Multiple-polylog in Loop Integrations

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ACAT 2010 @ Jaipur, India
26/Feb./2010

Mellin-Barnes Transformation
and
Multiple-polylog in Loop Integrations

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Motivation

- Loop integration is very important.
- Ultimate Goal
  - Any # of Loops
  - Any # of Legs
  - Any masses in Loops
  - Any Tensor Structure
  - Any Numerical Precision
  - Stable & Fast Calculation
  - Complete Automation

Methods

- Many methods are developed toward this goal
  - Sector Decomposition
    - T. Binoth, G. Heinrich, V.A. Smirnov ...
  - Integration by Parts
    - K.G. Chetyrkin, F.V. Tkachov, G. Passarino, ...
  - Differential Eqs.
    - A.V Kotikov, E. Remiddi,
      Z. Bern, L. Dixon, D.A. Kosower, ...
  - Direct Numerical Integration
    - E. de Doncker, F. Yuasa, ...

Sorry for an incomplete list

Methods

- Many methods are developed toward this goal
  - Hypergeometric Functions
    - E.E. Boose, A.I. Davydychev, YK, ...
  - Mellin-Barmes Transformation
    - N.I. Ussyukina, V.A. Smirnov,
      M. Czakon, J. Gluza, K. Kajda, T. Riemann, ...
  - Nested Sums/Harmonic-polylog/Multiple-polylog
    - S. Moch, P. Uwer, S. Weinzierl,
      E. Remiddi,J.A.M. Vermaseren, ...

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Calc. Flow of MB method

Feynman parametrization

↓

Momentum Integration

↓

Mellin-Barnes Transformation

↓

Feynman parameter integration

↓

ε-pole separation

↓

σ-Integration→Infinite summation

↓

Convert to known functions/Numerical Value

Cacl. Flow of MB method

Feynman parametrization

↓

Momentum Integration

↓

Mellin-Barnes Transformation

↓

Feynman parameter integration

↓

ε-pole separation

↓

σ-Integration→Infinite summation

↓

Convert to known functions/Numerical Value

Multiple MB variables

Mellin-Barnes Transformation

- Mellin-Barnes Transformation

$$\frac{1}{(\alpha_1+\alpha_2+\cdots+\alpha_n)^c}=\frac{1}{\Gamma(c)}\frac{1}{(2\pi i)^{n-1}}\int_{-i\infty}^{+i\infty}d\sigma_1\cdots\int_{-i\infty}^{+i\infty}d\sigma_{n-1}$$

$alpha_i=q^2

$$\times\Gamma(-\sigma_1)\cdots\Gamma(-\sigma_{n-1})\Gamma(\sigma_1+\cdots+\sigma_{n-1}+c)$$

$$\times\alpha^{\sigma_1}\cdots\alpha^{\sigma_{n-1}}\alpha_n^{-\sigma_1-\cdots-\sigma_{n-1}-c}$$

sum

multiplication

$$\alpha_i=q^2\xi_i(1-\xi_i)\xi_j\cdots \quad \rightarrow \quad \int_0^1 d\xi\,\xi^{a-1}(1-\xi)^{b-1}=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

Mellin-Barnes Transformation

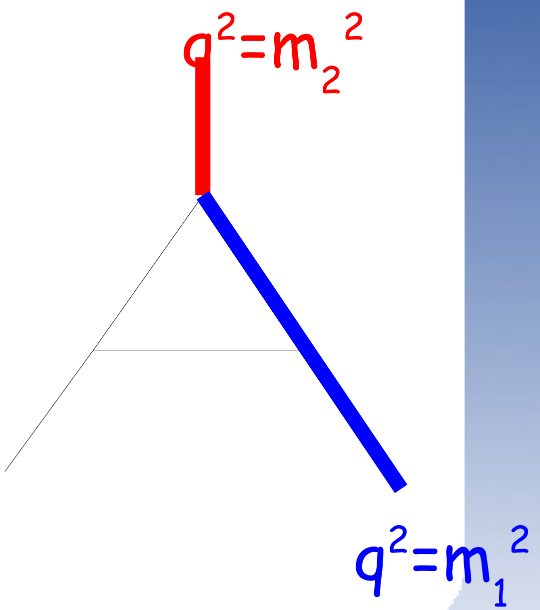
- Example

$$m_1^2(1-\xi_1)^2 - m_1^2(1-\xi_1)\xi_1(1-\xi_2) - m_2^2(1-\xi_1)\xi_1\xi_2$$

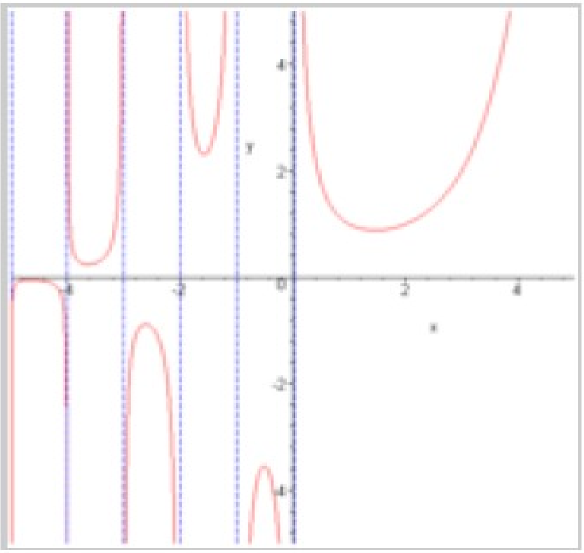
$$||$$

$$m_1^2(1-\xi_1)^2 - (m_1^2 - m_2^2)(1-\xi_1)\xi_1\xi_2$$

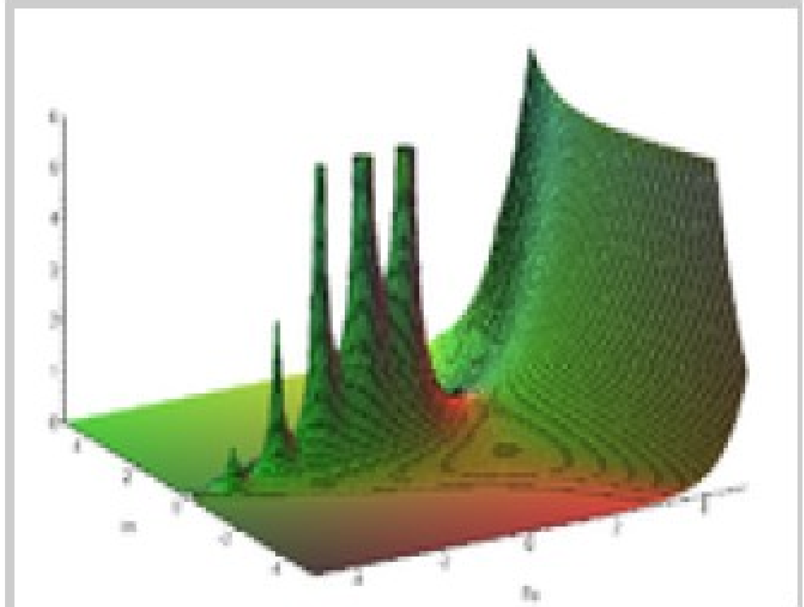
How to find the shortest form?



Gamma Function



$y=\Gamma(x)$

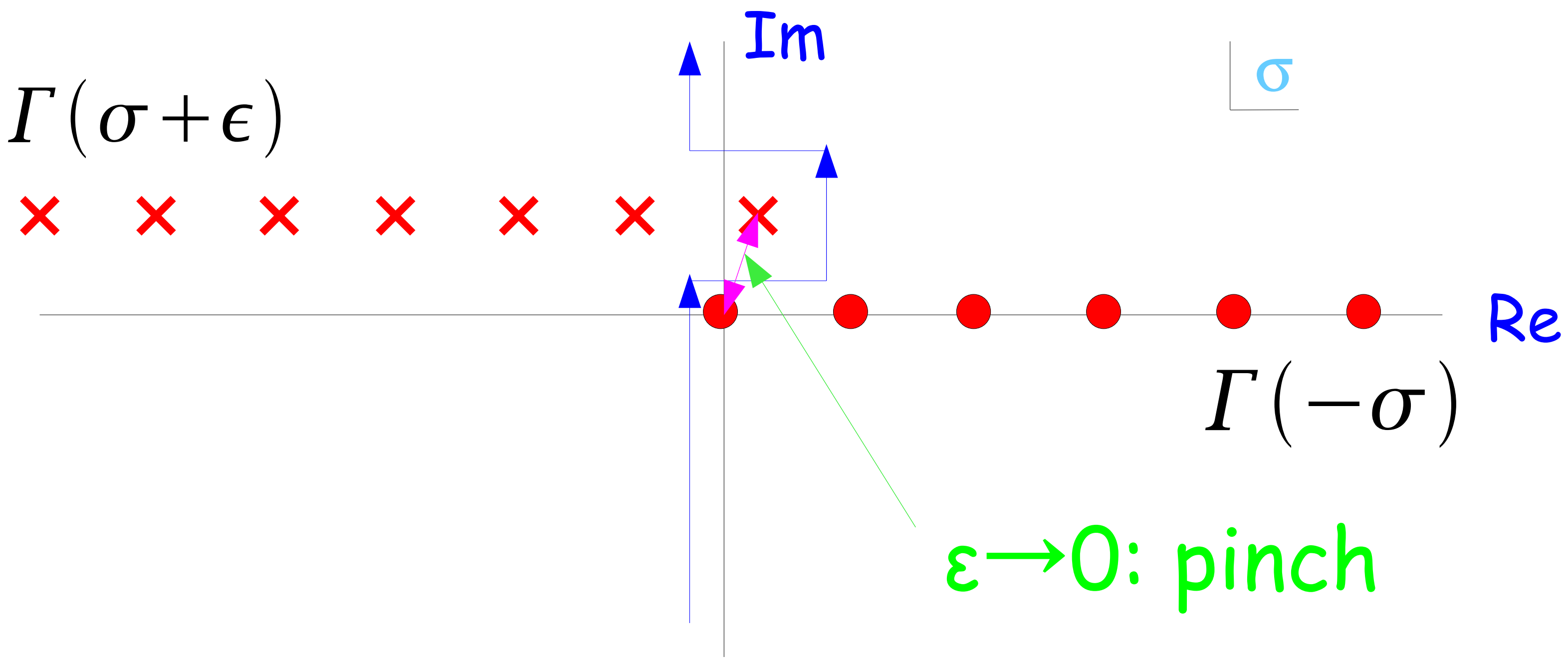


$$\mathrm{Res}(\Gamma, -n) = \frac{(-1)^n}{n!}$$

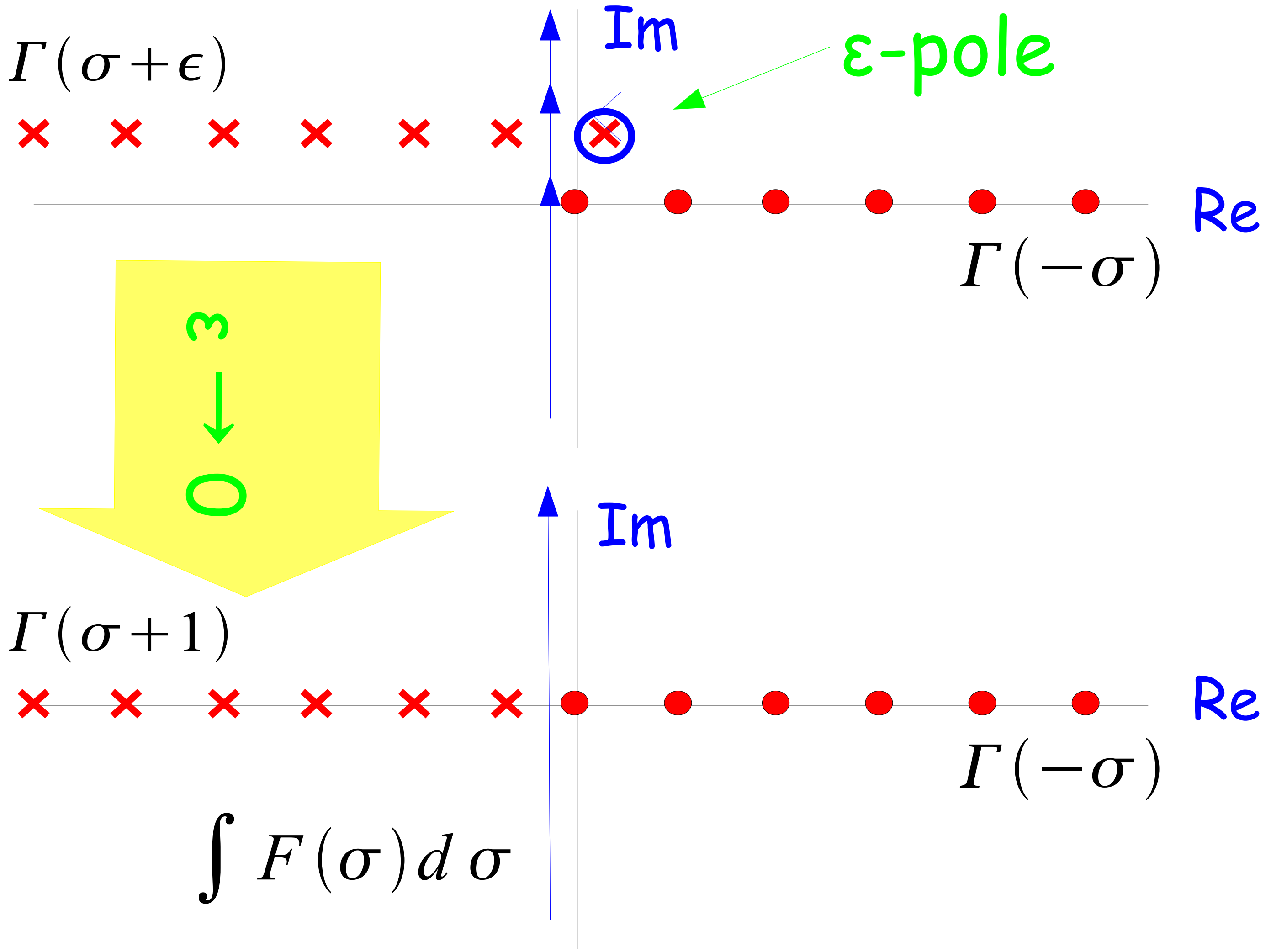
$$\Gamma(z+1) \approx \sqrt{2\pi z}\left(\frac{z}{e}\right)^z \qquad (|\arg z| < \pi, |z| \gg 0)$$

$$\lim_{z\to\infty} \frac{\Gamma(z+1)}{\sqrt{2\pi z}\left(\frac{z}{e}\right)^z} = 1 \qquad (|\arg z| < \pi)$$

ε-pole Separation



ε-pole Separation

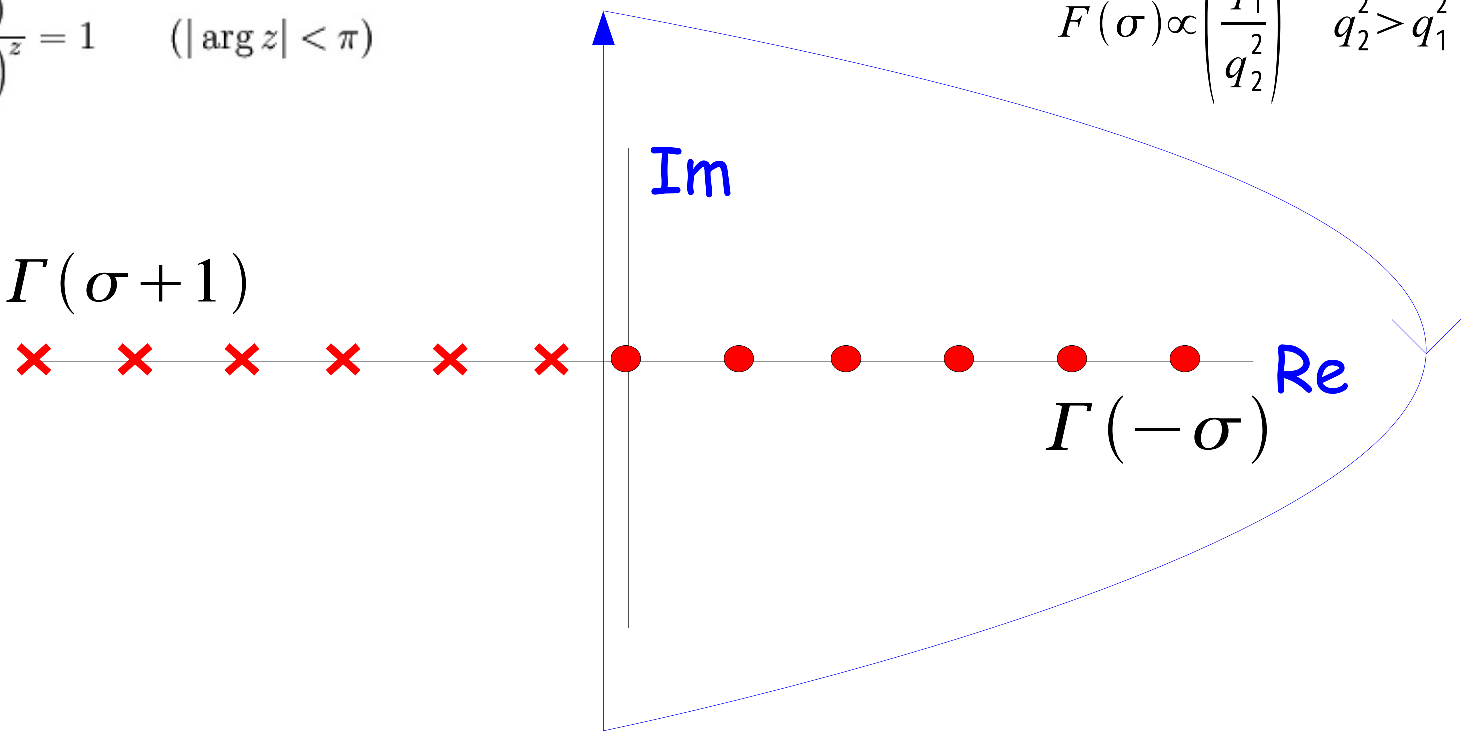


Contour Integration to infinite sum

$$\Gamma(z+1) \approx \sqrt{2\pi z}\left(\frac{z}{e}\right)^z \qquad (|\arg z| < \pi, |z| \gg 0)$$

$$\lim_{z\to\infty} \frac{\Gamma(z+1)}{\sqrt{2\pi z}\left(\frac{z}{e}\right)^z} = 1 \qquad (|\arg z| < \pi)$$

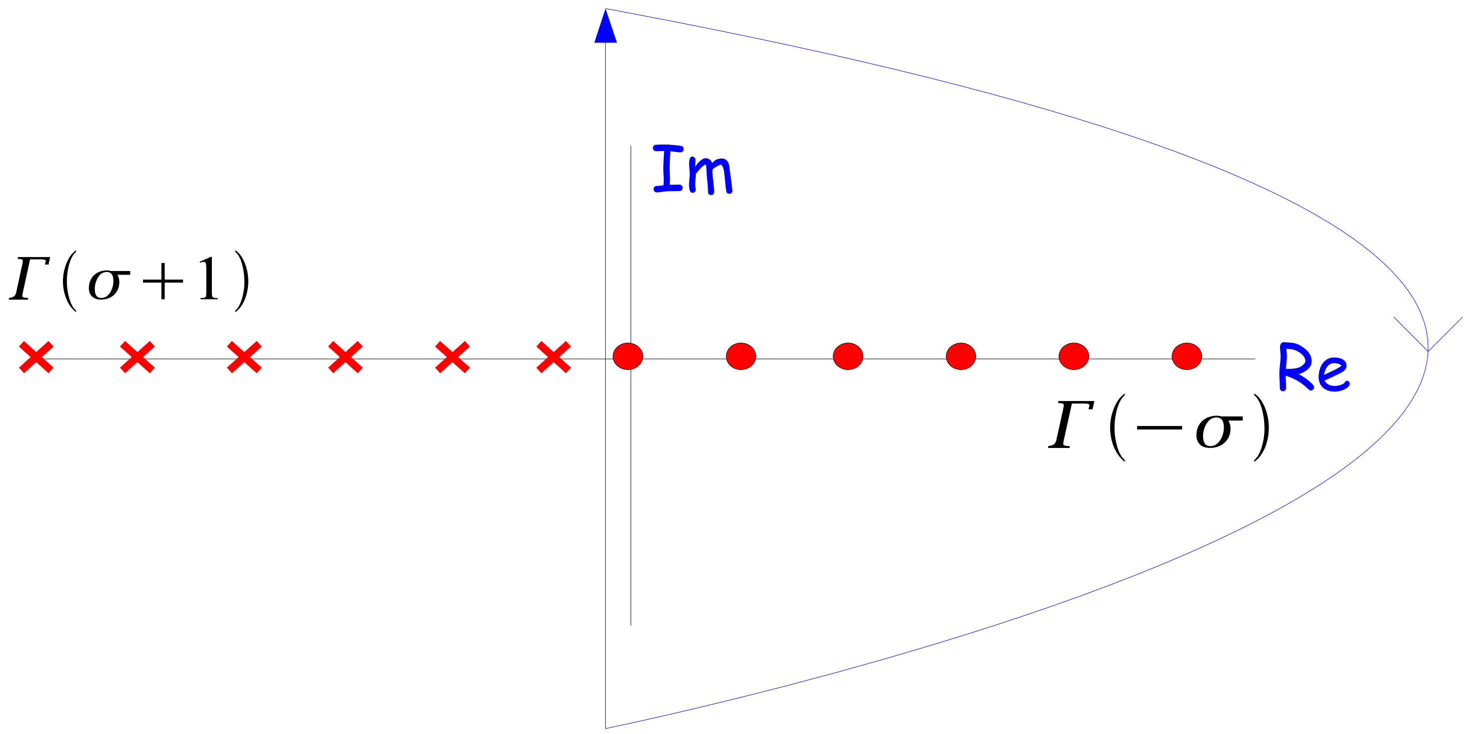
$$F(\sigma) \propto \left(\frac{q_1^2}{q_2^2}\right)^{\sigma} \quad q_2^2 > q_1^2$$



Cauchy's integral formula :

$$\int F(\sigma)\, d\sigma \rightarrow A(q_i^2) \sum_{\sigma \geqslant 0 \in \mathbb{N}} 2\pi i\, Res(\sigma)$$

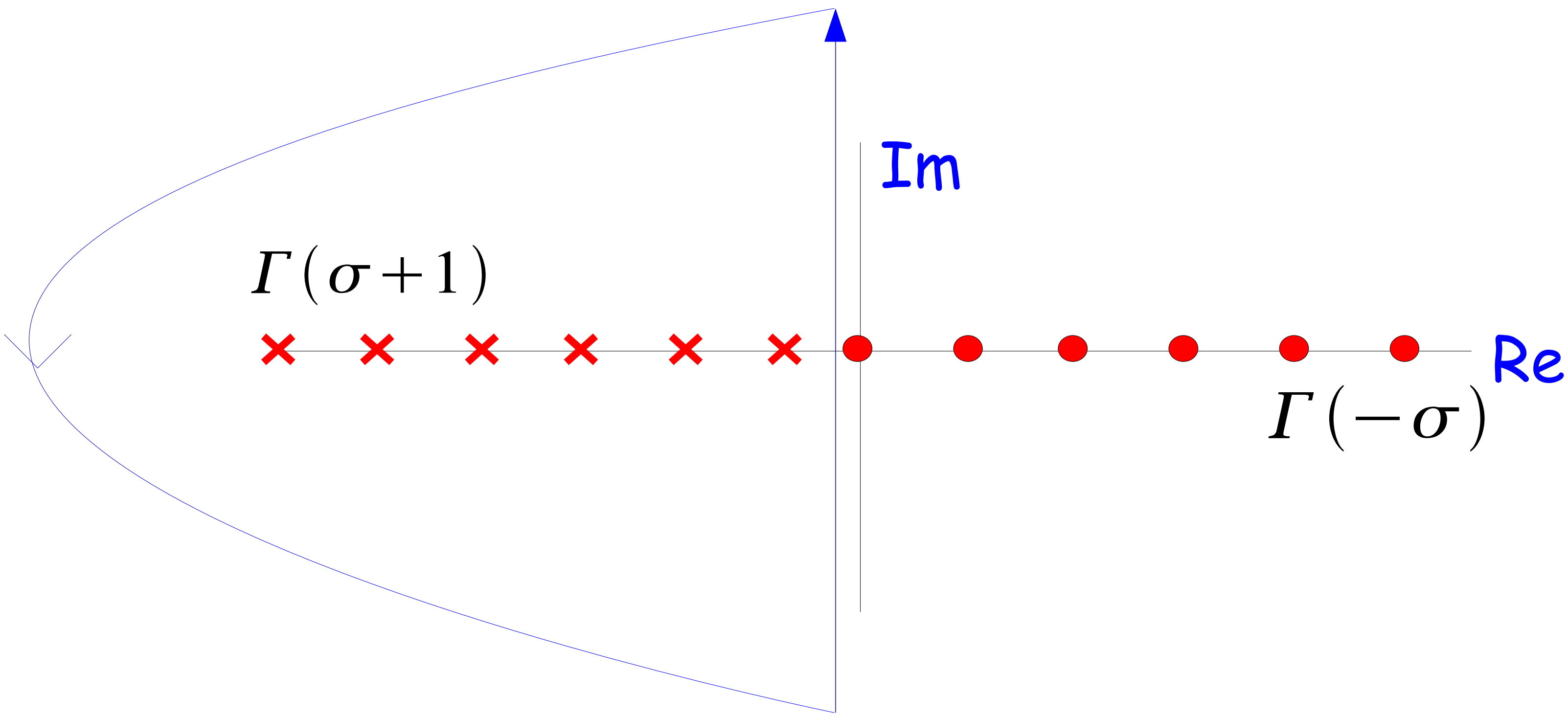
Contour Integration to infinite sum



$$Q^2 > q^2$$

$$I \propto \sum_n Res\left[\left(\frac{q^2}{Q^2}\right)^{\sigma} \Gamma[-\sigma]\Gamma[\sigma+1], -\sigma = -n\right] \to \sum_n \left(\frac{q^2}{Q^2}\right)^n \cdots$$

Contour Integration to infinite sum



$q^2 > Q^2$

$$I \propto \sum_n Res\left[\left(\frac{Q^2}{q^2}\right)^{\sigma} \Gamma[-\sigma]\Gamma[\sigma+1], \sigma+1=-n\right] \to \sum_n \left(\frac{q^2}{Q^2}\right)^n \cdots$$

Contour Integration to infinite sum

- Convergence Series

$$Q^2 > q^2$$

$$I \propto \sum_n Res\left[\left(\frac{q^2}{Q^2}\right)^{\sigma} \Gamma[-\sigma]\Gamma[\sigma+1], -\sigma = -n\right] \rightarrow \sum_n \left(\frac{q^2}{Q^2}\right)^n \cdots$$

$$I \propto \sum_n Res\left[\left(\frac{Q^2}{q^2}\right)^{\sigma} \Gamma[-\sigma]\Gamma[\sigma+1], \sigma+1 = -n\right] \rightarrow \sum_n \left(\frac{q^2}{Q^2}\right)^n \cdots$$

We can expect to get a convergence series.

Infinite sum to known functions/number

- Convert infinite-sum to known functions
  - *Mathematica* for a easy case
  - Generalized Hypergeometric function
    - Appel-Lauricella Type
    - GKZ Type
  - Nested-Sum Algorithm → Multiple Polylog
    - Single summs
    - Double/Multiple summs
  - Direct sums up to appropriate number of terms

Example 1

Massive Vertex integral

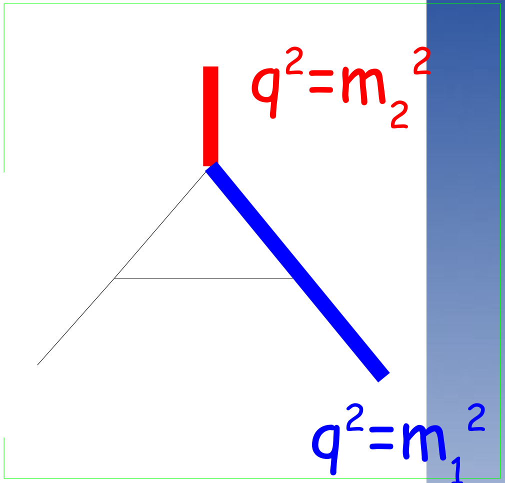
After Fyenmann Integration

$$\left(-i\, 2^{-2\epsilon-5} \pi^{-\epsilon-3} \frac{m12^{\sigma_1} (m12-m22)^{\epsilon-\sigma_1-1} \Gamma(\epsilon-\sigma_1) \Gamma(-\sigma_1) \Gamma(-\epsilon+\sigma_1+1) \Gamma(\epsilon+\sigma_1)}{\Gamma(2\epsilon+1)} \quad \{\sigma_1\} \quad \epsilon\text{-pole} \right)$$

Infinite Summs

$$-i\, 2^{-2\epsilon-5} \pi^{-\epsilon-3} \left\{ \frac{2 i (-1)^{-n1} m12^{-n1+\epsilon-1} (m12-m22)^{n1} \pi \Gamma(n1+1) \Gamma(n1-\epsilon+1) \Gamma(-n1+2\epsilon-1)}{n1!\, \Gamma(2\epsilon+1)}, \{n1, 0, \infty, 1\} \right\}$$

$$-i\, 2^{-2\epsilon-5} \pi^{-\epsilon-3} \left\{ \frac{2 i (-1)^{-n1-1} m12^{-n1-\epsilon-1} (m12-m22)^{n1+2\epsilon} \pi \Gamma(-n1-2\epsilon) \Gamma(n1+\epsilon+1) \Gamma(n1+2\epsilon+1)}{(n1+1)!\, \Gamma(2\epsilon+1)}, \{n1, 0, \infty, 1\} \right\}$$



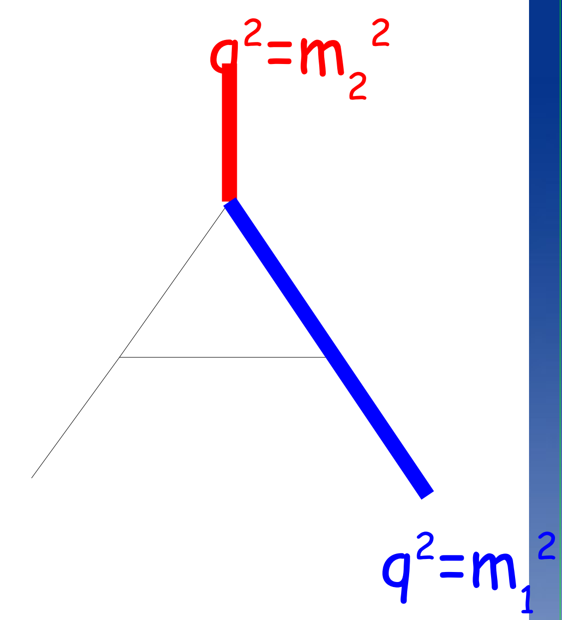
Example 1

Massive Vertex integral

Final result

$$\frac{1}{32\,(\mathrm{m12}-\mathrm{m22})\,\pi^2}\,\frac{1}{\epsilon^2}+-\frac{\mathrm{Log}\left[\frac{\mathrm{m12}}{\mu 2}\right]-2\,\mathrm{Log}\left[\frac{\mathrm{m12}-\mathrm{m22}}{\mu 2}\right]}{32\,(\mathrm{m12}-\mathrm{m22})\,\pi^2}\,\frac{1}{\epsilon}+$$

$$\frac{1}{384\,\pi^2\,(\mathrm{m12}-\mathrm{m22})}\left(-24\,\mathrm{Li}_2\left(1-\frac{\mathrm{m22}}{\mathrm{m12}}\right)+12\log\left(\frac{\mathrm{m22}}{\mathrm{m12}}\right)\left(-2\log\left(\frac{\mathrm{m12}-\mathrm{m22}}{\mu 2}\right)+\log\left(\frac{\mathrm{m12}}{\mu 2}\right)+\log\left(\frac{\mathrm{m22}}{\mu 2}\right)\right)\right.$$
$$\left.+6\left(\log\left(\frac{\mathrm{m12}}{\mu 2}\right)-2\log\left(\frac{\mathrm{m12}-\mathrm{m22}}{\mu 2}\right)\right)^2-12\log^2\left(\frac{\mathrm{m22}}{\mathrm{m12}}\right)+5\,\pi^2\right)$$



Comparison with independent calculation

$$\{1.097454175\times10^{-8},\ 1.3761796561\times10^{-7}+(0.\times10^{-33})\,i,\ 8.725862453\times10^{-7}+(0.\times10^{-32})\,i\}$$

$$\{1.097454175\times10^{-8},\ 1.3761796561\times10^{-7}+(0.\times10^{-33})\,i,\ 8.725862453\times10^{-7}+(0.\times10^{-32})\,i\}$$

Tensor Integral

- Scalar → Tensor is trivial

Scalar Tensor

$$\xi_i^{\epsilon+\sigma+\cdots} \rightarrow \xi_i^{\epsilon+\sigma+\cdots+k}$$

$$\Gamma(n+\epsilon) \rightarrow \Gamma(n+\epsilon+k)$$

Tensor Integral

- Scalar → Tensor is trivial

Scalar | **Tensor**

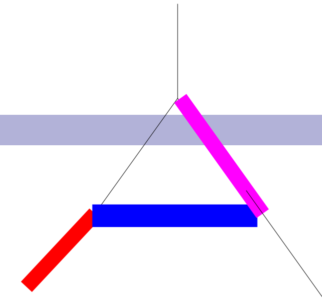
$$\xi_i^{\epsilon+\sigma+\cdots} \to \xi_i^{\epsilon+\sigma+\cdots+k}$$

$$\Gamma(n+\epsilon) \to \Gamma(n+\epsilon+k)$$

IR divergent | IR finite

Example 2

Massive Vertex integral



Final result Tensor Integration : $\xi_1\xi_2$

$$
\frac{1}{16\pi^2 \tilde{m}_{\chi 2}^2 (\tilde{m}_{d2} - \tilde{m}_{\chi 2})}
$$

$$
\Big((\tilde{m}_{d2} - \tilde{m}_{g2})(\tilde{m}_{d2} - \tilde{m}_{\chi 2}) \mathrm{Li}_2\left(\frac{\tilde{m}_{d2}}{\tilde{m}_{g2}}\right) + (\tilde{m}_{g2} - \tilde{m}_{d2})(\tilde{m}_{d2} - \tilde{m}_{\chi 2}) \mathrm{Li}_2\left(\frac{\tilde{m}_{d2} - \tilde{m}_{\chi 2}}{\tilde{m}_{g2}}\right) + \tilde{m}_{d2}^2 \log(\tilde{m}_{g2}) \log\left(\frac{(\tilde{m}_{d2} - \tilde{m}_{\chi 2})(-\tilde{m}_{d2} + \tilde{m}_{g2} + \tilde{m}_{\chi 2})}{\tilde{m}_{g2}}\right) - \tilde{m}_{d2}\tilde{m}_{g2} \log(\tilde{m}_{g2})
$$

$$
\log\big((\tilde{m}_{d2} - \tilde{m}_{\chi 2})(-\tilde{m}_{d2} + \tilde{m}_{g2} + \tilde{m}_{\chi 2})\big) + \tilde{m}_{d2}\tilde{m}_{\chi 2} \log(\tilde{m}_{g2}) \big(-\log\big((\tilde{m}_{d2} - \tilde{m}_{\chi 2})(-\tilde{m}_{d2} + \tilde{m}_{g2} + \tilde{m}_{\chi 2})\big) + \log(\tilde{m}_{g2}) + 1\big) - \tilde{m}_{d2}^2 \log(\tilde{m}_{d2} - \tilde{m}_{\chi 2}) \log(-\tilde{m}_{d2} + \tilde{m}_{g2} + \tilde{m}_{\chi 2}) +
$$

$$
\tilde{m}_{\chi 2}^2 \left(\log(\tilde{m}_{d2} - \tilde{m}_{\chi 2}) + \log\left(\frac{1}{\tilde{m}_{g2}}\right) - 1\right) + \tilde{m}_{d2}\tilde{m}_{g2} \log(\tilde{m}_{d2} - \tilde{m}_{\chi 2}) \log(-\tilde{m}_{d2} + \tilde{m}_{g2} + \tilde{m}_{\chi 2}) + \tilde{m}_{d2}\tilde{m}_{\chi 2} \log\left(1 - \frac{\tilde{m}_{d2}}{\tilde{m}_{g2}}\right) \log\left(\frac{\tilde{m}_{g2}}{\tilde{m}_{d2}}\right) +
$$

$$
\tilde{m}_{d2}\tilde{m}_{\chi 2} \log(\tilde{m}_{d2} - \tilde{m}_{\chi 2}) \big(\log(-\tilde{m}_{d2} + \tilde{m}_{g2} + \tilde{m}_{\chi 2}) + 12\big) + \tilde{m}_{g2}\tilde{m}_{\chi 2} \log\left(1 - \frac{\tilde{m}_{d2}}{\tilde{m}_{g2}}\right) \log\left(\frac{\tilde{m}_{d2}}{\tilde{m}_{g2}}\right) - \tilde{m}_{g2}\tilde{m}_{\chi 2} \log\left(\frac{\tilde{m}_{g2}}{\tilde{m}_{d2} - \tilde{m}_{\chi 2}}\right) \log\left(\frac{\tilde{m}_{g2}}{-\tilde{m}_{d2} + \tilde{m}_{g2} + \tilde{m}_{\chi 2}}\right) + \tilde{m}_{d2}\tilde{m}_{g2} \log^2(\tilde{m}_{g2}) +
$$

$$
\tilde{m}_{d2}^2 \log\left(1 - \frac{\tilde{m}_{d2}}{\tilde{m}_{g2}}\right) \log\left(\frac{\tilde{m}_{d2}}{\tilde{m}_{g2}}\right) + \tilde{m}_{d2}\tilde{m}_{g2} \log\left(1 - \frac{\tilde{m}_{d2}}{\tilde{m}_{g2}}\right) \log\left(\frac{\tilde{m}_{g2}}{\tilde{m}_{d2}}\right) + \tilde{m}_{d2}\tilde{m}_{\chi 2} + \tilde{m}_{d2}^2 \log\left(1 - \frac{\tilde{m}_{\chi 2}}{\tilde{m}_{d2}}\right) - 13\, \tilde{m}_{d2}\tilde{m}_{\chi 2} \log(\tilde{m}_{d2}) - 14\, \tilde{m}_{d2}\tilde{m}_{\chi 2} \log\left(1 - \frac{\tilde{m}_{\chi 2}}{\tilde{m}_{d2}}\right)\Big)
$$

Comparison with numerical integration

$1.03930569358118516 75 \times 10^{-5}$

$1.0393056935813659209 \times 10^{-5}$

-1.73917×10^{-13}

Box Integral

All massless box integral

After Fyenmann Integration

$$\left\{i\, 2^{2\epsilon-5}\pi^{\epsilon-3},\ \frac{s\,(-t)^{\sigma 1}\Gamma(-\sigma 1)\,\Gamma(\sigma 1+1)^2\,(-s)^{-\epsilon-\sigma 1-3}\,\Gamma(-\epsilon-\sigma 1-1)^2\,\Gamma(\epsilon+\sigma 1+2)}{\Gamma(-2\epsilon)},\ \{\sigma 1\}\right\}$$

Separte ε-pole

$$\left\{-\frac{1}{\Gamma(-2\epsilon)}2^{2\epsilon-4}\pi^{\epsilon-2}\,(-s)^{-\epsilon-2}\,\Gamma(-\epsilon)^2\,\Gamma(\epsilon+1)\left(\frac{t}{s}\right)^{-\epsilon-1}\left(\log\left(\frac{t}{s}\right)+\frac{1}{\epsilon}+2\pi\cot(\pi\epsilon)+\psi^{(0)}(\epsilon)+\gamma\right),\ 1,\ \{-\epsilon-\sigma 1-1\}\right\}$$

Contour Integration

$$\frac{1}{4\pi^2 s t}\ \frac{1}{\epsilon^2}+-\frac{\mathrm{Log}\left[\frac{t}{s}\right]+2\,\mathrm{Log}\left[-\frac{s}{\mu 2}\right]}{8\pi^2 s t}\ \frac{1}{\epsilon}+\frac{3\log\left(-\frac{s}{\mu 2}\right)\left(\log\left(\frac{t}{s}\right)+\log\left(-\frac{s}{\mu 2}\right)\right)-2\pi^2}{24\pi^2 s t}$$

Comprison with known result

$$\{-2.533029591\times 10^{-6}+(0.\times 10^{-26})\,i,\ 1.1665032353\times 10^{-5}+(3.978873577\times 10^{-6})\,i,\ -1.181146387\times 10^{-5}-(9.16169499\times 10^{-6})\,i\}$$

$$\{-2.533029591\times 10^{-6}+(0.\times 10^{-15})\,i,\ -1.1665032353\times 10^{-5}-(3.978873577\times 10^{-6})\,i,\ -1.181146387\times 10^{-5}-(9.16169499\times 10^{-6})\,i\}$$

Multiple Summs

- We have to be careful !

$$\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty} f(\sigma_{1,}\sigma_2)\,d\sigma_1\,d\sigma_2$$

$$=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} f_{n_{1,}n_2}$$

Multiple Summs

- We have to be careful !

$$\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty} f(\sigma_1,\sigma_2)\,d\sigma_1\,d\sigma_2$$

$$=\int_{-i\infty}^{+i\infty}\left(\sum_{n_1=0}^{\infty} f_{n_1}(\sigma_2)\right)d\sigma_2$$

$$\overset{?}{=}\sum_{n_1=0}^{\infty}\left(\int_{-i\infty}^{+i\infty} f_{n_1}(\sigma_2)\,d\sigma_2\right)$$

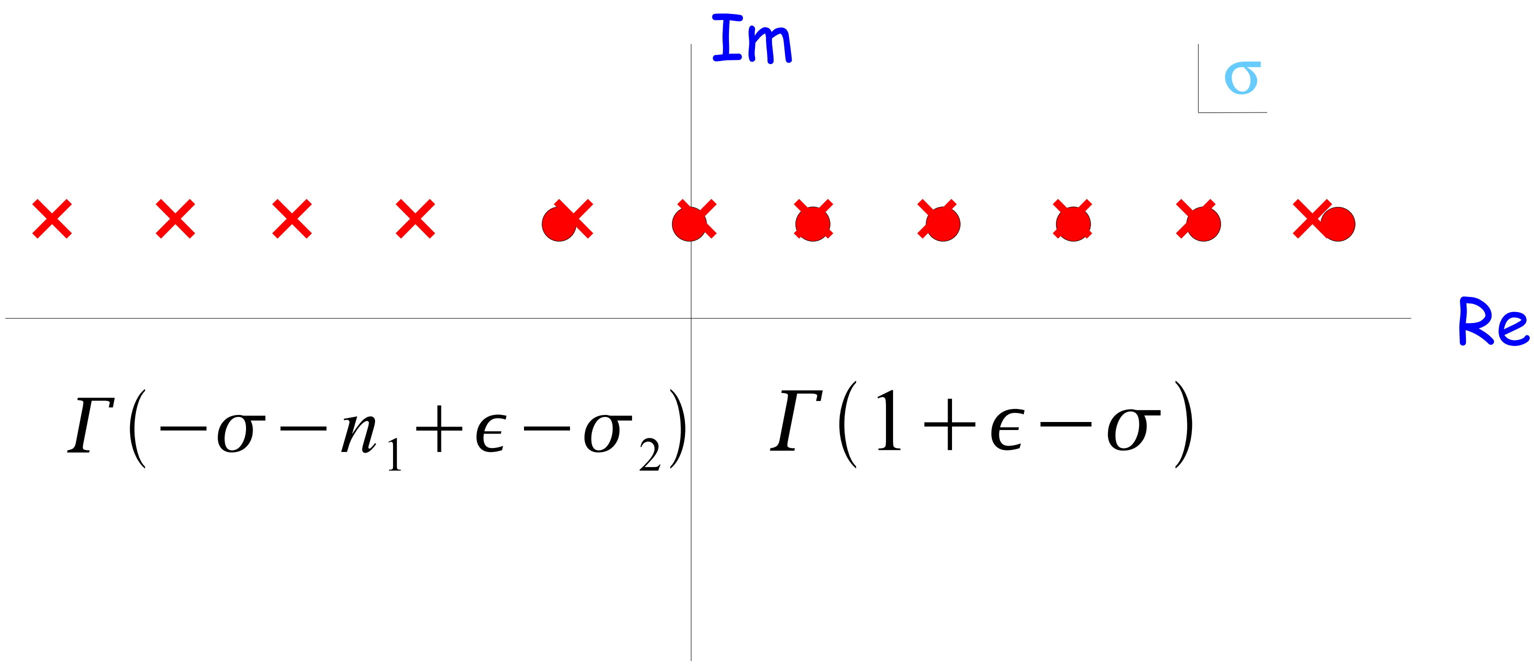
$$=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} f_{n_1,n_2}$$

Multiple Summs

- Example

pinch

$$\Gamma(-\sigma_2)\Gamma(-1+\epsilon-\sigma_2)\Gamma(-1-n_1+\epsilon-\sigma_2)$$
$$\Gamma(1+\sigma_2)\Gamma(1+n_1+\sigma_2)\Gamma(2+n_1-\epsilon+\sigma_2)$$

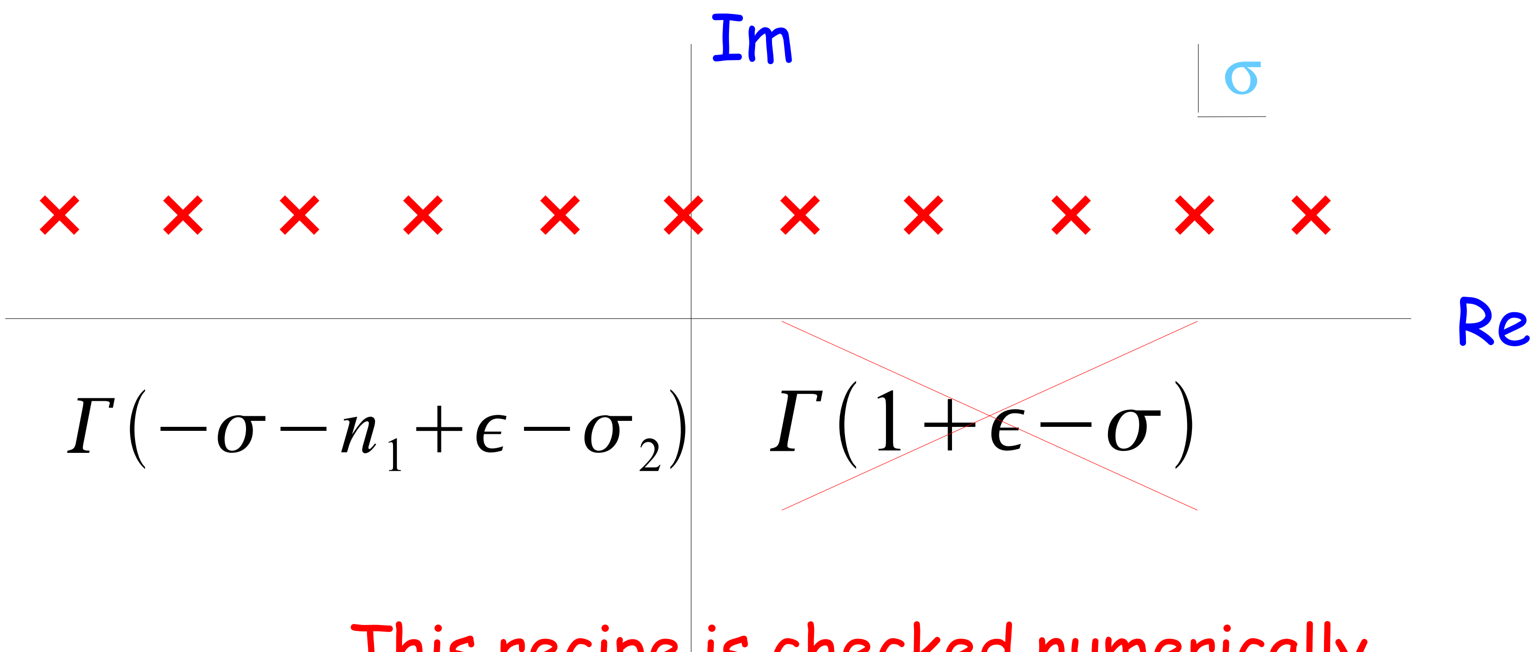


Multiple Summs

- Example

pinch

$$\Gamma(-\sigma_2)\Gamma(-1+\epsilon-\sigma_2)\Gamma(-1-n_1+\epsilon-\sigma_2)$$
$$\Gamma(1+\sigma_2)\Gamma(1+n_1+\sigma_2)\Gamma(2+n_1-\epsilon+\sigma_2)$$



This recipe is checked numerically.

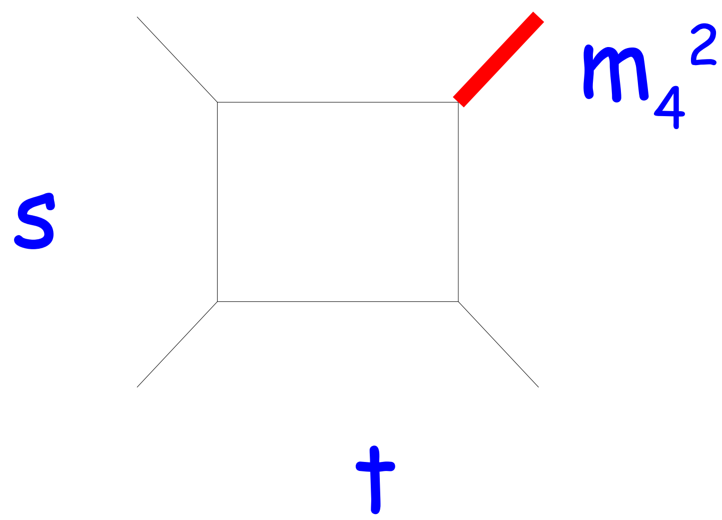
Example 3

All massless box integral (one External mass)

After Fyenmann Integration

$$\Big(-\frac{4^{-\epsilon-3}\,\pi^{-\epsilon-4}}{\Gamma(2\epsilon)}\;(-1)^{\epsilon}\, m42^{\sigma 1}\, s^{\epsilon-\sigma 1-\sigma 2-2}\, t^{\sigma 2}\,\Gamma(-\sigma 1)\,\Gamma(\epsilon-\sigma 2-1)\,\Gamma(\epsilon-\sigma 1-\sigma 2-1)$$

$$\Gamma(-\sigma 2)\,\Gamma(\sigma 2+1)\,\Gamma(\sigma 1+\sigma 2+1)\,\Gamma(-\epsilon+\sigma 1+\sigma 2+2)$$



Example 3

$$
\begin{pmatrix}
\frac{(-1)^{\epsilon} 2^{-2\epsilon-4} \pi^{-\epsilon-2} t^{\epsilon-1} \Gamma(1-\epsilon) \Gamma(\epsilon)^2 (\pi \cot(\pi\epsilon) + \log(m42) - \log(t))}{s \Gamma(2\epsilon)} & 1 & (0 \;\; 0 \;\; 0) \\
-\frac{(-1)^{\epsilon-2n1} 2^{-2\epsilon-4} m42^{n1} \pi^{-\epsilon-2} s^{-n1-1} t^{\epsilon-1} \Gamma(n1+1) \Gamma(1-\epsilon) \Gamma(\epsilon) \Gamma(n1+\epsilon) (-\pi \cot(\pi\epsilon) - \log(s) + \log(t) + \psi^{(0)}(n1+\epsilon) + \gamma)}{(n1!)^2 \Gamma(2\epsilon)} & 1 & (n1 \;\; 0 \;\; \infty) \\
-\frac{(-1)^{\epsilon-n1} 2^{-2\epsilon-4} m42^{n1+\epsilon+1} \pi^{-\epsilon-2} s^{-n1-2} \Gamma(-n1-\epsilon-1) \Gamma(\epsilon) \Gamma(n1+\epsilon+1)}{t \Gamma(2\epsilon)} & 1 & (n1 \;\; 0 \;\; \infty) \\
-\frac{(-1)^{\epsilon} (4\pi)^{-\epsilon-2} \Gamma(\epsilon)}{s \Gamma(2\epsilon)} & \frac{(-1)^{-n2} m42^{n2+1} t^{-n2+\epsilon-2} \Gamma(n2+1) \Gamma(n2-\epsilon+2) \Gamma(-n2+\epsilon-1)}{(n2+1)!} & (n2 \;\; 0 \;\; \infty) \\
\frac{(-1)^{\epsilon} (4\pi)^{-\epsilon-2} \Gamma(\epsilon)}{s \Gamma(2\epsilon)} & \frac{(-1)^{-n2} m42^{n2+\epsilon} t^{-n2-1} \Gamma(n2+1) \Gamma(-n2-\epsilon) \Gamma(n2+\epsilon)}{n2!} & (n2 \;\; 0 \;\; \infty) \\
\frac{(-1)^{\epsilon} (4\pi)^{-\epsilon-2} s^{\epsilon}}{\Gamma(2\epsilon)} & \frac{(-1)^{-n1-n2} m42^{n1} s^{-n1-n2-2} t^{n2} \Gamma(n2+1) \Gamma(n1+n2+1) \Gamma(n1+n2-\epsilon+2) \Gamma(-n2+\epsilon-1) \Gamma(-n1-n2+\epsilon-1)}{n1!\, n2!} & \begin{pmatrix} n1 & 0 & \infty \\ n2 & 0 & \infty \end{pmatrix} \\
\frac{(4\pi)^{-\epsilon-2}}{\Gamma(2\epsilon)} & \frac{(-1)^{-n1-n2+\epsilon} m42^{n1+n2+\epsilon+2} s^{-n1-2} t^{-n2-2} \Gamma(n1+2) \Gamma(n2+2) \Gamma(-n1-n2-\epsilon-2) \Gamma(n1+\epsilon+1) \Gamma(n2+\epsilon+1)}{(n1+1)!\, (n2+1)!} & \begin{pmatrix} n1 & 0 & \infty \\ n2 & 0 & \infty \end{pmatrix} \\
\frac{(4\pi)^{-\epsilon-2} \log(s)}{\Gamma(2\epsilon)} & \frac{(-1)^{-2n1-2n2+\epsilon} m42^{n1} s^{-n1-n2-2} t^{n2+\epsilon} \Gamma(n1+n2+2) \Gamma(-n2-\epsilon) \Gamma(n2+\epsilon+1) \Gamma(n1+n2+\epsilon+1)}{n1!\, (n2+1)!\, (n1+n2+1)!} & \begin{pmatrix} n1 & 0 & \infty \\ n2 & 0 & \infty \end{pmatrix} \\
-\frac{(4\pi)^{-\epsilon-2} \log(t)}{\Gamma(2\epsilon)} & \frac{(-1)^{-2n1-2n2+\epsilon} m42^{n1} s^{-n1-n2-2} t^{n2+\epsilon} \Gamma(n1+n2+2) \Gamma(-n2-\epsilon) \Gamma(n2+\epsilon+1) \Gamma(n1+n2+\epsilon+1)}{n1!\, (n2+1)!\, (n1+n2+1)!} & \begin{pmatrix} n1 & 0 & \infty \\ n2 & 0 & \infty \end{pmatrix} \\
\frac{(4\pi)^{-\epsilon-2}}{\Gamma(2\epsilon)} & \frac{(-1)^{-2n1-2n2+\epsilon} m42^{n1} s^{-n1-n2-2} t^{n2+\epsilon} \Gamma(n1+n2+2) \Gamma(-n2-\epsilon) \Gamma(n2+\epsilon+1) \Gamma(n1+n2+\epsilon+1) \psi^{(0)}(n2+2)}{n1!\, (n2+1)!\, (n1+n2+1)!} & \begin{pmatrix} n1 & 0 & \infty \\ n2 & 0 & \infty \end{pmatrix} \\
\frac{(4\pi)^{-\epsilon-2}}{\Gamma(2\epsilon)} & \frac{(-1)^{-2n1-2n2+\epsilon} m42^{n1} s^{-n1-n2-2} t^{n2+\epsilon} \Gamma(n1+n2+2) \Gamma(-n2-\epsilon) \Gamma(n2+\epsilon+1) \Gamma(n1+n2+\epsilon+1) \psi^{(0)}(-n2-\epsilon)}{n1!\, (n2+1)!\, (n1+n2+1)!} & \begin{pmatrix} n1 & 0 & \infty \\ n2 & 0 & \infty \end{pmatrix} \\
-\frac{(4\pi)^{-\epsilon-2}}{\Gamma(2\epsilon)} & \frac{(-1)^{-2n1-2n2+\epsilon} m42^{n1} s^{-n1-n2-2} t^{n2+\epsilon} \Gamma(n1+n2+2) \Gamma(-n2-\epsilon) \Gamma(n2+\epsilon+1) \Gamma(n1+n2+\epsilon+1) \psi^{(0)}(n2+\epsilon+1)}{n1!\, (n2+1)!\, (n1+n2+1)!} & \begin{pmatrix} n1 & 0 & \infty \\ n2 & 0 & \infty \end{pmatrix} \\
-\frac{(4\pi)^{-\epsilon-2}}{\Gamma(2\epsilon)} & \frac{(-1)^{-2n1-2n2+\epsilon} m42^{n1} s^{-n1-n2-2} t^{n2+\epsilon} \Gamma(n1+n2+2) \Gamma(-n2-\epsilon) \Gamma(n2+\epsilon+1) \Gamma(n1+n2+\epsilon+1) \psi^{(0)}(n1+n2+\epsilon+1)}{n1!\, (n2+1)!\, (n1+n2+1)!} & \begin{pmatrix} n1 & 0 & \infty \\ n2 & 0 & \infty \end{pmatrix}
\end{pmatrix}
$$

Infinite Summs to MPL

S. Moch, P. Uwer, S. Weimzierl, *J. Math. Phys.* **43**, 3363 (2002)

- Algorithms (Nested Summs Algorithm)
  - Z-sum

    Z[n,{},{}]={1 for n≥1, 0 for n<0}

    $Z[n,\{m_1,\ldots,m_k\},\{x_1,\ldots,x_k\}]=\Sigma_{i=1}^{n}\,(x_1/i^{m1})Z[i-1,\{m_2,\ldots,m_k\},\{x_2,\ldots,x_k\}]$
  - MPL

    $L_i[\{m_k,\ldots,m_1\},\{x_k,\ldots,x_1\}]=Z[\infty,\{m_1,\ldots,m_k\},\{x_1,\ldots,x_k\}]$
  - Series expansion of Γ-function w.r.t. ε

    $\Gamma[\pm n+\varepsilon]=\Sigma_{i=(-1),0,1,\ldots}\,\varepsilon^i f_i(Z[n-1,\{1,\ldots1\},\{1,\ldots1\}])$
  - $Z \times Z \rightarrow Z$

Multiple Polylog

- Iterative Integration Representation of MPL

$$\beta_j = \frac{1}{x_1 \cdots x_j}$$

$$\omega_0 = \frac{d\xi}{\xi}, \qquad \omega_j = \frac{d\xi}{\xi - \beta_j}$$

$$L_i\left[\{m_k, \cdots, m_1\}, \{x_k, \cdots, x_1\}\right] = \prod_{j=k}^{1} \int_\gamma \omega_j \omega_0^{m_j - 1}$$

$$L_i\left[\{m_k, \cdots, m_1\}, \{x_k, \cdots, x_1\}\right] = \int_\gamma L_i\left[\{m_k, \cdots, m_1 - 1\}, \{x_k, \cdots, \xi\}\right] \omega_0$$

$$L_i\left[\{m_k, \cdots, 0\}, \{x_k, \cdots, x_1\}\right] = \left(\xi \partial_\xi L_i\left[\{m_k, \cdots, 1\}, \{x_k, \cdots, \xi\}\right]\right)_{\xi = x_1}$$

Infinite Sum to MPL

- Additional algorithm
  - Li[{1,…,1,0},{1,…,1,ξ}]

    $\Sigma_{j=1}^{n} (q^2/Q^2)^j$ Z[j-1,{1,…,1},{1,…,1}]

    =Z[n,{0,1,…,1},{q^2/Q^2,1,…,1}]

    → Li[{1,…,1,0},{1,…,1,q^2/Q^2}] → Integration Form
  - PolyGamma : ψ[n]

    ψ[n] = Γ'[n]/Γ[n]

    = Z[n-1,{1},{1}]-γ_E

    ψ[n]' =-Z[n-1,{2},{1}]+$\pi^2/6$

    ψ[n]''= …

All tools are checked numerically.

Summary

- Open Questions
  - How can we get the most compact expressions for MB transformation?

    Hint! on-shell condition
  - Multiple Summs → MPL

    Direct sum may work
  - Multiple Γ-series → Hypergeometric functions

- Future
  - Multi-Loop
  - Multi-Leg