Some variations of the reduction of one-loop Feynman tensor integrals

Tord Riemann

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Outline

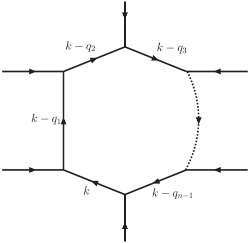
Introduction

n-point tensor integrals of rank *R*: (n,R)-integrals

$$I_n^{\mu_1\cdots\mu_R} = \int \frac{d^dk}{i\pi^{d/2}} \frac{\prod_{r=1}^{R} k^{\mu_r}}{\prod_{j=1}^{n} c_j^{\nu_j}}, \tag{1}$$

$d = 4 - 2\epsilon$ and denominators c_j have *indices* ν_j and *chords* q_j

$$c_j = (k - q_j)^2 - m_j^2 + i\varepsilon \tag{2}$$



tensor integrals due to:

- fermion propagators
- three-gauge boson couplings
- e.g. unitary gauge propagators

Reduction of tensor integrals ⇒ express them by a (very) small set of scalar integrals

Presently needed for massive processes: $n \leq 6$ and rank $R \leq n$
For box diagrams and simpler ones: Use of the 'conventional' Passarino-Veltman reduction [Passarino:1978 [1]]
Examples:

- LO (Lowest order) of e.g. $Z \to e + \mu$ is one-loop [Riemann:1981,Mann:1983 [2][3]]
- NLO: one-loop corrections to e.g. $H \to \tau^+\tau^-, WW, ZZ$ [Fleischer:1980 [4]]
- NNLO: e.g. radiative loop corrections $e^+e^- \to e^+e^-\gamma$ (here with 5-point functions)

Status of opensource packages - hopefully complete

- package FF [vanOldenborgh:1990 [5]] ,
- package LoopTools/FF [Hahn:1998,2006 [6]] – covers also 5-point functions, rank $R \leq 4$ $1/\epsilon^2$ not covered, and we observed sometimes problems in certain configurations with light-like external particles
- package Golem95 [Binoth:2008 [7]] for $n \leq 6$, but only massless propagators
- Mathematica package hexagon.m [Diakonidis:2008 [8, 9]] for $n \leq 6, rankR \leq 4$
- package for all $n \leq 4$ scalar integrals: QCDloop [Ellis:2007 [10]]
- see also: review A.Denner, DESY TH workshop 2009

So, if you want to evaluate something less trivial, you have to create your own tensor reduction library.

This talk: derive efficient reduction formulae in the algebraic Fleischer-Davydychev-Tarasov approach

The original Passarino-Veltman reduction allows to express tensor integrals by a small set of scalar 4-,3-,2-,1-point functions integrals in *d* dimensions.

- Extensions: Need of reduction of *n*-point functions with $n > 4$
- Improvements: Avoid the break-down in certain kinematical configurations

A crucial step was to avoid for $n = 5$ – pentagons – the towers of inverse Gram determinants.
They may get small or vanishing and may appear by reducing algebraically the tensors to scalars.

Crucial contributions [of course, list is incomplete . . .] ⇒

- [Campbell:1996 [11]]
- [Denner:2002,2005 [12, 13]]
- [Binoth:1999,2005 [14, 15]]
- [Bern:1993 [16]]
- [Ossola:2006 [17]]

In the following, I will describe recent developments in the Fleischer-Davydychev-Tarasov approach.

- [Davydychev:1991,Tarasov:1996,Fleischer:1999,Diakonidis:2008,2009 [18, 19, 20, 9, 21]]

- get tensor reduction
- kill pentagon-Gram det's
- treat sub-Gram det's

Recursions for hexagons I

Express any hexagon by pentagons

[Fleischer:1999,Binoth:2005,Denner:2005,Diakonidis:2008 [20, 15, 13, 9]]

$$I_6^{\mu_1\cdots\mu_{R-1}\rho} = -\sum_{s=1}^{6} I_5^{\mu_1\cdots\mu_{R-1},s}\,\bar{Q}_s^{\rho}. \tag{3}$$

auxiliary vectors

$$\bar{Q}_s^{\rho} \quad = \quad \sum_{i=1}^{6} q_i^{\rho}\,\frac{\binom{0s}{0i}_6}{\binom{0}{0}_6} \quad , \quad s = 1\ldots 6. \tag{4}$$

Scalar case [N=5: Melrose:1965 [22]]

$$I_N = -\sum_{s=1}^{6} \frac{\binom{0}{s}_N}{\binom{0}{0}_N} I_{N-1}^{s}, \quad N = 5, 6 \tag{5}$$

Notations: $I_{n-1}^{\{\mu_1,\cdots\},s}$ etc.

$$I_{n-1,ab}^{\{\mu_1,\cdots\},s}$$

is obtained from

$$I_n^{\{\mu_1,\cdots\}}$$

by

- shrinking line s
- raising the powers of inverse propagators a, b.

Notations: modified Cayley determinant [Melrose:1965]

Modified Cayley determinant $()_N$ of a diagram with N internal lines and chords q_j:

$$()_N \equiv \begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & Y_{11} & Y_{12} & \dots & Y_{1N} \\ 1 & Y_{12} & Y_{22} & \dots & Y_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & Y_{1N} & Y_{2N} & \dots & Y_{NN} \end{vmatrix}, \tag{6}$$

with matrix elements

$$Y_{ij} = -(q_i - q_j)^2 + m_i^2 + m_j^2, \quad (i, j = 1 \dots N) \tag{7}$$

Gram determinant G_n: $G_n = |2q_iq_j|, i, j = 1, \dots n$

For a choice $q_n = 0$, both determinants are related: $()_N = -G_{N-1}$

$\Rightarrow$ The determinant $()_N$ does not depend on the masses.

Notations: signed minors [Melrose:1965]

We also need signed minors of $()_N$, constructed by deleting m rows and m columns from $()_N$, and multiplying with a sign factor:

$$\begin{pmatrix} j_1 & j_2 & \cdots & j_m \\ k_1 & k_2 & \cdots & k_m \end{pmatrix}_N \equiv$$

$$\equiv (-1)^{\sum_l (j_l+k_l)} \mathrm{sgn}_{\{j\}}\, \mathrm{sgn}_{\{k\}} \begin{vmatrix} \text{rows } j_1 \cdots j_m \text{ deleted} \\ \text{columns } k_1 \cdots k_m \text{ deleted} \end{vmatrix} \quad (8)$$

where $\mathrm{sgn}_{\{j\}}$ and $\mathrm{sgn}_{\{k\}}$ are the signs of permutations that sort the deleted rows $j_1 \cdots j_m$ and columns $k_1 \cdots k_m$ into ascending order.

Dimensional shifts and recurrence relations for pentagons I

'Naive', direct approach.
Following [Davydychev:1991 [18]]
replace tensors by scalar integrals in higher dimensions:
Example $R = 3$:

$$I_5^{\mu\nu\lambda} = \int \frac{d^{4-2\epsilon}k}{\pi^{d/2}} \prod_{r=1}^{5} c_r^{-1} k^\mu k^\nu k^\lambda \tag{9}$$

$$= -\sum_{i,j,k=1}^{4} q_i^\mu q_j^\nu q_k^\lambda n_{ijk} I_{5,ijk}^{[d+]^3} + \frac{1}{2}\sum_{i=1}^{n-1}(g^{\mu\nu}q_i^\lambda + g^{\mu\lambda}q_i^\nu + g^{\nu\lambda}q_i^\mu) I_{n,i}^{[d+]^2},$$

and $n_{ijk} = (1+\delta_{ij})(1+\delta_{ik}+\delta_{jk})$.

Following [Tarasov:1996,Fleischer:1999 [19, 20]]
apply recurrence relations, some of them relating scalar integrals of different dimensions, in order to get rid of the dimensionalities $D = d - 2\epsilon + 2l$:

$$\nu_j \mathbf{j}^+ I_5^{[d+]} = \frac{1}{()_5}\left[-\binom{j}{0}_5 + \sum_{k=1}^{5}\binom{j}{k}_5 \mathbf{k}^-\right] I_5 \tag{10}$$

$$(d - \sum_{i=1}^{n}\nu_i + 1) I_n^{[d+]} = \frac{1}{()_n}\left[\binom{0}{0}_n - \sum_{k=1}^{n}\binom{0}{k}_n \mathbf{k}^-\right] I_n, \quad n = 5 here \tag{11}$$

$$\binom{0}{0}_5 \nu_j \mathbf{j}^+ I_5 = \sum_{k=1}^{5}\binom{0j}{0k}_5 \left[d - \sum_{i=1}^{5}\nu_i(\mathbf{k}^-\mathbf{i}^+ + 1)\right] I_5 \tag{12}$$

where the operators $\mathbf{i}^\pm, \mathbf{j}^\pm, \mathbf{k}^\pm$ act by shifting the indices ν_i, ν_j, ν_k by ± 1.

At the end one may arrange a representation of e.g. $I_5^{\mu\nu\lambda}$ in terms of a collection of scalar 1-point to 4-point functions in generic dimension d: A_0, B_0, C_0, D_0.
A problem for applications are the powers of the inverse Gram determinants
$1/()_5$
in recursions with dimensional shifts.

Alternative: Recursions for pentagons I

Express any $(5, R)$ pentagon by a $(5, R-1)$ pentagon plus $(4, R-1)$ boxes [Fleischer et al., Diakonidis:2010 [23]]

$$I_5^{\mu_1\cdots\mu_{R-1}\mu} = I_5^{\mu_1\cdots\mu_{R-1}} Q_0^{\mu} - \sum_{s=1}^{5} I_4^{\mu_1\cdots\mu_{R-1},s} Q_s^{\mu} \tag{13}$$

auxiliary vectors with inverse Gram determinants

$$Q_s^{\mu} = \sum_{i=1}^{5} q_i^{\mu} \frac{\binom{s}{i}_5}{()_5}, \quad s = 0, \ldots, 5 \tag{14}$$

For e.g. $R = 3$, again $[1/()_5]^3$ will occur.

Interlude: Further recursion steps

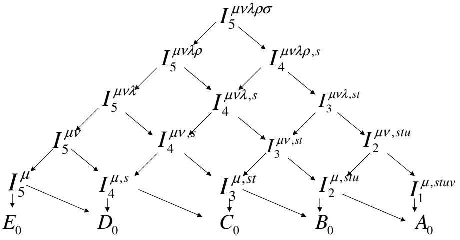
The recursions may be set forth, see [Diakonidis:2009 [21]] ; they get additional terms, e.g.:

$$I_4^{\mu\nu\lambda\rho} \quad = \quad I_4^{\mu\nu\lambda} Q_0^{\rho} - \sum_{t=1}^{4} I_3^{\mu\nu\lambda,t} Q_t^{\rho} - G^{\mu\rho} T^{\nu\lambda} - G^{\nu\rho} T^{\mu\lambda} - G^{\lambda\rho} T^{\mu\nu}, \qquad (15)$$

with the additional tensor and vector components:

$$T^{\mu\nu} = I_4^{\mu,[d+]} Q_0^{\nu} - \sum_{t=1}^{4} I_3^{\mu,[d+],t} Q_t^{\nu} - G^{\mu\nu} I_4^{[d+]^2} \qquad (16)$$

$$G^{\mu\lambda} = \frac{1}{2} g^{\mu\lambda} - \sum_{i,j=1}^{4} q_i^{\mu} q_j^{\lambda} \frac{\binom{i}{j}_4}{()_4}$$



Fleischer's triangle

Algebraic simplifications, 1st step

With the identity

$$\binom{0}{0}_5 \binom{s}{i}_5 = \binom{0s}{0i}_5 ()_5 + \binom{0}{i}_5 \binom{s}{0}_5 \tag{17}$$

we eliminate the inverse Gram determinant from all terms with exclusion of Q_0^μ:

$$I_5^{\mu_1\cdots\mu_{R-1}\mu} = \left[I_5^{\mu_1\cdots\mu_{R-1}} - \sum_{s=1}^{5} \frac{\binom{s}{0}_5}{\binom{0}{0}_5} I_4^{\mu_1\cdots\mu_{R-1},s} \right] Q_0^\mu - \sum_{s=1}^{5} I_4^{\mu_1\cdots\mu_{R-1},s} \overline{Q}_s^\mu \tag{18}$$

The auxiliary vectors $\overline{Q}_s^\mu$ were introduced already for $n = 6$:

$$Q_0^\mu = \sum_{i=1}^{5} q_i^\mu \frac{\binom{0}{i}_5}{()_5} \quad \text{while} \quad Q_0^\mu = \sum_{i=1}^{5} q_i^\mu \frac{\binom{0}{i}_5}{()_5} \overline{Q}_s^\mu = \sum_{i=1}^{5} q_i^\mu \frac{\binom{0s}{0i}_5}{\binom{0}{0}_5} \tag{19}$$

Algebraic simplifications, 2nd step I

Have to show for the product $T^{\mu_1\cdots\mu_{R-1}} \times Q_0^{\mu}$ that the Gram determinant cancels.
This came out to be a complicated task.

$$T^{\mu_1\cdots\mu_{R-1}} = \left[\binom{0}{0}_5 I_5^{\mu_1\ldots\mu_{R-1}} - \sum_{s=1}^{5} \binom{s}{0}_5 I_4^{\mu_1\ldots\mu_{R-1},s} \right] \quad (20)$$

Example: For $R = 3$ pentagons need rank 2 tensor:

$$T^{\mu\nu} = \left[\binom{0}{0}_5 I_5^{\mu\nu} - \sum_{s=1}^{5} \binom{s}{0}_5 I_4^{\mu\nu} \right] \quad (21)$$

Algebraic simplifications, 2nd step I

Example $R = 3$: the building blocks are here $I_5^{\mu\nu}$ and $I_4^{\mu\nu}$:

$$
\begin{aligned}
I_5^{\mu\nu} &= \sum_{i,j=1}^{5} q_i^\mu q_j^\nu \left[(1+\delta_{ij}) I_{5,ij}^{[d+]^2}\right] + g^{\mu\nu}\left[-\frac{1}{2} I_5^{[d+]}\right] \\
\overset{\text{work!!!}}{\Rightarrow} & \sum_{i,j=1}^{4} q_i^\mu q_j^\nu \frac{1}{\binom{0}{0}_5} \sum_{s=1}^{5}\left[\binom{0i}{sj}_5 I_4^{[d+],s} + \binom{0s}{0j}_5 I_{4,i}^{[d+],s}\right] \\
& + g^{\mu\nu}\left[-\frac{1}{2}\frac{1}{\binom{0}{0}_5}\sum_{s=1}^{5}\binom{s}{0}_5 I_4^{[d+],s}\right] \qquad (22)
\end{aligned}
$$

See: The $I_5^{\mu\nu}$ had already been made free of $1/()_5$.

Davydychev's higher dimensional integrals

The second term with $I_4^{\mu\nu}$ is a typical example of [Davydychev:1991 [18]] : tensor $\Rightarrow$ scalars in $d+2l$

$$I_4^{\mu\nu} = \sum_{i,j=1}^{4} q_i^\mu q_j^\nu \left[(1+\delta_{ij}) I_{4,ij}^{[d+]^2}\right] + g^{\mu\nu}\left[-\frac{1}{2} I_4^{[d+]}\right] \quad (23)$$

Further, at this point, we have to reduce the scalar integrals $I_{4,ij}^{[d+]^2}$ etc. to generic dimension d with Tarasov's recurrence relations, see next slide.
The $I_4^{\mu\nu}$ is naturally free of $1/()_5$.

Algebraic simplifications, 2nd step

Work out the red part in

$$\binom{0}{0}_5 I_5^{\mu\nu\lambda} = \left[\binom{0}{0}_5 I_5^{\mu\nu} - \sum_{s=1}^{5} \binom{s}{0}_5 I_4^{\mu\nu,s}\right] Q_0^{\lambda} - \sum_{s=1}^{5} I_4^{\mu\nu,s} \overline{Q}_s^{0,\lambda}$$

⇒ work!!! **Use of identities for the determinants**

$$\binom{0}{0}_5 \binom{s}{i}_5 = \binom{0s}{0i}_5 ()_5 + \binom{0}{i}_5 \binom{s}{0}_5 \tag{24}$$

$$\binom{s}{i}_5 \frac{\binom{0}{j}_5}{()_5} = -\binom{0i}{sj}_5 + \binom{s}{0}_5 \frac{\binom{i}{j}_5}{()_5}, \qquad g^{\mu\nu} = 2 \sum_{i,j=1}^{4} \frac{\binom{i}{j}_5}{()_5} q_i^{\mu} q_j^{\nu} \tag{25}$$

$$\binom{s}{0}_5 \binom{0s}{is}_5 = \binom{s}{i}_5 \binom{0s}{0s}_5 - \binom{s}{s}_5 \binom{0s}{0i}_5 \tag{26}$$

$$\binom{s}{0}_5 \binom{ts}{js}_5 = \binom{s}{j}_5 \binom{ts}{0s}_5 - \binom{s}{s}_5 \binom{ts}{0j}_5 \tag{27}$$

Algebraic simplifications, 2nd step

$\Rightarrow$ work!!! Use of identities for the determinants

$$\binom{s}{0}_5 \binom{is}{js}_5 = \binom{s}{i}_5 \binom{0s}{js}_5 + \binom{s}{s}_5 \binom{0i}{sj}_5 \qquad (28)$$

$$\binom{s}{s}_5 \binom{0st}{0st}_5 = \binom{0s}{0s}_5 \binom{st}{st}_5 - \binom{ts}{0s}_5^2 \qquad (29)$$

$$\left[\binom{ts}{0s}_5 \binom{ust}{jst}_5 - \binom{ts}{js}_5 \binom{ust}{0st}_5\right] \binom{s}{s}_5 = \left[\binom{ts}{0s}_5 \binom{us}{js}_5 - \binom{ts}{js}_5 \binom{us}{0s}_5\right] \binom{st}{st}_5 \qquad (30)$$

$$\sum_{t=1}^{5} \binom{ts}{is}_5 = 0 \qquad (31)$$

Express pentagons I_5^{μ}, $I_5^{\mu\nu}$, $I_5^{\mu\nu\lambda}$ etc. by d-shifted scalar boxes I

Intermediate result with $I_4^{[d+],s}$, $I_{4,ij}^{[d+]^2,s}$ etc.

$$I_5^{\mu} = -\sum_{i=1}^{4}\left[\frac{1}{\binom{0}{0}_5}\sum_{s=1}^{5}\binom{0i}{0s}_5 I_4^s\right] q_i^{\mu} \qquad (32)$$

$$I_5^{\mu\,\nu} = \sum_{i,j=1}^{4} q_i^{\mu}\, q_j^{\nu}\, E_{ij} + g^{\mu\nu} E_{00} \qquad (33)$$

$$E_{ij} = \frac{1}{\binom{0}{0}_5}\sum_{s=1}^{5}\left[\binom{0i}{sj}_5 I_4^{[d+],s} + \binom{0s}{0j}_5 I_{4,i}^{[d+],s}\right] \qquad (34)$$

$$E_{00} = -\frac{1}{2}\frac{1}{\binom{0}{0}_5}\sum_{s=1}^{5}\binom{s}{0}_5 I_4^{[d+],s} \qquad (35)$$

Express pentagons I_5^{μ}, $I_5^{\mu\nu}$, $I_5^{\mu\nu\lambda}$ etc. by d-shifted scalar boxes II

$$I_5^{\mu\,\nu\,\lambda} = \sum_{i,j,k=1}^{4} q_i^{\mu} q_j^{\nu} q_k^{\lambda} E_{ijk} + \sum_{k=1}^{4} g^{[\mu\nu} q_k^{\lambda]} E_{00k} \tag{36}$$

$$E_{ijk} = -\frac{1}{\binom{0}{0}_5} \sum_{s=1}^{5} \left\{ \left[\binom{0j}{sk}_5 I_{4,i}^{[d+]^2,s} + (i \leftrightarrow j) \right] + \binom{0s}{0k}_5 \nu_{ij} I_{4,ij}^{[d+]^2,s} \right\} \tag{37}$$

$$E_{00j} = \frac{1}{\binom{0}{0}_5} \sum_{s=1}^{5} \left[\frac{1}{2} \binom{0s}{0j}_5 I_4^{[d+],s} - \frac{d-1}{3} \binom{s}{j}_5 I_4^{[d+]^2,s} \right] \tag{38}$$

These presentations are evidently free of inverse Gram determinants.

Isolation of inverse sub-Gram det^s $()_4$ I

We have now two kinds of objects in higher dim^s to be evaluated:

$$I_4^s,\ I_4^{[d+],s},\ I_4^{[d+]^2,s} \qquad \text{boxes} \tag{39}$$

$$I_{4,i}^{[d+],s},\ I_{4,i}^{[d+]^2,s} \quad I_{4,ij}^{[d+]^2,s} \qquad \text{boxes with higher indices} \tag{40}$$

Application of dimension-shifting recurrence relations produces powers of $1/()_4$.
They are the unwanted sub-Gram-determinants $()_4$.
Simplest:
In I_5^μ (32) appears I_4^s.

Isolation of inverse sub-Gram det^s $()_4$ II

One of Tarasov's recurrence relations (11) applies ($n = 4$):

$$()_n (d-3) I_4^{[d+]} \;=\; \left[\binom{0}{0}_4 - \sum_{k=1}^{4} \binom{0}{k}_4 \mathbf{k}^- \right] I_4 \tag{41}$$

So that we express:

$$I_4^D = \sum_{k=1}^{4} \frac{\binom{k}{0}_4}{\binom{0}{0}_4} I_3^{D,k}, \quad \text{for} ()_n = 0 \tag{42}$$

If the dimension $D = d + 2l > d$, one has to reduce the dimension of $I_3^{D,k}$ by Tarasov's dim-recurrences, but now without $1/()_4$, because now $1/()_3$ appear.

Isolation of inverse sub-Gram det^s $()_4$ III

For the other boxes, with higher indices, the game is more involved.
Try to have inverse Gram determinants only in front of boxes I_4, and hold the I_3, I_2, I_1 free of them
Introduce:

$$\lim_{\binom{s}{s}_5 \to 0} I_4^{d,s} = \sum_{t=1}^{5} \frac{\binom{ts}{0s}_5}{\binom{0s}{0s}_5} I_3^{d,st} \equiv Z_4^{d,s} \tag{43}$$

A typical example appearing in (34) is $I_{4,i}^{[d+],s}$.
With the aid of relations like

$$\binom{0s}{is}_5 \binom{ts}{0s}_5 - \binom{0s}{0s}_5 \binom{ts}{is}_5 = -\binom{s}{s}_5 \binom{0st}{0si}_5, \tag{44}$$

Isolation of inverse sub-Gram det^s $()_4$ IV

one may cancel out the factor $\binom{s}{s}_5$ from the combination

$$\frac{\binom{0s}{is}_5}{\binom{s}{s}_5} Z_4^s - \sum_{t=1}^{5} \frac{\binom{ts}{is}_5}{\binom{s}{s}_5} I_3^{st} = \cdots = -\frac{1}{\binom{0s}{0s}_5} \sum_{t=1}^{5} \binom{0st}{0si}_5 I_3^{st} \tag{45}$$

This allows to rewrite the recurrence relation

$$I_{4,i}^{[d+],s} = -\frac{\binom{0s}{is}_5}{\binom{s}{s}_5} I_4^s + \sum_{t=1}^{4} \frac{\binom{ts}{is}_5}{\binom{s}{s}_5} I_3^{st}$$

into one with $1/\binom{s}{s}_5 = 1/()_4$ ONLY in front of boxes:

$$I_{4,i}^{[d+],s} = -\binom{0s}{is}_5 \frac{\left[I_4^s - Z_4^s\right]}{\binom{s}{s}_5} + \frac{1}{\binom{0s}{0s}_5} \sum_{t=1}^{5} \binom{0st}{0si}_5 I_3^{st} \tag{46}$$

Isolation of inverse sub-Gram det^s $()_4$ V

This is a nice expression, because it contains an explicit difference quotient for the case that $\binom{s}{s}_5$ becomes zero or if it becomes small but finite.
The numerical strategy will be:

- The $()_4 \neq 0$: Just evaluate what you like.
- The $()_4 = 0$: Replace

$$\frac{\left[I_4^s - Z_4^s\right]}{\binom{s}{s}_5} \Rightarrow \frac{d-3}{\binom{0}{0}_4} I_4^{[d+],s}$$

and the $I_4^{[d+],s}$ may be written, in that case $()_4 = 0$, by $\sum_t c_k I_3^{[d+],st}$.
Further reducing to scalars in d dimensions, no $1/()_4$ will re-appear.

Isolation of inverse sub-Gram det^s $()_4$ VI

This is, what one may do if not explicitely work on expansions of the scalar integrals for small $()_n$.

Example: 4-point tensor of rank 3 I

Following Davydychev, [18], one gets

$$I_4^{\mu\nu\lambda} = \int^d \frac{k^\mu k^\nu k^\lambda}{\prod_{r=1}^n c_r} = -\sum_{i,j,k=1}^n q_i^\mu q_j^\nu q_k^\lambda \nu_{ijk} I_{n,ijk}^{[d+]^3} + \frac{1}{2}\sum_{i=1}^n g^{[\mu\nu} q_i^{\lambda]} I_{n,i}^{[d+]^2} \quad (47)$$

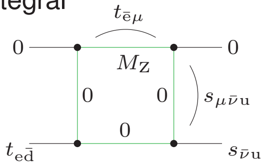
We identify the tensor coefficient D_{111}:

$$D_{111} \sim \nu_{ijk} I_{4,ijk}^{[d+]^3} \text{ for } ijk = 222$$

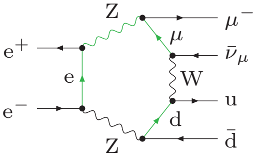
An example from A. Denner: D_{111}

Next figures copied from: A.Denner, plenary talk DESY Theory Workshop 2009, p.69 (backup transparency)

box integral



appears, e.g., in subgraph of diagram

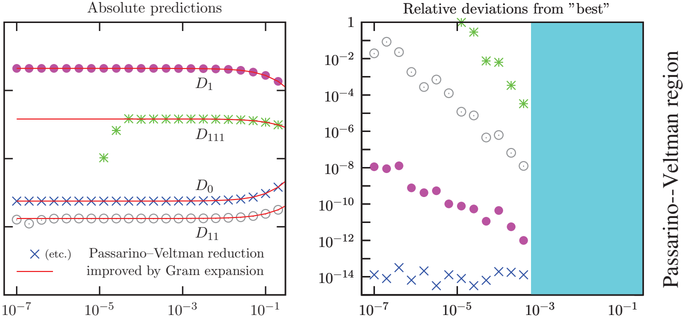


Gram det.: $\Delta^{(N)} \to 0$ if $t_{e\bar{d}} \to t_{crit} \equiv \dfrac{s_{\mu\bar{\nu}u}(s_{\mu\bar{\nu}u} - s_{\bar{\nu}u} + t_{\bar{e}\mu})}{s_{\mu\bar{\nu}u} - s_{\bar{\nu}u}}$

The figure demonstrates the effects of careful treatment of vanishing Gram determinant $()_4$.

Gram det.: $\Delta^{(N)} \to 0$ if $t_{\mathrm{e\bar{d}}} \to t_{\mathrm{crit}} \equiv \dfrac{s_{\mu\bar{\nu}\mathrm{u}}(s_{\mu\bar{\nu}\mathrm{u}} - s_{\bar{\nu}\mathrm{u}} + t_{\bar{\mathrm{e}}\mu})}{s_{\mu\bar{\nu}\mathrm{u}} - s_{\bar{\nu}\mathrm{u}}}$

numerical comparison: maximal tensor rank $=$ 6 (similar to $\mathrm{ee} \to 4f$ application)



$$x \equiv \frac{t_{\mathrm{e\bar{d}}}}{t_{\mathrm{crit}}} - 1$$

$$\begin{aligned} s_{\mu\bar{\nu}\mathrm{u}} &= +2\times10^4\ \mathrm{GeV}^2 \\ s_{\bar{\nu}\mathrm{u}} &= +1\times10^4\ \mathrm{GeV}^2 \\ t_{\bar{\mathrm{e}}\mu} &= -4\times10^4\ \mathrm{GeV}^2 \\ t_{\mathrm{crit}} &= -6\times10^4\ \mathrm{GeV}^2 \end{aligned}$$

PV reduction breaks down,
but Gram exp. stable
for $\Delta^{(N)} \to 0$!

D_{111} – numerics with our reductions

Now rewrite:

$$\nu_{ijk} I_{4,ijk}^{[d+]^3} = -\frac{\binom{0}{k}_4}{()_4} I_{4,ij}^{[d+]^2} + \sum_{t=1,t\neq i,j}^{4} \frac{\binom{t}{k}_4}{()_4} I_{3,ij}^{[d+]^2,t} + \frac{\binom{i}{k}_4}{()_4} I_{4,j}^{[d+]^2} + \frac{\binom{j}{k}_4}{()_4} I_{4,i}^{[d+]^2} \tag{48}$$

$$\begin{aligned}\nu_{ij}\nu_{ijk} I_{4,ijk}^{[d+]^3,s,reg} &= -\frac{\binom{0s}{ks}_5}{\binom{s}{s}_5}\nu_{ij}\left[I_{4,ij}^{[d+]^2,s} - Z_{4,ij}^{[d+]^2,s}\right] \\ &+ \frac{1}{\binom{0s}{0s}_5}\left[\binom{0si}{0sk}_5 I_{4,j}^{[d+]^2,s} + \binom{0sj}{0sk}_5 I_{4,i}^{[d+]^2,s} + \sum_{t=1,t\neq i,j}^{5}\binom{0st}{0sk}_5 \nu_{ij} I_{3,ij}^{[d+]^2,st}\right]\end{aligned} \tag{49}$$

$$\begin{aligned}\nu_{ij}\nu_{ijk} I_{4,ijk}^{[d+]^3,s} &= -\frac{\binom{0}{k}}{\binom{0}{0}}\left[d\frac{\binom{0}{i}}{\binom{0}{0}}\frac{\binom{0}{j}}{\binom{0}{0}}(d-3)(d+1)I_4^{[d+]^3} + (d-3)\frac{1}{\binom{0}{0}}\binom{0i}{0j}I_4^{[d+]^2} + 2\nu_{ij}I_{4,ij}^{[d+]^3}\right. \\ &\left. -\frac{5d-16}{\binom{0}{0}}\frac{\binom{0}{j}}{\binom{0}{0}}\sum_{t=1}^{4}\binom{0t}{0i}I_3^{[d+]^2,t} + \frac{d-3}{\binom{0}{0}}\sum_{t=1}^{4}\binom{0t}{0j}I_{3,i}^{[d+]^2,t}\right] \\ &+ \frac{1}{\binom{0}{0}}\left[\binom{0i}{0k}I_{4,j}^{[d+]^2} + \binom{0j}{0k}I_{4,i}^{[d+]^2} + \sum_{t=1,t\neq i,j}^{4}\binom{0t}{0k}\nu_{ij}I_{3,ij}^{[d+]^2,t}\right]\end{aligned} \tag{50}$$

Numerics: LoopTools versus our approach

Using LoopTools call and our math numerics (preliminary):

```
x      D111

 -7 : -0.007106204244698895     +0.004653980785027332 5 I  D0i[dd111]
      -3.15345811639208  -10    -3.318373348243635`-10 I  Z4d30,Z4d20,I4id20

 -6 : -3.2313079078584034-06    -2.8963160014947846-06 I D0i[dd111]
      -3.1479286753545824-10    -3.318332145498356 -10 I Z4d30,Z4d20,I4id20

 -5 : -5.5231182028025025-09    +3.4832284324178667-09 I D0i[dd111]
      -3.0926394107374516-10    -3.3179201270079527-10 I Z4d30,Z4d20,I4id20

 x< -4:        LoopTools dies out

 -4 : -3.1544928789869657-10    -3.33218368329059  -10 I D0i[dd111]
      -3.0798250216856066-10    -3.3447698103297804-10 I   flei

 x < -3:       loss of accuracy

 -3 : -3.153742175665908 -10    -3.31639655233478  -10 I D0i[dd111]
      -3.1537481925176414-10    -3.3164147721227693-10 I   flei

 -2 : -3.1500799889469005-10    -3.29915924109457  -10 I D0i[dd111]
      -3.150080001830792 -10    -3.2991592067243136-10 I   flei

 -1 : -3.112267506942415 -10    -3.135823319774082 -10 I D0i[dd111]
      -3.1122675069507063-10    -3.1358233197649007-10 I   flei
```

Summary

- Recursive treatment of pnetagon tensors integrals of rank R in terms of pentagons and boxes of rank $R - 1$
- Systematic derivation of expressions which are explicitely free of inverse Gram determinants $()_5$ until pentagons of rank $R = 5$
- Properly isolation of inverse Gram determinants of subdiagrams of the type $\binom{s}{s}_n$, which cannot be completely avoided.
- Some [preliminary] numerics, so far Mathematica, and C++ under way

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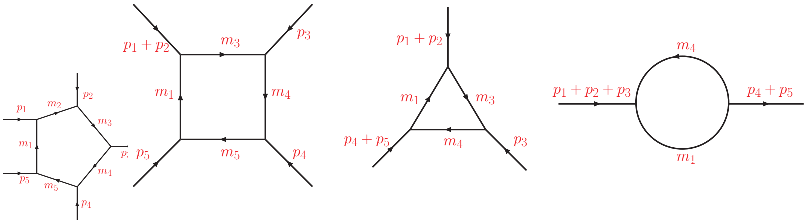
Numbers (I) – Pentagons

Randomly chosen phase space point with massive and massless internal particles

p_1	5.00000000000 E+00	0.0000000000 E+00	0.00000000000 E+00	4.00000000000 E+00
p_2	5.00000000000 E+00	0.0000000000 E+00	0.00000000000 E+00	-4.00000000000 E+00
p_3	– 0.30770034895 E+01	0.5359484673 E+00	–0.37447035150 E+00	– 0.20120057390 E+00
p_4	– 0.34048537280 E+01	0.2184763540 E-01	– 0.10479394969 E+01	0.12224460727 E+01
p_5	– 0.35181427825 E+01	– 0.5577961027 E+00	0.14224098484 E+01	– 0.10212454988 E+01
$m_1 = 0.0, \quad m_2 = 2.0, \quad m_3 = 3.0, \quad m_4 = 4.0, \quad m_5 = 5.0$				



Selected pentagon components

Shown are the constant terms of the tensor components

	Pentagon.F
E^2	(2.80450709388539E-05 , -1.08461817406464E-05)
E^{12}	(-5.41333978667301E-06 , 6.26985967678899E-06)
E^{232}	(-1.20374858970726E-04 , 4.07974751672555E-04)
E^{0321}	(-9.11194535703727E-06 , 4.39187998675819E-05)
E^{01230}	(4.37928367160152E-05 , -2.18183151665913E-04)

Box.F	*LoopTools*
(6.81403420828588E-03 , -5.74298462683219E-03)	(6.8140342082847463E-03 , -5.7429846268324187E-03)
(2.40138809967981E-03 , 1.11591328775015E-02)	(2.4013880996803092E-03,1.1159132877500448E-02)
(-1.69702786278243E-03 , -2.83731121595478E-03)	(-1.6970278627700630E-03,-2.8373112159962330E-03)
(-1.92190388316994E-04 , -4.04730302413490E-04)	(-1.9219038693301300E-04,-4.0473030187772325E-04)

	Triangle.F	*LoopTools*
C^2	(2.44757827793318E-04 , -7.50688449850356E-03)	(2.4475782779342707E-04,-7.5068844985030472E-03)
C^{01}	(-1.28259813172255E-02 , -6.73809718907549E-02)	(-1.2825981317215014E-02,-6.7380971890795340E-02
C^{133}	(-7.00360822297110E-02 , 7.24628606014397E-02)	(-7.0036082229746830E-02,7.2462860601566081E-02)

	Bubble.F	*LoopTools*
B^3	(-0.141525070262337E+00 , 0.138870631815383E+00)	(-0.1415250702623366,0.1388706318153829)
B^{12}	(0.102490343329085E+00 , -6.12154531068256E-02)	(0.1024903433290848,-6.1215453106825706E-02)

here some text 1.

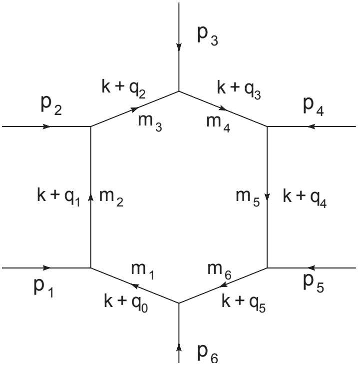


Figure: Momenta flow for the massive six-point topology.

here some text 2.

Numbers (II) – Hexagons

p_1	0.21774554 E+03	0.0	0.0	0.21774554 E+03
p_2	0.21774554 E+03	0.0	0.0	– 0.21774554 E+03
p_3	– 0.20369415 E+03	– 0.47579512 E+02	0.42126823 E+02	0.84097181 E+02
p_4	– 0.20907237 E+03	0.55215961 E+02	– 0.46692034 E+02	– 0.90010087 E+02
p_5	– 0.68463308 E+01	0.53063195 E+01	0.29698267 E+01	– 0.31456871 E+01
p_6	– 0.15878244 E+02	– 0.12942769 E+02	0.15953850 E+01	0.90585932 E+01
$m_1 = 110.0,\ m_2 = 120.0,\ m_3 = 130.0,\ m_4 = 140.0,\ m_5 = 150.0,\ m_6 = 160.0$				

		F_0
		– 0.223393 E–18 – i 0.396728 E–19
μ		F^{μ}
0		0.192487 E–17 + i 0.972635 E–17
1		– 0.363320 E–17 – i 0.11940 E–17
2		0.365514 E–17 + i 0.106928 E–17
3		0.239793 E–16 + i 0.341928 E–17
μ	ν	$F^{\mu\nu}$
0	0	0.599459 E–14 – i 0.114601 E–14
0	1	0.323869 E–15 + i 0.423754 E–15
0	2	– 0.294252 E–15 – i 0.375481 E–15
0	3	– 0.255450 E–14 – i 0.195640 E–14
1	1	– 0.164562 E–14 – i 0.993796 E–16
1	2	0.920944 E–16 + i 0.706487 E–17
1	3	0.347694 E–15 – i 0.127190 E–16
2	2	– 0.163339 E–14 – i 0.994148 E–16
2	3	– 0.341773 E–15 + i 0.818678 E–17
3	3	– 0.413909 E–14 + i 0.670676 E–15

μ	ν	λ	$F^{\mu\nu\lambda}$
0	0	0	– 0.227754 E–11 – i 0.267244 E–12
0	0	1	0.140271 E–13 – i 0.119448 E–12
0	0	2	– 0.201270 E–13 + i 0.101968 E–12
0	0	3	0.102976 E–12 + i 0.624467 E–12
0	1	1	0.183904 E–12 + i 0.142429 E–12
0	1	2	– 0.131028 E–13 – i 0.610343 E–14
0	1	3	– 0.543316 E–13 – i 0.158809 E–13
0	2	2	0.181352 E–12 + i 0.141686 E–12
0	2	3	0.506408 E–13 + i 0.163568 E–13
0	3	3	0.600542 E–12 + i 0.130733 E–12
1	1	1	– 0.563539 E–13 + i 0.178403 E–13
1	1	2	0.210641 E–13 – i 0.584990 E–14
1	1	3	0.120482 E–12 – i 0.574688 E–13
1	2	2	– 0.201182 E–13 + i 0.620591 E–14
1	2	3	– 0.686164 E–14 + i 0.205457 E–14
1	3	3	– 0.447329 E–13 + i 0.193180 E–13
2	2	2	0.582201 E–13 – i 0.163889 E–13
2	2	3	0.119659 E–12 – i 0.570084 E–13
2	3	3	0.457464 E–13 – i 0.181141 E–13
3	3	3	0.557081 E–12 – i 0.374359 E–12

Table: Tensor components for a massive rank $R = 3$ six-point function

μ	ν	λ	ρ	$F^{\mu\nu\lambda\rho}$
0	0	0	0	0.666615 E–09 + i 0.247562 E–09
0	0	0	1	– 0.200049 E–10 + i 0.294036 E–10
0	0	0	2	0.200975 E–10 – i 0.237333 E–10
0	0	0	3	0.645477 E–10 – i 0.162236 E–09
0	0	1	1	– 0.116956 E–10 – i 0.516760 E–10
0	0	1	2	0.160357 E–11 + i 0.222284 E–11
0	0	1	3	0.792692 E–11 + i 0.729502 E–11
0	0	2	2	– 0.111838 E–10 – i 0.513133 E–10
0	0	2	3	– 0.681086 E–11 – i 0.708933 E–11
0	0	3	3	– 0.804454 E–10 – i 0.801909 E–10
0	1	1	1	0.100498 E–10 – i 0.151735 E–13
0	1	1	2	– 0.348984 E–11 – i 0.195436 E–12
0	1	1	3	– 0.211111 E–10 + i 0.295212 E–11
0	1	2	2	0.357455 E–11 + i 0.662809 E–14
0	1	2	3	0.121595 E–11 – i 0.807388 E–13
0	1	3	3	0.825803 E–11 – i 0.142086 E–11
0	2	2	2	– 0.958961 E–11 – i 0.585948 E–12
0	2	2	3	– 0.209232 E–10 + i 0.289031 E–11
0	2	3	3	– 0.802359 E–11 + i 0.994701 E–12
0	3	3	3	– 0.102576 E–09 + i 0.378476 E–10
1	1	1	1	– 0.246426 E–10 + i 0.276326 E–10
1	1	1	2	0.915670 E–12 – i 0.660629 E–12
1	1	1	3	0.303529 E–11 – i 0.287480 E–11
1	1	2	2	– 0.822697 E–11 + i 0.919635 E–11
1	1	2	3	– 0.116294 E–11 + i 0.100024 E–11
1	1	3	3	– 0.146918 E–10 + i 0.183799 E–10
1	2	2	2	0.908296 E–12 – i 0.654735 E–12
1	2	2	3	0.109510 E–11 – i 0.100875 E–11
1	2	3	3	0.717342 E–12 – i 0.557293 E–12
1	3	3	3	0.450661 E–11 – i 0.485065 E–11
2	2	2	2	– 0.245154 E–10 + i 0.274313 E–10
2	2	2	3	– 0.318500 E–11 + i 0.279750 E–11
2	2	3	3	– 0.146317 E–10 + i 0.182912 E–10
2	3	3	3	– 0.477335 E–11 + i 0.477368 E–11
3	3	3	3	0.730168 E–10 + i 0.112865 E–09

p_1	0.5	0.0	0.0	0.5
p_2	0.5	0.0	0.0	– 0.5
p_3	– 0.19178191	– 0.12741180	– 0.08262477	– 0.11713105
p_4	– 0.33662712	0.06648281	0.31893785	0.08471424
p_5	– 0.21604814	0.20363139	– 0.04415762	– 0.05710657
$p_6 = -(p_1 + p_2 + p_3 + p_4 + p_5)$				

Table: Phase space point of massless six-point functions taken from [Binoth:2008 [7]] . Golem95: Binoth, Guillet, Heinrich, Pilon, Reiter [arXiv:hep-ph/0810.0992]

Shown are only the constant terms of the tensor components.

	Hexagon.F	*Golem95*
F^{03121}	(0.158428986740235E+00 , 0.416706979843194E-01)	(0.158428980552600E+00 , 0.416706995132716E-01)
F^{11020}	(-0.143913859903552E+01 , -0.164647048275408E+00)	(-0.143913852754709E+01 , -0.164647075385477E+00)
F^{20200}	(0.242928799509288E+02 , 0.555041844207877E+02)	(0.242928775936564E+02 , 0.555041824180155E+02)
F^{22130}	(0.225563941055782E+00 , 0.231928571404353E+00)	(0.225563949300093E+00 , 0.231928509918651E+00)
F^{33333}	(0.244568134868438E+00 , 0.740146041525474E+00)	(0.244568138432017E+00 , 0.740146095196997E+00)