Sector Decomposition via Computational Geometry

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arXiv:0908.2897 [hep-ph]

Contents

- Introduction : Sector decomposition
- Geometric method
- Test implementation
- Conclusion

Sector decomposition

- References
  - T. Binoth and G. Heinrich, Nucl. Phys. B585 (2000) 741
  - see also: G. Heinrich, Int. J. Mod. Phys. A23 (2008) 1457
- Purpose
  - Separation of IR divergent parts from multi-loop integration in $(4-2\varepsilon)$ space-time dimension:

    Extract poles of $1/\varepsilon$ with IR finite coefficients (numerical integration)

Sector decomposition

- Example of separation of a divergence

$$I = \int_0^1 dx\, x^{-1-\epsilon} f(x) \qquad \text{for} \quad f(0) : \text{finite}, \quad \epsilon < 0, \quad \epsilon \to -0,$$
$$= \int_0^1 dx\, x^{-1-\epsilon} f(0) + \int_0^1 dx\, x^{-1-\epsilon}(f(x) - f(0))$$
$$= -\frac{f(0)}{\epsilon} + \int_0^1 dx\, (x^{-\epsilon} f'(x) + \frac{1}{2} x^{1-\epsilon} f''(x) + \cdots)$$

  The first term has a pole. The second term is finite.

- Problems :
  - How IR diverging part is factored out for multi-dimensional integration?
- Answer : Sector Decomposition Method
  - Decomposes the integration domain into several sectors.
  - Finds good variables in each sector for the integration.
  - The integrand is factorized in each sector.

Sector decomposition

- Procedure
  1. Start from Feynman parameter representation of a loop integration.
  2. Primary sector decomposition: integration with δ-function.
  3. Factorization of the integrand
     - Iterated sector decomposition :
       ➔ iteration of sector decomposition finding new variables
     - Our problem
  4. Separation of $1/\varepsilon$ poles (expansion and analytical integration)
  5. Integration of coefficients (usually numerical method)

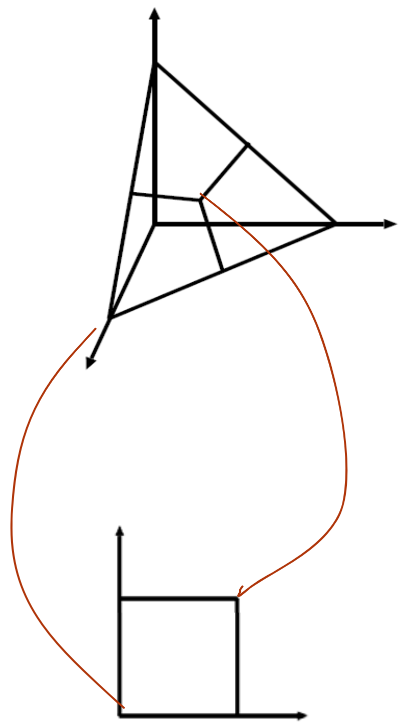
Sector Decomposition

1. Feynman parameter representation

$$G = (-1)^{N_\nu} \frac{\Gamma(N_\nu - LD/2)}{\prod_{j=1}^{N} \Gamma(\nu_j)} \int_0^\infty d^N x \, x^{\nu - \mathbb{I}} \delta \left(1 - \sum_{l=1}^{N} x_l \right) \frac{\mathcal{U}^{N_\nu - (L+1)D/2}}{\mathcal{F}^{N_\nu - LD/2}},$$

- L is the number of loops,
- $D = 4 - 2\epsilon$ is the dimension of the space-time,
- N is the dimension of the integration,
- ν_j $(j = 1, \cdots, N)$ is the power of the propagator corresponding to the Feynman parameter x_j,
- $N_\nu := \sum_{j=1}^{N} \nu_j$,
- $\mathcal{U}$ is a homogeneous polynomial of $\{x_j\}$ of degree L, and all the coefficients of the monomials of $\mathcal{U}$ are equal to 1.
- $\mathcal{F}$ is a homogeneous polynomial of $\{x_j\}$ of degree $L+1$, and the coefficients of the monomials of $\mathcal{F}$ consist of kinematic and mass parameters.

Sector decomposition



2. Primary sector decomposition

- (Sector l) = $\{(x_1, x_2, \cdots, x_N) \mid x_j \le x_l, \ \forall j \ne l\}$
- Change variables

$$x_j = x_l \times \begin{cases} t_j & \text{for} \quad j < l \\ 1 & \text{for} \quad j = l \\ t_{j-1} & \text{for} \quad j > l \end{cases}$$

- Integrate over x_l with δ-function
- Results

$$G = (-1)^{N_\nu} \frac{\Gamma(N_\nu - LD/2)}{\prod_{j=1}^{N} \Gamma(\nu_j)} \sum_{l=1}^{N} G_l,$$

$$G_l = \int_0^1 d^{N-1}t \, t^{\nu' - \mathbb{I}} \, \mathcal{U}_l^{\gamma}(t) \, \mathcal{F}_l^{\beta}(t),$$

$$\beta = -(N_\nu - LD/2),$$
$$\gamma = N_\nu - (L+1)D/2,$$

- IR singularities : boundary ➔ some of variables = 0.

$\mathcal{U}_l(t) = 0$ or $\mathcal{F}_l = 0$ for some of $t_j = 0$.

Sector decomposition

3. Factorization of the integrand

Find a sector decomposition satisfying the following conditions (let us call it a polynomial sector decomposition)

1. Divide integration domain into finite number of sectors
2. In each sector, new variables are defined $z = (z_1, \cdots, z_{N-1})$.
   1. An original variable is expressed by a monomial of a new variable.
   2. Jacobean consists of one term (new polynomial does not emerge)
   3. Integration domain for the new variable is a unit cube.
3. In each sector, polynomials in the integrand are expressed by

$$\mathcal{U}_l = C_a z^{b_a} (1 + H_a(z)), \qquad H_a(0) = 0$$
$$\mathcal{F}_l = C'_a z^{b'_a} (1 + H'_a(z)), \qquad H'_a(0) = 0$$

Sector decomposition

3. Factorization of the integrand (cont.)

Once polynomial sector decomposition is obtained:

$$G_l = \sum_a C''_a \int_0^1 d^{N-1}z \; z^{c_a + b'_a \beta + b_a \gamma} \; (1 + H_a(z))^\gamma \, (1 + H'_a(z))^\beta$$

IR diverging factors are separated.

4. Separation of $1/\varepsilon$ poles : similar to the first example

Expand $(1 + H_a(z))^\gamma \, (1 + H'_a(z))^\beta$ in terms of z

(γ and β are liner function of ε).

$\Rightarrow$ integrand = (IR diverging term) + (IR finite part)

Integrate (IR diverging term) analytically.

$\Rightarrow$ Separation of poles

Sector decomposition

How can we construct a polynomial sector decomposition?

- Iterated sector decomposition :

  Repeat the following steps while S is not empty:

1. Determine a minimal set of parameters, say $\mathcal{S} = \{t_{\alpha_1}, \ldots, t_{\alpha_r}\}$, such that $\mathcal{U}_l$, respectively $\mathcal{F}_l$, vanish if the parameters of S are set to zero. S is generally not unique.
2. Decompose the corresponding r-cube into r subsectors:

$$\prod_{j=1}^{r} \theta(1 \geqslant t_{\alpha_j} \geqslant 0) = \sum_{k=1}^{r} \prod_{\substack{j=1 \\ j \neq k}}^{r} \theta(t_{\alpha_k} \geqslant t_{\alpha_j} \geqslant 0)$$

3. Remap the variables to the unit cube in each new subsector by substituting

$$t_{\alpha_j} \rightarrow \begin{cases} t_{\alpha_k} t_{\alpha_j}, & j \neq k, \\ t_{\alpha_k}, & j = k. \end{cases}$$

Jacobean $= t_{\alpha_k}^{r-1}$.

Sector decomposition

- This procedure may fall into infinite loop.
- Improvement : strategy for selecting a set of parameters in each iteration.

  Avoid infinite loop of iterations
  - C. Bogner and S. Weinzierl, CPC. 178 (2008) 596
  - A.V. Smirnov and M.N. Tentyukov, CPC. 180 (2009) 735
  - A.V. Smirnov and V.A. Smirnov, JHEP 0905 (2009) 004
  - A.V. Smirnov, V.A. Smirnov and M.N. Tentyukov, arxiv:0912.0158 [hp-ph]
- How many sectors? Depends on the strategy of iterated sector decomposition.

Diagram	A	B	C	S	X
Box	12	12	12	12	12
Double box	755	586	586	362	293
Triple box	M	114256	114256	22657	10155
D420	8898	564	564	180	F

(A.V. Smirnov and M.N. Tentyukov)

$\Rightarrow$ Integration should be repeated for all decomposed sectors : efficiency

Geometric method

- Observations

$$\mathcal{F}_l \quad = \quad C'_a z^{b'_a} \left(1 + H'_a(z)\right)$$

  - Diverging factor corresponds to one of terms of the polynomial. Example:

$$\mathcal{F}_l(t) = -s_{23}t_2t_3 - s_{12}t_1 - s_4t_1t_3 = t_2t_3\left(-s_{23} - s_{12}\frac{t_1}{t_2t_3} - s_4\frac{t_1}{t_2}\right)$$
$$= z_2z_3(-s_{23} - s_{12}z_1 - s_4z_1z_3)$$
$$t_2 = z_2, \quad t_3 = z_3 \quad t_1 = z_1z_2z_3,$$

  - Such a term dominates other terms for the limit $t \to 0$ on some path in the integration domain.

➔ There will be a sub-domain (possibly empty) in which a specific term dominates others around 0.

Geometric method

- Asymptotic behavior of polynomial $P(t)$ around $t \to 0$
  - Change variable $t_j = e^{-y_j}$
  - Non-negative integer vector of powers of a monomial

$$t^b = t_1^{b_1} t_2^{b_2} \cdots t_{N-1}^{b_{N-1}} = e^{-(b,y)} \quad \Rightarrow \quad b = (b_1, b_2, \cdots, b_{N-1})$$

  - Let Z^P be the set of such vectors corresponding to the terms of a polynomial $P(t)$. That is:

$$P(t) = \sum_{b \in Z^P} a_b t^b = \sum_{b \in Z^P} a_b e^{-(b,y)}$$

Geometric method

$$P(t) = \sum_{b \in Z^P} a_b t^b = \sum_{b \in Z^P} a_b e^{-(b,y)}$$

- Consider a limit $\lambda \to +\infty$ for $y_j = \lambda u_j \to +\infty$ with non-negative real fixed vector u.

  A term with power vector b dominates others in this limit when

$$(b, y) \leq (c, y), \ \forall c \in Z^P$$

- Conversely, let us fix b and vary y (or u).

  This term with b dominates others in the sub-domain:

$$\Delta_b^P \ := \ \{y \in \mathbb{R}_{\geq 0}^{N-1} \mid (b, y) \leq (c, y), \ \forall c \in Z^P\}$$

$$P(t(y)) \ = \ \sum_{b \in Z^P} \theta(y \in \Delta_b^P) \, e^{-(b,y)} \left[a_b + \sum_{c \in Z^P \setminus \{b\}} a_c e^{-(c-b,y)} \right]$$

Geometric method

- Our integral is (with $\Delta_{bb'} := \Delta_b^{\mathcal{U}_l} \cap \Delta_{b'}^{\mathcal{F}_l}$, pick up non-empty set)

$$G_l = \sum_{b \in Z^{\mathcal{U}_l}} \sum_{b' \in Z^{\mathcal{F}_l}} \int_0^\infty d^{N-1}y\ \theta(y \in \Delta_{bb'})\ e^{-(\nu' + \gamma b + \beta b', y)} \times \left[1 + \sum_{c \in Z^{\mathcal{U}_l} \setminus \{b\}} e^{-(c-b, y)}\right]^\gamma \left[a_{b'} + \sum_{c' \in Z^{\mathcal{F}_l} \setminus \{b'\}} a_{c'} e^{-(c'-b', y)}\right]^\beta$$

- We call this decomposition an exponential sector decomposition.
  - This decomposition classifies the asymptotic behavior of polynomials around the origin of t-space.

    ➔ polynomial sector decomposition should have more sectors.

Geometric method

- How can we construct ?

$$\Delta_b^P \quad := \quad \{y \in \mathbb{R}_{\geq 0}^{N-1} \mid (b, y) \leq (c, y),\ \forall c \in Z^P\}$$

➔Combinatorial or computational geometry.

- Convex polyhedral cone for a finite set S in n-dimensional Euclidean space:

$$C(S) := \{\textstyle\sum_{v \in S} r_v v \in \mathbb{R}^n \mid r_v \geq 0,\ \forall v \in S\}$$

- Dual cone of a convex polyhedral cone C:

$$C(S)^\vee := \{y \in \mathbb{R}^n \mid (v, y) \geq 0,\ \forall v \in C(S)\}$$

A dual cone is also a convex polyhedral cone.

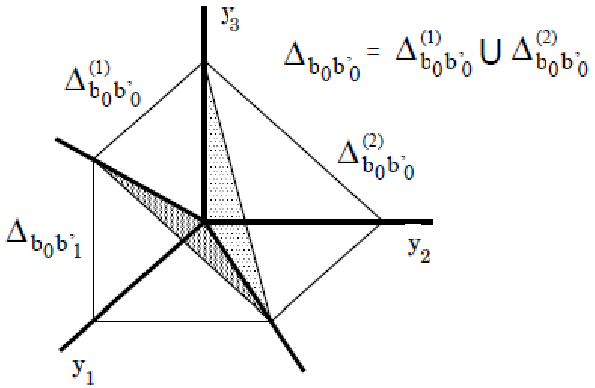
- Our case:

$$\Delta_b^P := C(Z_b^P)^\vee \cap \mathbb{R}_{\geq 0}^{N-1}, \quad \text{with} \quad Z_b^P := \{c - b \in \mathbb{Z}^{N-1} \mid c \in Z^P\}$$

$$\Delta_{bb'} = C(Z_b^{\mathcal{U}_l})^\vee \cap C(Z_{b'}^{\mathcal{F}_l})^\vee \cap \mathbb{R}_{\geq 0}^{N-1}$$

These objects are obtained by algorithms in computational geometry.

Geometric method



- Find new variables ➔ Triangulation
- Example from one-loop box:

$$\mathcal{U}_l(t) = 1 + t_1 + t_2 + t_3,$$
$$\mathcal{F}_l(t) = -s_{12}t_1 - s_{23}t_2t_3 - s_4t_1t_3.$$

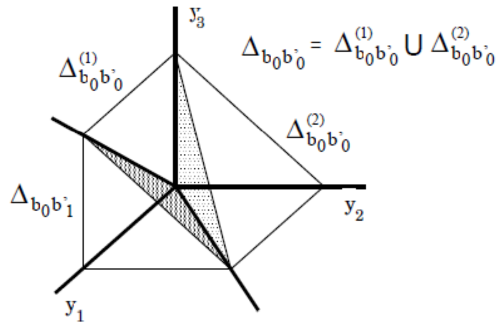
$$Z^{\mathcal{U}_l} = \{b_0 = (0,0,0),\ b_1 = (1,0,0),\ b_2 = (0,1,0),\ b_3 = (0,0,1)\},$$
$$Z^{\mathcal{F}_l} = \{b'_0 = (1,0,0),\ b'_1 = (0,1,1),\ b'_2 = (1,0,1)\}.$$

$$\Delta_{b_0b'_0} = \{x_1(1,1,0) + x_2(1,0,1) + x_3(0,1,0) + x_4(0,0,1) \mid x_1, x_2, x_3, x_4 \geq 0\},$$
$$\Delta_{b_0b'_1} = \{x_1(1,0,0) + x_2(1,1,0) + x_3(1,0,1) \mid x_1, x_2, x_3 \geq 0\},$$
$$\Delta_{b_0b'_1} = \emptyset.$$

- We want to change variable form y to x. However $\Delta_{b_0b'_0}$ has 4 parameters in 3 dimensional space

Too many parameters ➔ triangulate it (not unique).

Geometric method



- Integration domain is divided into 3 sector

$$\Delta^{(1)}_{b_0b'_0} = \{x_1(1,1,0) + x_2(1,0,1) + x_4(0,0,1) \mid x_1, x_2, x_4 \geq 0\}$$

$$\Delta^{(2)}_{b_0b'_0} = \{x_1(1,1,0) + x_3(0,1,0) + x_4(0,0,1) \mid x_1, x_3, x_4 \geq 0\}$$

$$\Delta_{b_0b'_1} = \{x_1(1,0,0) + x_2(1,1,0) + x_3(1,0,1) \mid x_1, x_2, x_3 \geq 0\}$$

- Change variables :

$$t_j \Rightarrow y_j = -\log t_j \Rightarrow x_i \Rightarrow z_i = e^{-x_i}$$

  - Integration domain of *z* is the unit cube.
- We obtain polynomial sector decomposition:

$$\begin{array}{llll}
\Delta^{(1)}_{b_0b'_0} & : \quad t_1 = z_1z_2,\ t_2 = z_1,\ t_3 = z_2z_4 & : & \mathcal{F}_l(t) = z_1z_2(-s_{12} - s_{23}z_4 - s_4z_2z_4) \\
\Delta^{(2)}_{b_0b'_0} & : \quad t_1 = z_1,\ t_2 = z_1z_3,\ t_3 = z_4 & : & \mathcal{F}_l(t) = z_1(-s_{12} - s_{23}z_3z_4 - s_4z_4), \\
\Delta_{b_0b'_1} & : \quad t_1 = z_1z_2z_3,\ t_2 = z_2,\ t_3 = z_3 & : & \mathcal{F}_l(t) = z_2z_3(-s_{12}z_1 - s_{23} - s_4z_1z_3)
\end{array}$$

Geometric method : triangulation

- Convex polyhedral cone $C(V)$ is simplicial when V is a set of $(N-1)$ linear independent vectors: $V=\{v_1,\cdots,v_{N-1}\}$

  Set V is regarded as a matrix with element $V_{ij}=(v_j)_i$.
- Cone $\Delta_{bb'}$ is triangulated when
$$\Delta_{bb'}=\bigcup_{V\in S_{bb'}} C(V)$$
  where $S_{bb'}$ is a finite set of $(N-1)$ linear independent vectors.
- Our integral is

$$G_l=\sum_{b\in Z^{\mathcal{U}_l}}\sum_{b'\in Z^{\mathcal{F}_l}}\sum_{V\in S_{bb'}}\int_0^\infty d^{N-1}y\;\theta(y\in C(V))\;e^{-(\nu'+\gamma b+\beta b',y)}$$
$$\times\left[1+\sum_{c\in Z^{\mathcal{U}_l}\setminus\{b\}} e^{-(c-b,y)}\right]^\gamma\left[a_{b'}+\sum_{c\in Z^{\mathcal{F}_l}\setminus\{b'\}} a_c e^{-(c-b',y)}\right]^\beta.$$

Geometric method : triangulation

- A point $y = (y_i) \in C(V)$ parameterized with barycentric coordinate: $y_i = \sum_{j=1}^{N-1} (v_j)_i x_j, \qquad (x_j \in \mathbb{R}_{\geq 0})$
- Change variable : $y \Rightarrow x$ (Jacobian $= |\det V|$) $\Rightarrow z_j = e^{-x_j}$
- Original variable : $t_j = e^{-y_j} = e^{-(Vx)_j} = \prod_k z_k^{(v_k)_j}$
- Our polynomial sector decomposition is:

$$G_l = \sum_{b \in Z^{\mathcal{U}_l}} \sum_{b' \in Z^{\mathcal{F}_l}} \sum_{V \in S_{bb'}} |\det V| \int_0^1 d^{N-1}z \prod_{j=1}^{N-1} z_j^{(\nu' + \gamma b + \beta b', v_j) - 1}$$
$$\times \left[1 + \sum_{c \in Z^{\mathcal{U}_l} \setminus \{b\}} \prod_{j=1}^{N-1} z_j^{(c-b, v_j)}\right]^\gamma \left[a_{b'} + \sum_{c \in Z^{\mathcal{F}_l} \setminus \{b'\}} a_c \prod_{j=1}^{N-1} z_j^{(c-b', v_j)}\right]^\beta .$$

- Set $S_{bb'}$ (result of triangulation, set of matrices) carries all the information about decomposed sectors.

Geometric method : summary

- Identity

$$\Delta_{bb'} = C(Z_b^{\mathcal{U}_l})^\vee \cap C(Z_{b'}^{\mathcal{F}_l})^\vee \cap \mathbb{R}_{\geq 0}^{N-1} = C(Z_b^{\mathcal{U}_l} \cup Z_{b'}^{\mathcal{F}_l} \cup E^{N-1})^\vee,$$

$$E^{N-1} = \{(1,0,\cdots,0),(0,1,\cdots,0),\ldots,(0,0,\cdots,1)\},$$

$$\mathbb{R}^{N-1} = C(E^{N-1}) = C(E^{N-1})^\vee$$

- Our algorithm

1. Construct point sets: $Z^{\mathcal{U}_l}$, $Z^{\mathcal{F}_l}$, $\{Z_b^{\mathcal{U}_l}\}$, $\{Z_{b'}^{\mathcal{F}_l}\}$, $Z_b^{\mathcal{U}_l} \cup Z_{b'}^{\mathcal{F}_l} \cup E^{N-1}$
2. Construct cone : $C(Z_b^{\mathcal{U}_l} \cup Z_{b'}^{\mathcal{F}_l} \cup E^{N-1})$
3. Construct dual cone $\Delta_{bb'}$
4. Triangulate $\Delta_{bb'}$: $S_{bb'}$ is obtained.

   ➔ Sectors and new variables are obtained.

Test implementation

- Convex hull algorithm (eliminate points inside a cone, analize the structure of a cone)
  - Incremental algorithm
  - Compared with qhull package
- Dual cone
  - Construct normal vectors.
- Triangulation
  - Our own simple algorithm. Neither unique, nor optimum.
  - Check : computation of the integration volume
- Implementation
  - Python program: arbitrary long integer arithmetic
  - Input : obtained by a package by T. Ueda and J. Fujimoto (last ACAT)
  - Output : integrand (factorized polynomials), passed to the same package.
  - Check : integrated value for some processes.

Test implementation

- Result

Diagram	A	B	C	S	X	H	This method	Exponential S.D.
Bubble	2	2	2	2*	2		2	2
Triangle	3	3	3	3*	3		3	3
Box	12	12	12	12	12		12	8
Tbubble	58	48	48	48*	48		48	36
Double box, $p_i^2 = 0$	775	586	586	362	293	282	266	106
Double box, $p_4^2 \neq 0$	543*	245*	245*	230*	192*	197	186	100
Double box, $p_i^2 = 0$ nonplanar	1138	698	698	441*	395		360	120
D420	8898	564	564	180	F		168	100
3 loop vertex (A8)	4617*	1196*	1196*	871*	750*	684	684	240
Triple box	M	114256	114256	22657	10155		6568	856

- "H" : G. Heinrich, Int. J. Mod. Phys. A23 (2008) 1457.
- "A" , "B", "C", "S", "X" (without "*") is cited from Borger & Weinzierl, and Smirnov &Tentyukov.
- "*" added by private communication with A.V. Smirnov.
- "Exponential S.D." : before triangulation.

Conclusion

- We propose a geometric method which corresponds to iterative sector decompositions.
- Based on the classification of the asymptotic behavior of polynomials around 0.
- It is realized by employing algorithms developed in computational geometry.
- It is guaranteed not to fall into an infinite loop.
- The number of produced sectors depends on triangulation method.
- A test implementation shows that less number of sectors is produced than currently available algorithms of iterative sector decomposition.