Cosmological constant + Ricci scalar + Lovelock terms with scalar field conformally coupled to higher dimensional Euler densities.

[7] Oliva, J. and Ray, S. *Class. Quant. Grav.* **29** (2012) 205008 [arXiv:1112.4112].

Infinitely many critical points

Recall the necessary condition for a critical point,

$$\frac{\partial p}{\partial v}\bigg|_{\alpha,h,q} = \frac{\partial^2 p}{\partial v^2}\bigg|_{\alpha,h,q} = 0\,.$$

Consider the particular choice of α, h, q given by

$$\alpha = \sqrt{5/3}\,, \quad h = \frac{4(2d-5)(d-2)^2(15^{1/4})^{d-6}}{\pi d(d-4)}\,, q = \pm\sqrt{\frac{2(d-1)(d-2)(15^{1/4})^{2d-10}}{\pi(d-4)}}\,.$$

Then the critical point condition admits infinitely many solutions:

$$\begin{aligned} v_c &= 15^{1/4}\,, \\ p_c &= \left[\frac{8}{225}(15)^{\frac{3}{4}}\right] t_c + \frac{\sqrt{15}(11d-40)(d-1)(d-2)}{900\pi d}\,, \\ t_c &\in \mathbb{R}\,. \end{aligned}$$

Results I: Gravitational Equations of Motion & Ghosts

Considered linearized equations of motion about a constant scalar field background in AdS, $g_{ab} = g^{[0]}_{ab} + h_{ab}$:

$$-\frac{1}{2}\left(1 - 2\frac{\alpha_2}{L^2}F_\infty + 3\frac{\alpha_3}{L^4}F_\infty^2 + 16\pi b_1 \Phi^{d-2} - \frac{32\pi(d-3)(d-4)b_2 F_\infty \Phi^{d-4}}{L^2}\right)\Big[\Box h_{ab}$$
$$+\nabla_b\nabla_a h^c{}_c - \nabla_c\nabla_a h_b{}^c - \nabla_c\nabla_b h_a{}^c + g^{[0]}_{ab}\left(\nabla_d\nabla_c h^{cd} - \Box h^c{}_c\right) + \frac{(d-1)F_\infty}{L^2} g^{[0]}_{ab} h^c{}_c\Big]$$
$$+\left[\frac{F_\infty(d-1)}{L^2}\left(1 - 2\frac{\alpha_2}{L^2}F_\infty + 3\frac{\alpha_3}{L^4}F_\infty^2 + 8\pi d b_1 \Phi^{d-2} - \frac{8\pi(d-4)(d-3)(d+2)b_2 F_\infty \Phi^{d-4}}{L^2}\right) - 8\pi b_0 \Phi^d\right] h_{ab}$$
$$= 8\pi G T_{ab}$$

The red terms must be positive, while respecting the EOM of the scalar field. EOM of scalar field (not shown) restrict allowed couplings (b_n's).

For a non-zero scalar field couplings for spherical black holes in Gauss-Bonnet gravity, cannot meet these criteria in $d < 7$ (*can* be saved in $D = 5$). Hyperbolic black holes and black holes in cubic Lovelock are fine.

Results II: Solution and Thermodynamics

$$ds^2 = -fdt^2 + f^{-1}dr^2 + r^2 d\Sigma^2_{(\sigma)d-2}$$

$$\sum_{k=0}^{k_{\max}} \alpha_k \left(\frac{\sigma - f}{r^2}\right)^k = \frac{16\pi GM}{(d-2)\Sigma^{\sigma}_{d-2} r^{d-1}} + -\frac{8\pi G}{(d-2)(d-3)} \frac{Q^2}{r^{2d-4}} + \boxed{\frac{H}{r^d}}$$

$$T = \frac{1}{4\pi r_+ D(r_+)} \left[\sum_k \sigma\alpha_k (d-2k-1)\left(\frac{\sigma}{r_+^2}\right)^{k-1} + \boxed{\frac{H}{r_+^{d-2}}} - \frac{8\pi G Q^2}{(d-2)r_+^{2(d-3)}}\right]$$

$$S = \frac{\Sigma^{(\sigma)}_{d-2}}{4G}\left[\sum_{k=1}^{k_{\max}} \frac{(d-2)k\sigma^{k-1}\alpha_k}{d-2k} r_+^{d-2k} - \boxed{\frac{d}{2\sigma(d-4)}H}\right] \quad \text{if } b_k = 0 \ \forall k > 2.$$

Hairy term

$$\boxed{H = \sum_{k=0}^{k_{\max}} \frac{(d-3)!}{(d-2(k+1))!} b_k \sigma^k N^{d-2k}}, \quad \phi = \frac{N}{r}, \quad D(r_+) = \sum_{k=1}^{k_{\max}} k\alpha_k (\sigma r_+^{-2})^{k-1}$$

Results III: Critical Behaviour

Triple point:

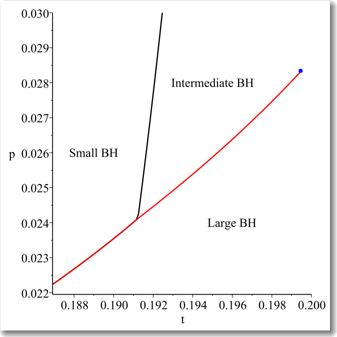


Figure: Triple point for $h = 0.064,\ q = 0.1,\ \sigma = +1, d = 5$ Gauss-Bonnet.

Double reentrant phase transition:

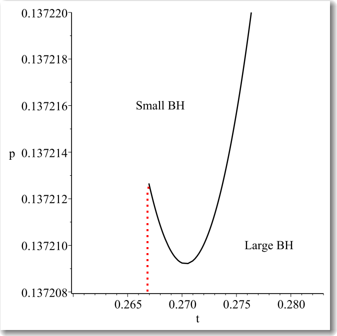


Figure: Double reentrant phase transition for $h = -1, q = 0, \alpha = 4, \sigma = -1, d = 7$ Cubic Lovelock

Results IV: Virtual Triple Point

The hair parameter allows for a much broader class of critical behaviour.

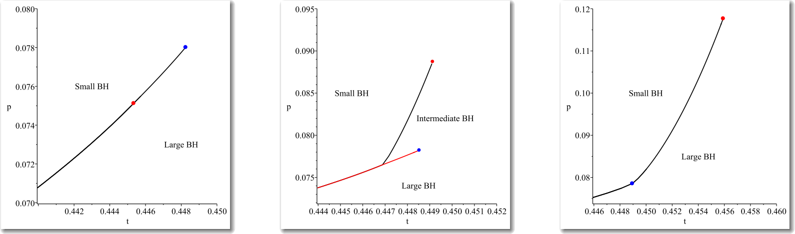


Figure: Representative virtual triple point phase diagram: Occurs for both Gauss-Bonnet and cubic Lovelock black holes. *Left*: $q = 0.05$, $h = -0.00869$. The critical point lies precisely on the coexistence curve. *Center*: $q = 0.05$, $h = -0.007$. *Right*: $q = 0.05$, $h \approx -0.0046$.

h increases from left to right: varying h continuously deforms the coexistence curves in this manner.

Results V: Isolated Critical Points I

Nearly all of critical behaviour found in black hole systems are described by mean field theory critical exponents,

$$\alpha = 0\,, \quad \beta = \frac{1}{2}\,, \quad \gamma = 1\,, \quad \delta = 3\,.$$

Different critical exponents have been found for black holes in higher order ($\geq$ cubic) Lovelock gravity at *isolated critical points*:[8]

- The only examples of non-mean field theory critical exponents for a gravitational system.
- Require finely tuned coupling constants; occur at a "thermodynamic singularity".

$$\alpha = 0,\ \beta = 1,\ \gamma = 2,\ \delta = 3$$

[8] Dolan, B. *et al. Class. Quant. Grav.* **31** (2014) [arXiv:1407.4783].

Results V: Isolated Critical Points II

For hairy black holes in cubic Lovelock gravity there is a family of isolated critical points with h set to (in $d = 7$):

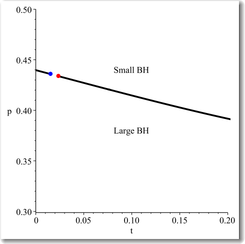
$$h_\epsilon = \frac{10}{7\pi}\left[\frac{225 - 945\alpha^2 + 540\alpha^4 - \epsilon(165\alpha - 180\alpha^3)\sqrt{9\alpha^2 - 15} + \pi q^2}{\left(3\alpha + \epsilon\sqrt{9\alpha^2 - 15}\right)^{3/2}}\right]$$

where $\epsilon = \pm 1$ and $\alpha = \alpha_2/\sqrt{\alpha_3}$.

Does not occur at thermodynamically singular point.

Occurs for wide range of parameters.

$$\alpha = 0,\ \beta = 1,$$
$$\gamma = 2,\ \delta = 3$$



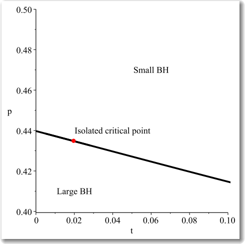


Figure: Cubic Lovelock $d = 7\,, \sigma = -1\,, q = 0.59$ Left: $h = -0.1367$, Right: $h \approx -0.136685$.

Results VI: Superfluid black holes

For

$$\alpha = \sqrt{5/3}, \quad \sigma = -1, \quad h = \frac{4(2d-5)(d-2)^2 v_c^{d-6}}{\pi d(d-4)}, \quad q^2 = \frac{2(d-1)(d-2)v_c^{2d-10}}{\pi(d-4)},$$

a line of critical points:

$$v_c = 15^{1/4}, \quad p_c = \left[\frac{8}{225}(15)^{\frac{3}{4}}\right] t_c + \frac{\sqrt{15}(11d-40)(d-1)(d-2)}{900\pi d}, \quad t_c \in \mathbb{R}$$

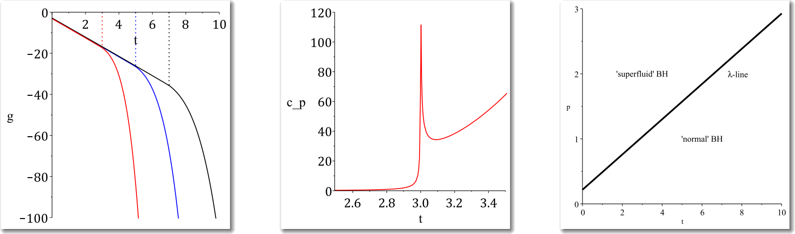


Figure: Example plots for $d = 7$. Right: Solid black line is locus of critical points.

To our knowledge, the first example of a "λ"-line in black hole thermodynamics.

The equations of motion of the scalar field, $\phi = N/r$:

$$\begin{aligned} \sum_{k=1}^{k_{\max}} k b_k \frac{(d-1)!}{(d-2k-1)!}\sigma^{k-1} N^{2-2k} &= 0\,, \\ \sum_{k=0}^{k_{\max}} b_k \frac{(d-1)!\left(d(d-1)+4k^2\right)}{(d-2k-1)!}\sigma^{k} N^{-2k} &= 0\,. \end{aligned} \tag{1}$$

$$\begin{aligned} 0 = d b_0 \Phi^{d-1} &- \frac{d(d-1)(d-2)F_\infty b_1 \Phi^{d-3}}{L^2} \\ &+ \frac{d(d-1)(d-2)(d-3)(d-4)F_\infty^2 b_2 \Phi^{d-5}}{L^4} \end{aligned}$$

$$\Phi = \epsilon \frac{\sqrt{2b_0 b_1 d(d-2)\left(d^2-d+2\lambda\sqrt{2d(d-1)}\right)F_\infty}}{2d|b_0|L} \tag{2}$$

Gauss-Bonnet dimensionless:

$$r_+ = v\sqrt{\alpha_2}\,, \quad T = \frac{t}{\sqrt{\alpha_2}(d-2)}\,, \quad Q = \frac{q}{\sqrt{2}}\alpha_2^{\frac{d-3}{2}}\,, \quad \frac{(d-1)(d-2)\alpha_0}{16\pi} = \frac{p}{4\alpha_2}\,,$$
$$H = \frac{4\pi h}{d-2}\alpha_2^{\frac{d-2}{2}}\,. \tag{3}$$

Cubic Lovelock dimensionless:

$$r_+ = v\alpha_3^{1/4}\,, \quad T = \frac{t\alpha_3^{-1/4}}{d-2}\,, \quad H = \frac{4\pi h}{d-2}\alpha_3^{\frac{d-2}{4}} \tag{4}$$
$$Q = \frac{q}{\sqrt{2}}\alpha_3^{\frac{d-3}{4}}\,, \quad m = \frac{16\pi M}{(d-2)\Sigma_{d-2}^{(\kappa)}\alpha_3^{\frac{d-3}{4}}}p = \frac{\alpha_0(d-1)(d-2)\sqrt{\alpha_3}}{4\pi}\,, \quad \alpha = \frac{\alpha_2}{\sqrt{\alpha_3}}\,. \tag{5}$$

$$
\begin{aligned}
M &= \frac{(d-2)\Sigma^{\sigma}_{d-2}}{16\pi G}\sum_{k=0}^{k_{\max}}\alpha_k\sigma^k r_+^{d-2k-1} - \frac{(d-2)\Sigma^{\sigma}_{d-2}H}{16\pi G r_+} + \frac{\Sigma^{\sigma}_{d-2}Q^2}{2(d-3)r_+^{d-3}} \\
T &= \frac{1}{4\pi r_+ D(r_+)}\left[\sum_k \sigma\alpha_k(d-2k-1)\left(\frac{\sigma}{r_+^2}\right)^{k-1} + \frac{H}{r_+^{d-2}} - \frac{8\pi G Q^2}{(d-2)r_+^{2(d-3)}}\right] \\
\Phi &= \frac{\Sigma^{\sigma}_{d-2}Q}{(d-3)r_+^{d-3}} \\
S &= \frac{(d-2)\Sigma^{(\sigma)}_{d-2}}{4G}\sum_{k=1}^{k_{\max}}k\sigma^{k-1}\left[\frac{\alpha_k}{d-2k}r_+^{d-2k} + \frac{b_k(d-3)!}{(d-2k)!}N^{d-2k}\right] \\
\Psi^{(k)} &= \frac{\Sigma^{(\sigma)}_{d-2}(d-2)}{16\pi G}\sigma^{k-1}r_+^{d-2k}\left[\frac{\sigma}{r_+} - \frac{4\pi k T}{d-2k}\right], \\
\mathcal{K}^{(k)} &= -\frac{\Sigma^{(\sigma)}_{d-2}(d-2)!}{16\pi G}\sigma^{k-1}N^{d-2k}\left[\frac{\sigma}{(d-2(k+1))!\,r_+} + \frac{4\pi k T}{(d-2k)!}\right] \qquad (6)
\end{aligned}
$$

where

$$
D(r_+) = \sum_{k=1}^{k_{\max}} k\alpha_k(\sigma r_+^{-2})^{k-1}. \qquad (7)
$$