Beam Physics Fundamentals

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Introduction

❑ Accelerator Physics

Special relativity → Beam dynamics

Electromagnetism → Beam dynamics

Classical mechanics → Beam dynamics

Beam dynamics → Longitudinal beam dynamics

Beam dynamics → Transverse beam dynamics

RF design | Non-linear dynamics | Magnet design

Optical design | Collective effects | Operations

+ many others

Lorentz force

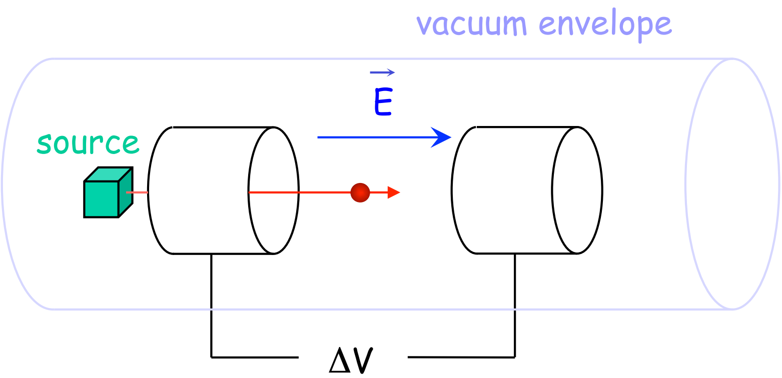
- A charged particle moving with velocity ***v*** through an electro-magnetic field experiences a force

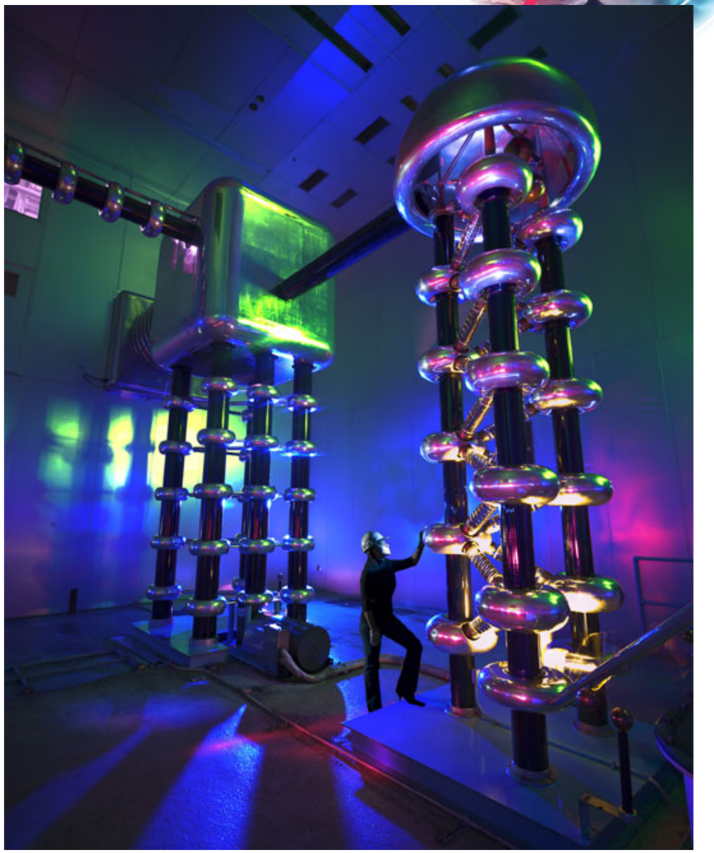
$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

- The second term is always perpendicular to the direction of motion, so it does not give any longitudinal acceleration and it does not increase the energy of the particle.

- Acceleration has to be done by an electric field E

Electrostatic acceleration





750 kV Cockroft-Walton generator at Fermilab (Proton source)

Electrostatic Field:

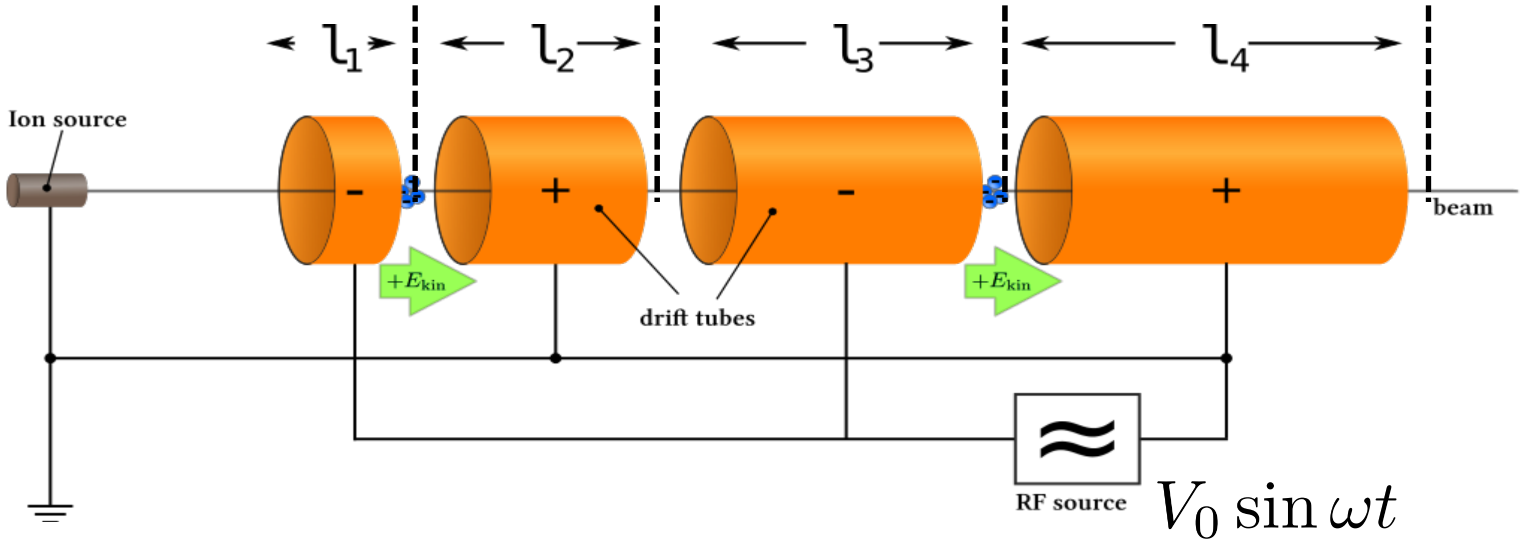
Energy gain: $W = e\,\Delta V$

Limitation: insulation problems
maximum high voltage (~ 10 MV)

Used for first stage of acceleration: particle sources, electron guns, x-ray tubes

Radio-Frequency (RF)

❑ Ising-Wideröe type structure



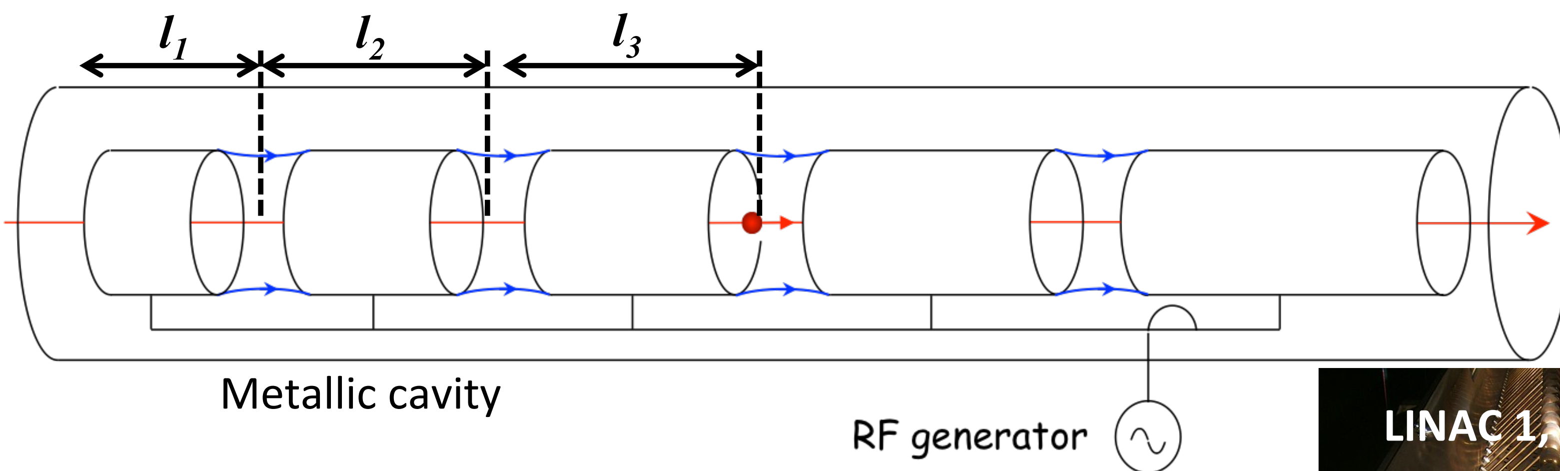
Cylindrical electrodes (drift tubes) separated by gaps and fed by a RF generator, as shown above, lead to an alternating electric field polarity

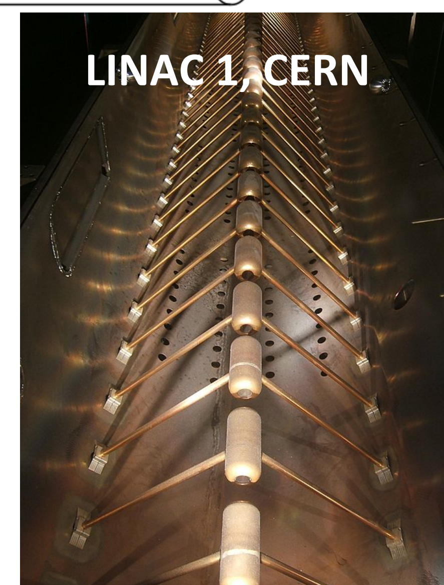
Synchronism condition $\boxed{l_i = \frac{v_i T_{RF}}{2} = \frac{\beta_i \lambda_{RF}}{2}}$

v_i = particle velocity
T_{RF} = RF period

Radio-Frequency (RF)

❑ Alvarez type structure





Synchronism condition

$$\boxed{l_i = v_i T_{RF} = \beta_i \lambda_{RF}}$$

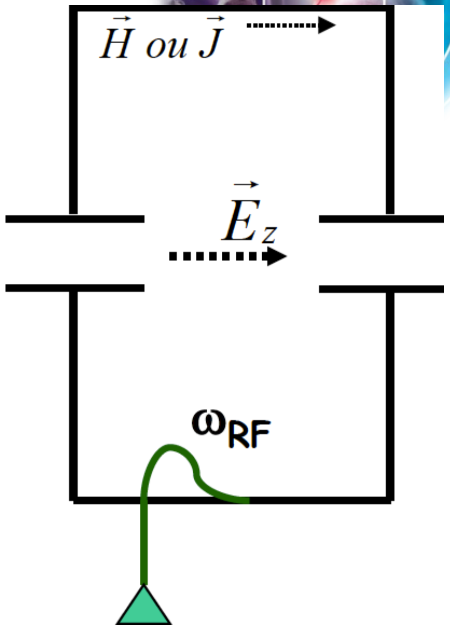
$$\omega_{RF} = 2\pi \frac{v_i}{l_i}$$

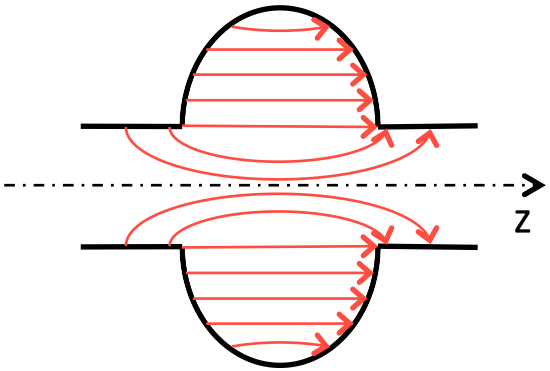
Used for proton and ions (50 – 200 MeV), $f_{RF} \sim 200$ MHz

Radio-Frequency (RF)

❑ Resonant Cavities

- Higher operating frequency.
- The power lost by radiation, due to circulating currents on the electrodes, is proportional to the RF frequency.
- Solution: enclosing the system in a cavity whose resonant frequency matches the RF generator frequency.
- Electromagnetic power constrained in the resonant volume
- Each such cavity can be independently powered from the RF generator
- Note however that joule losses will occur in the cavity walls (unless made of superconducting materials)

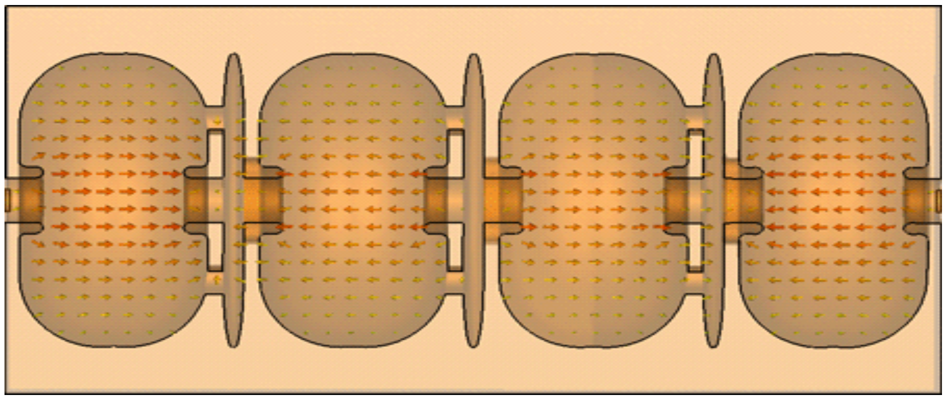




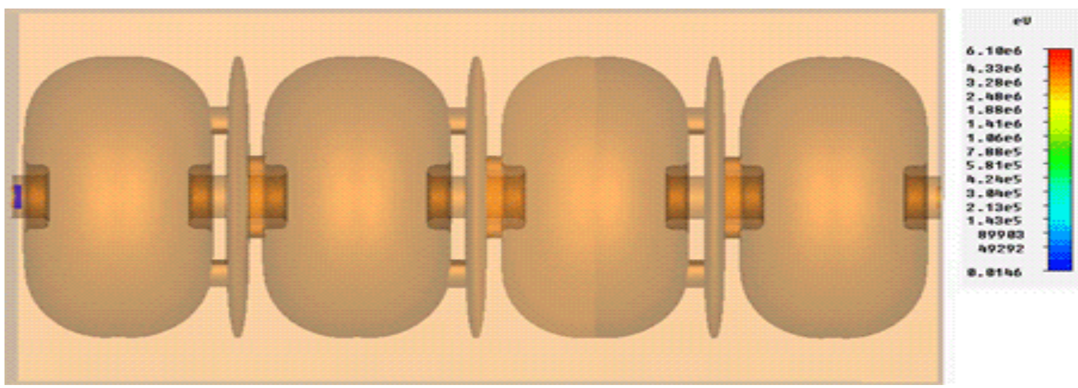
Radio-Frequency (RF)

❑ Resonant Cavities

Standing wave linear accelerator



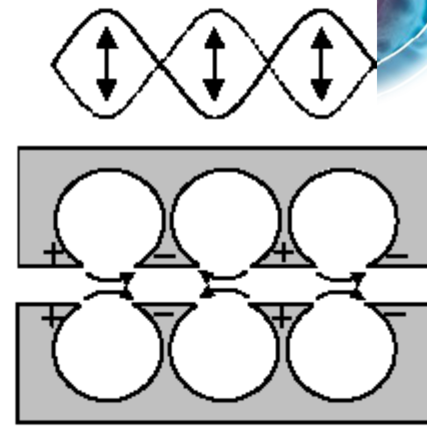
π/2 - mode of the E010 field pattern



Field gradient: 20 MeV/m

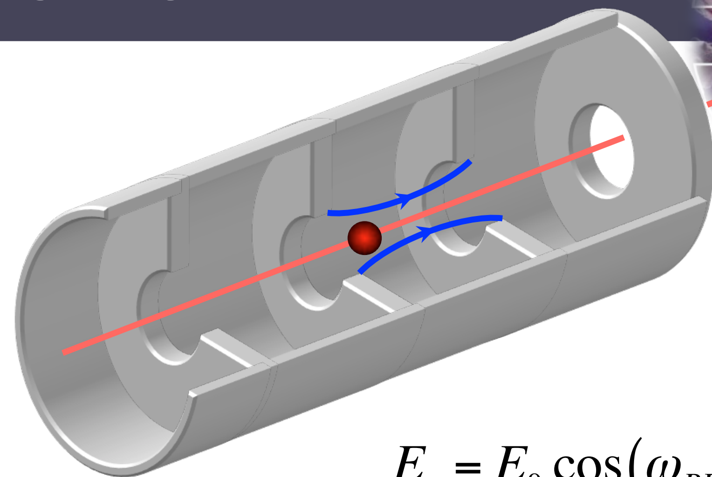
6 GeV

60 keV



CST® simulation examples: https//www.cst.com/Applications

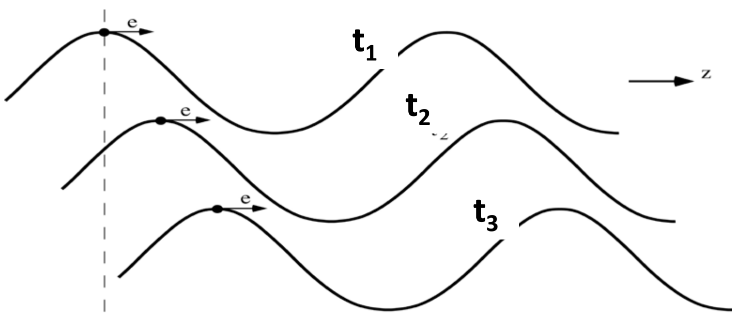
Radio-Frequency (RF)



❑ Resonant Cavities

Traveling wave

(ultra-relativistic particles)



The particle travels along with the wave, and k represents the wave propagation factor.

$$E_z = E_0 \cos\left(\omega_{RF} t - kz\right)$$

$$k = \frac{\omega_{RF}}{v_\varphi} \quad \text{wave number}$$

$$z = v(t - t_0)$$

v_φ = phase velocity

v = particle velocity

$$E_z = E_0 \cos\left(\omega_{RF} t - \omega_{RF} \frac{v}{v_\varphi} t - \phi_0\right)$$

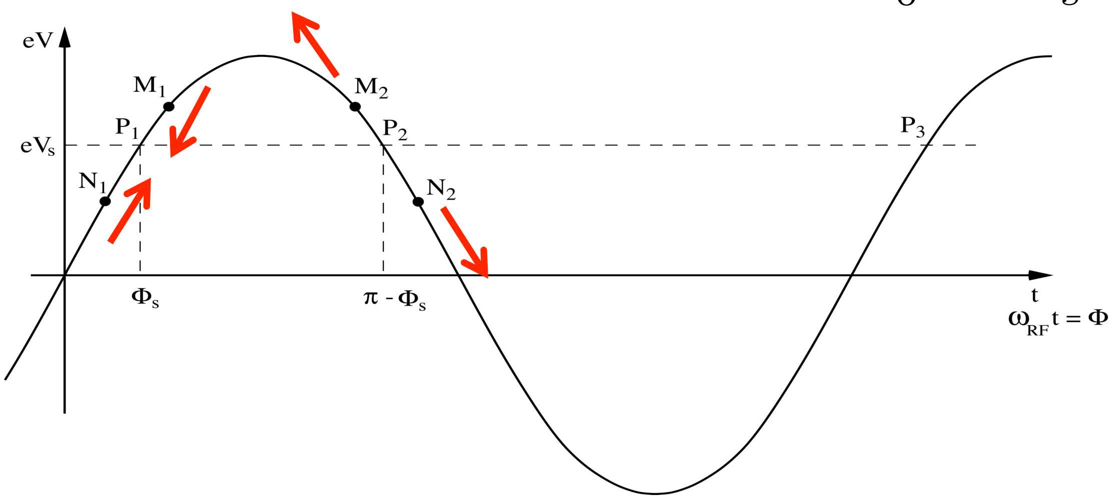
Synchronism: $\boxed{v = v_\varphi \quad \text{and} \quad E_z = E_0 \cos\phi_0}$

Φ_0: RF phase seen by the particle.

Synchronous phase

❑ Phase stability (linac)

- Cavity set up so that particle at the centre of bunch (synchronous particle) acquires just the right amount of energy $\Delta E = eV_0 \sin \Phi_s$



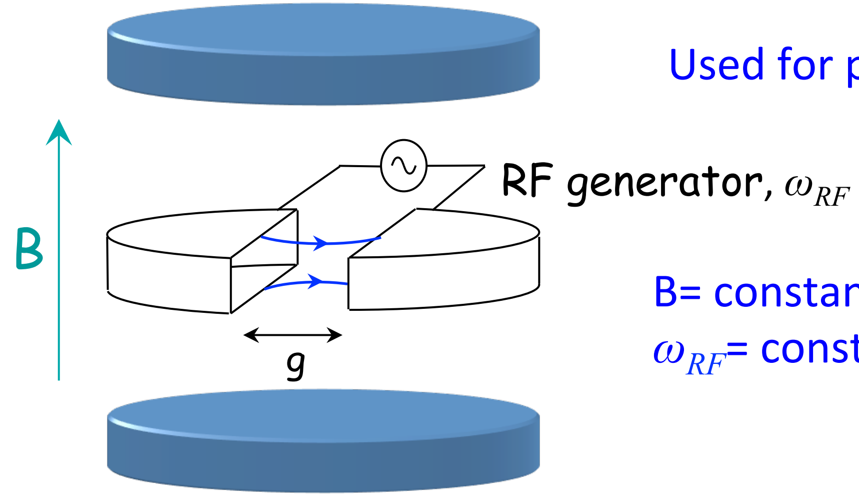
Stable phase region

$$0 < \Phi_s < \frac{\pi}{2}$$

- Particles arriving early (N_1) see $\Phi < \Phi_s$ and will gain less energy. In the next gap it will appear closer to particle P_1 (synchronous particle)

- Particles arriving late (M_1) see $\Phi > \Phi_s$ and reduce its delay compared to P_1

Cyclotron



Used for protons, ions

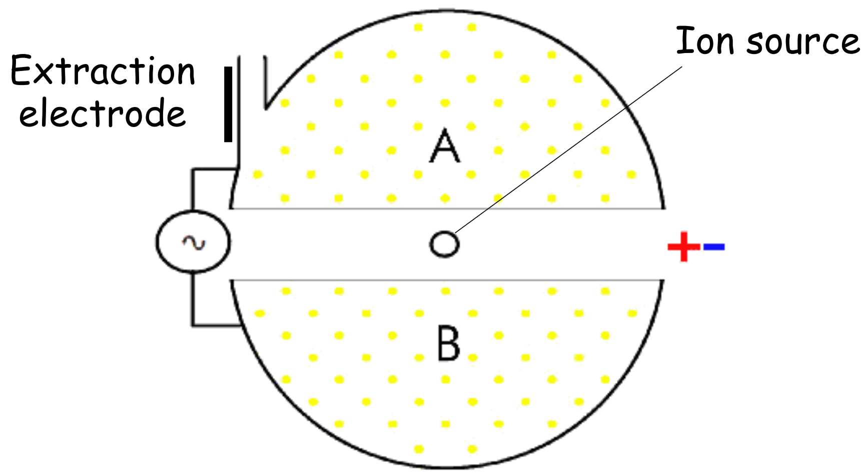
B= constant

ω_{RF}= constant

Synchronism condition

$$\boxed{\begin{aligned}\omega_s &= \omega_{RF} \\ 2\pi\,\rho &= v_s\,T_{RF}\end{aligned}}$$

$$\omega_s = \frac{qB}{m_0\gamma}$$



As long as $v_s \ll c$ and $\gamma \approx 1$ the synchronism conditions stays fulfilled, and the revolution frequency does not depend on the radius

For higher energies … ?

Synchrocyclotron

In order to keep the synchronism at higher energies, one has to decrease ω_{RF} during the acceleration cycle according to the relativistic $\Upsilon(t)$

$$\boxed{\omega_{RF}(t) = \omega_s(t) = \frac{qB}{m_0\gamma(t)}}$$

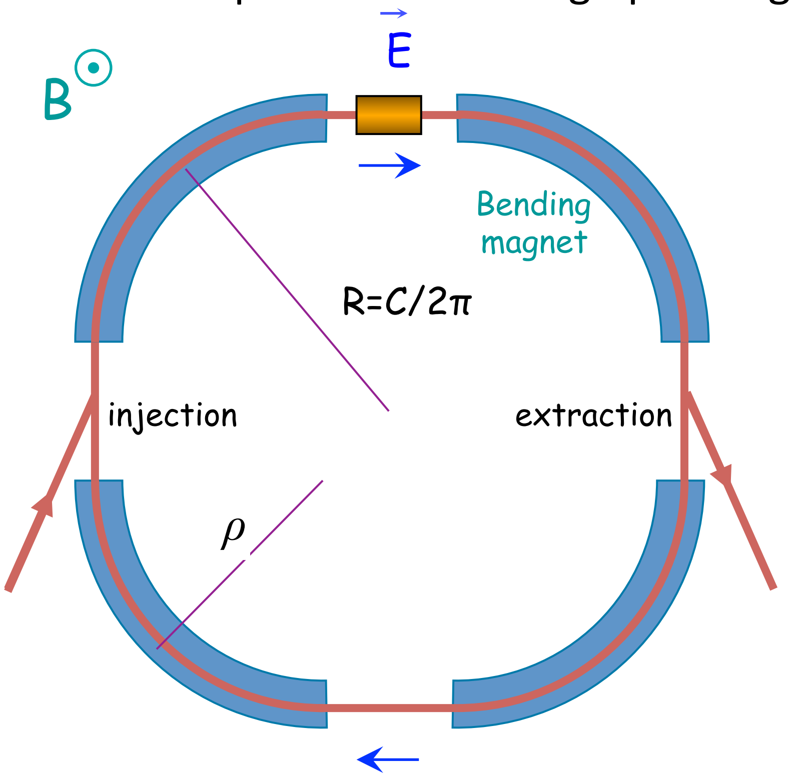
It can accelerate protons up to around 500 MeV.

Limitation due to the size of the magnet

Synchrotron

Synchronous accelerator since there is a synchronous RF phase for which the energy gain fits the increase of the magnetic field at each turn.

That implies the following operating conditions:



$e\hat{V}\sin\Phi \longrightarrow$ **Energy gain per turn**

$\Phi = \Phi_s = cte \longrightarrow$ **Synchronous particle**

$\omega_{RF} = h\omega_r \longrightarrow$ **RF synchronism (h - harmonic number)**

$\rho = cte \quad R = cte \longrightarrow$ **Constant orbit**

$B\rho = P/e \Rightarrow B \longrightarrow$ **Variable magnetic field**

If $v \approx c$, ω_r hence ω_{RF} remain constant (ultra-relativistic regime)

Recap.

- Remember the Lorentz force

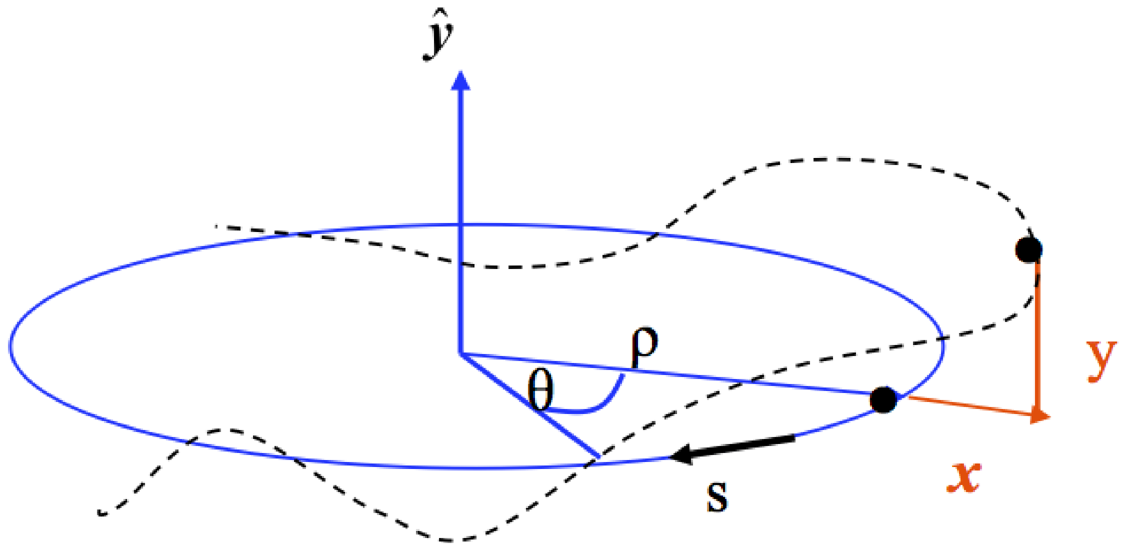
$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

  - We have used E to accelerate particles
  - Now we shall use B to deflect particles as we really benefit from the presence of the velocity in the Lorentz force

Coordinate system

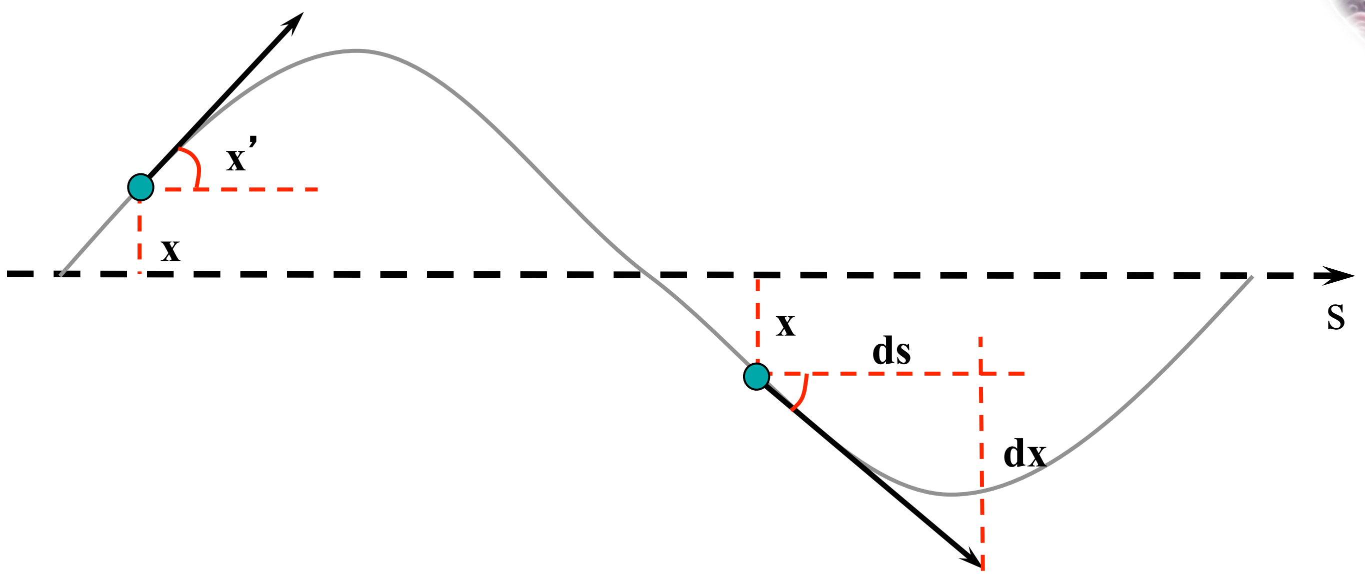
Coordinates w.r.t. the design orbit

Let's consider a local segment of one particle's trajectory



Paraxial approximation: we will assume the deviation of particle coordinates from the design orbit is small , so x, y << bending radius (ρ)

Coordinate system



The state of a particle (phase space) represented with a **6-D vector**

$$(x, x', y, y', z = s - \beta ct, \delta = \Delta p / p_0)$$

with

$$x' = \frac{dx}{ds} = \frac{dxdt}{dtds} = \frac{p_x}{p_z}; \quad y' = \frac{dy}{ds} = \frac{dydt}{dtds} = \frac{p_y}{p_z}$$

Magnetic field

❑ Taylor expansion:

$$B_y(x) = B_{y0} + \frac{dB_y}{dx}x + \frac{1}{2}\frac{d^2B_y}{dx^2}x^2 + \frac{1}{3!}\frac{d^3B_y}{dx^3}x^3 + \ldots$$

$$\frac{e}{p}B_y(x) = \boxed{\frac{1}{\rho} + kx} + \frac{1}{2}k_2x^2 + \frac{1}{3!}k_3x^3 + \ldots$$

- Paraxial approximation: taking into account linear terms in the deviation of the field ***B*** w.r.t. x and y:

  - **Dipole fields, $1/\rho$**
  - **Quadrupole fields, k**

- It is more practical to use "separate function" magnets, rather than combined ones: optimise them regarding their function: Bending, focusing, etc.

Bending

❑ Dipole

If we want to deflect particles

$$F = qvB$$

We equate this to the centripetal force

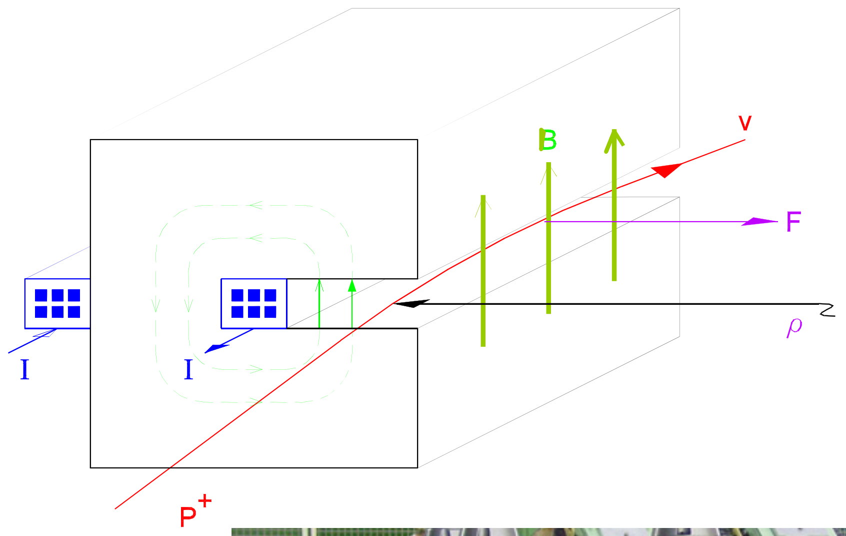
$$F = \frac{\gamma m v^2}{\rho}$$

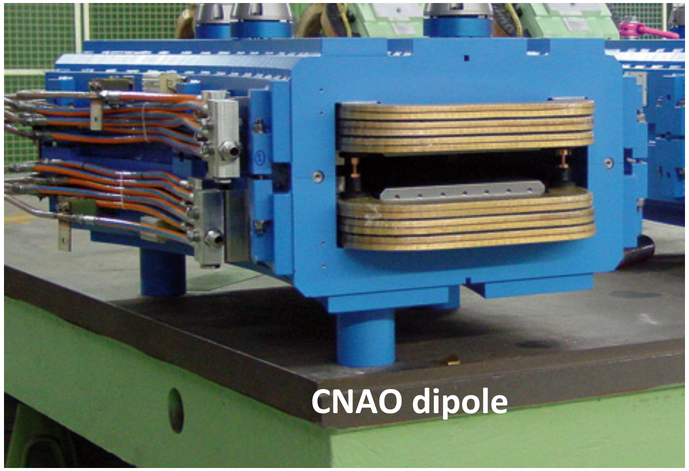
$$\boxed{\frac{1}{\rho}[\mathrm{m}^{-1}] = 0.2998 \frac{B[\mathrm{T}]}{p[\mathrm{GeV}/c]}}$$

(for q=e)

Rigidity: $B\rho = \dfrac{p}{q}$

$$\boxed{B\rho[\mathrm{T} \cdot m] = 3.3356 \cdot p[\mathrm{GeV}/c]}$$





Focusing/defocusing

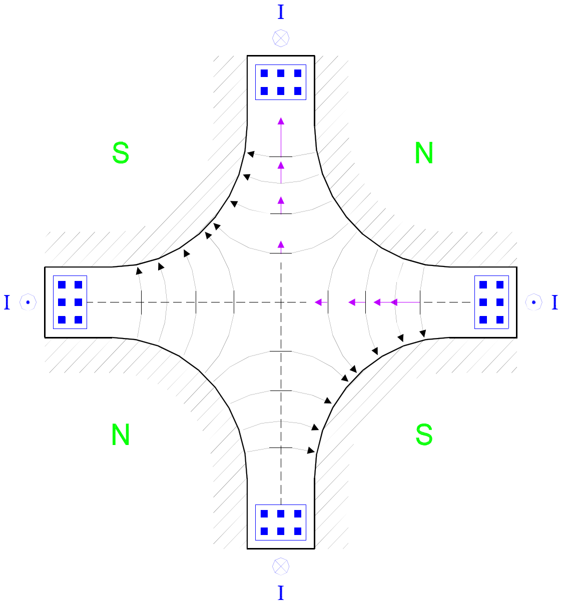
❑ Quadrupole

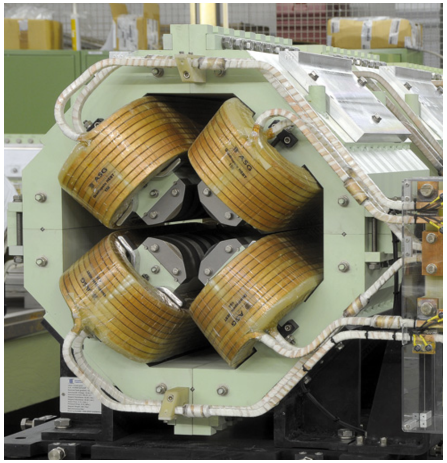
- Constant gradient $g = -\frac{dB_y}{dx}$

- Focusing forces increase linearly with displacement

- Magnetic lenses are focusing in one plane but are defocusing in the orthogonal plane

$$\boxed{k[\mathrm{m}^{-2}] = 0.2998\frac{g[\mathrm{T/m}]}{p[\mathrm{GeV}/c]}}$$





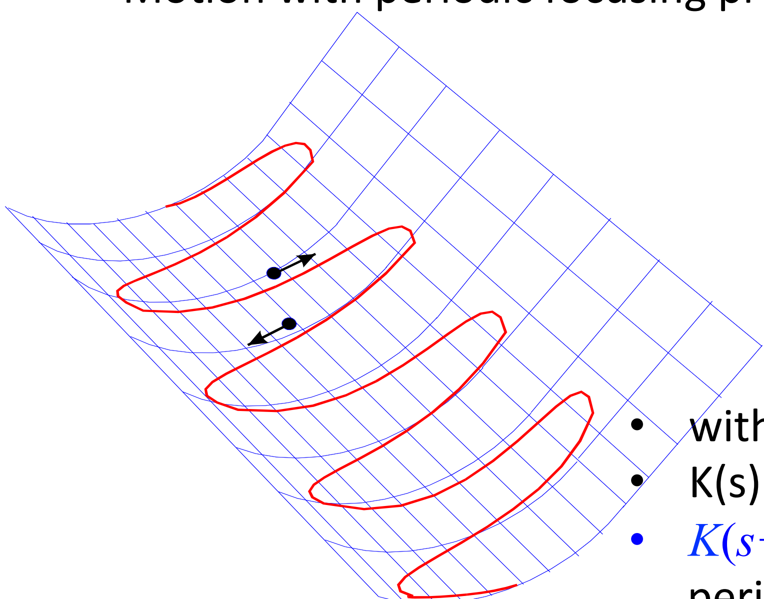
CNAO quadrupole

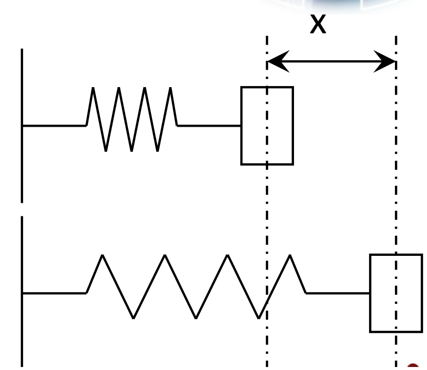
Hill's equation

- Basic equation of motion for a particle in an accelerator

$$\boxed{x''(s) + K(s)x(s) = 0}$$

Motion with periodic focusing properties





Compare with a simple harmonic oscillator with restoring force F=-kx

- with a restoring force $\neq$ const
- K(s) depends on longitudinal position s
- $K(s+L)=K(s)$ periodic function, where L is the "lattice period"

Focusing quadrupole

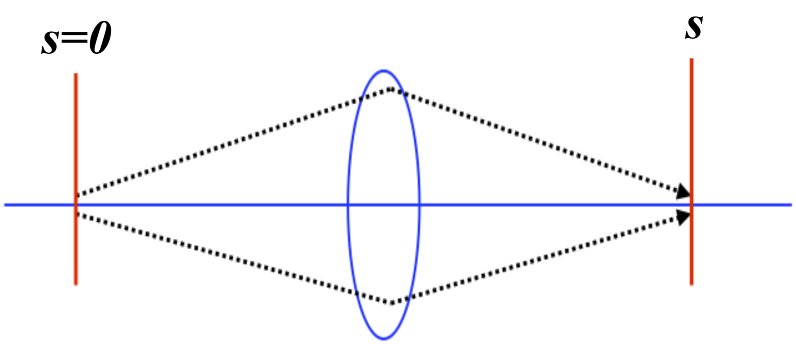
- For similarity with harmonic oscillator, considering K=const.

- Horizontal focusing quadrupole (defocusing in vertical plane), $\boldsymbol{K > 0}$**:**

$$\boxed{\begin{aligned} x(s) &= x(0)\cos(\sqrt{K}s) + \frac{x'(0)}{\sqrt{K}}\sin(\sqrt{K}s) \\ x'(s) &= -x(0)\sqrt{K}\sin(\sqrt{K}s) + x'(0)\cos(\sqrt{K}s) \end{aligned}}$$

- For convenience we can use a **matrix formalism:**

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{K}s) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}s) \\ -\sqrt{K}\sin(\sqrt{K}s) & \cos(\sqrt{K}s) \end{pmatrix} \begin{pmatrix} x(0) \\ x'(0) \end{pmatrix}$$



Defocusing quadrupole

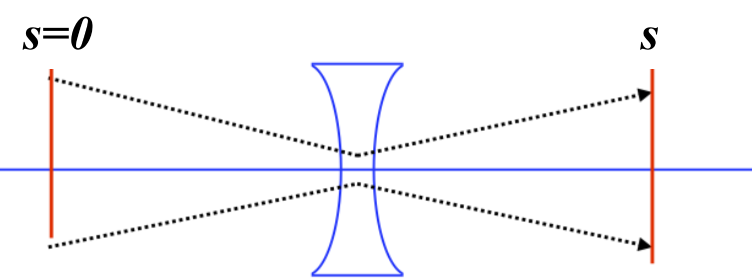
- For similarity with harmonic oscillator, considering K=const.

- Horizontal defocusing quadrupole (focusing in vertical plane), $\boldsymbol{K < 0}$**:**

$$\boxed{\begin{aligned} x(s) &= x(0)\cosh(\sqrt{|K|}s) + \frac{x'(0)}{\sqrt{|K|}}\sinh(\sqrt{|K|}s) \\ x'(s) &= x(0)\sqrt{|K|}\sinh(\sqrt{|K|}s) + x'(0)\cosh(\sqrt{|K|}s) \end{aligned}}$$

- **Matrix formalism:**

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} \cosh(\sqrt{|K|}s) & \frac{1}{\sqrt{|K|}}\sinh(\sqrt{|K|}s) \\ \sqrt{|K|}\sinh(\sqrt{|K|}s) & \cosh(\sqrt{|K|}s) \end{pmatrix} \begin{pmatrix} x(0) \\ x'(0) \end{pmatrix}$$



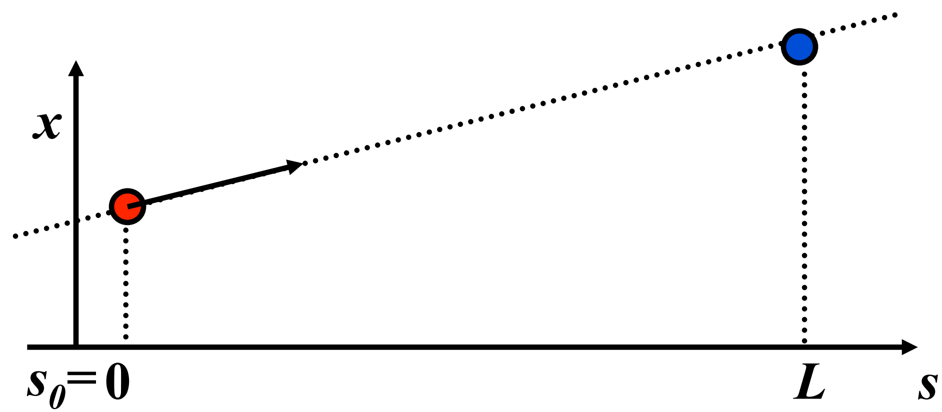
Drift

- For $K=0$, and no further magnetic elements, we have a drift space:

$$x(s) = x(0) + (s - s_0)x'(0) = x(0) + Lx'(0)$$
$$x'(s) = x'(0)$$

- Position changes if particle has a slope which remains unchanged.

$$M_{\mathrm{drift}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$



Thin lens approximation

- For a focusing quadrupole (K>0)

$$M_{\mathrm{QF}} = \begin{pmatrix} \cos(\sqrt{K}L) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}L) \\ -\sqrt{K}\sin(\sqrt{K}L) & \cos(\sqrt{K}L) \end{pmatrix}$$

- For a defocusing quadrupole (K<0)

$$M_{\mathrm{QD}} = \begin{pmatrix} \cosh(\sqrt{|K|}L) & \frac{1}{\sqrt{K}}\sinh(\sqrt{|K|}L) \\ \sqrt{|K|}\sinh(\sqrt{|K|}L) & \cosh(\sqrt{|K|}L) \end{pmatrix}$$

- In the limit $L \to 0$, and KL=const.

$$M_{\mathrm{QF,QD}} = \begin{pmatrix} 1 & 0 \\ -KL & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$

Focal length

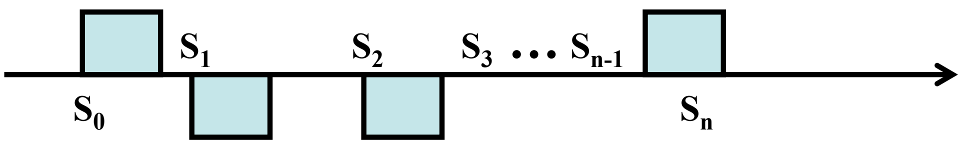
$$f = \frac{1}{KL}$$

- **Note** that the **sign** of K or f is now absorbed inside the symbol
- In the other plane, focusing becomes defocusing and vice versa

Joining elements

For an arbitrary number of transport elements, each with a constant, but different, K_n, we have

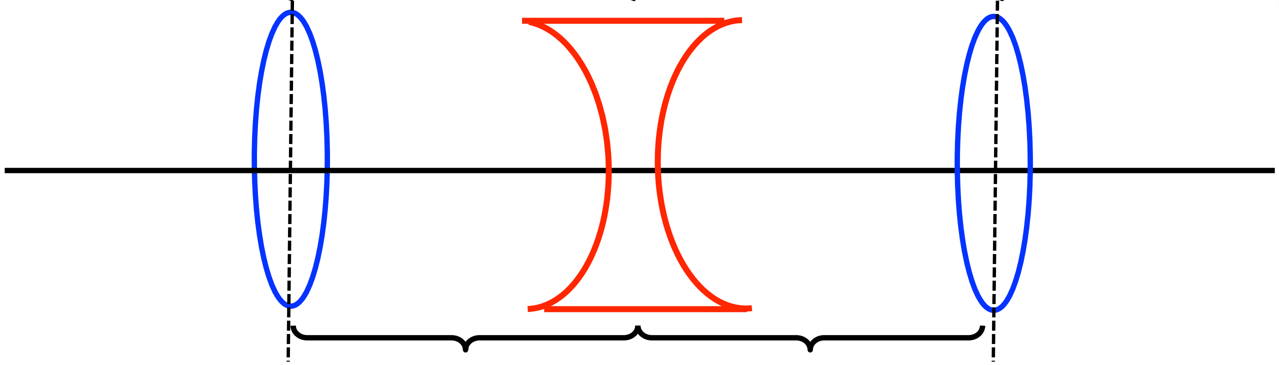
$$M(s_n|s_0) = \underbrace{M(s_n|s_{n-1}) \ldots \underbrace{M(s_3|s_2) \cdot \underbrace{M(s_2|s_1) \cdot \underbrace{M(s_1|s_0)}_{\text{from } \mathbf{s_0} \text{ to } \mathbf{s_1}}}_{\text{from } \mathbf{s_0} \text{ to } \mathbf{s_2}}}_{\text{from } \mathbf{s_0} \text{ to } \mathbf{s_3}}}_{\text{from } \mathbf{s_0} \text{ to } \mathbf{s_n}}$$



$$\boxed{\begin{pmatrix} x_n \\ x'_n \end{pmatrix} = M(s_n|s_0) \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}}$$

Thus by breaking up the parameter $K(s)$ into piecewise constant chunks, $K(s)=\{K1, K2, \ldots Kn\}$, we have found a useful method for finding the particle transport equation through a long section of beamline with many elements.

Basic blocks: FODO Cell



Symmetric transfer matrix from center to center of focusing quads

$$\mathcal{M}_{\mathrm{FODO}} = \mathcal{M}_{\mathrm{HQF}} \cdot \mathcal{M}_{\mathrm{drift}} \cdot \mathcal{M}_{\mathrm{QD}} \cdot \mathcal{M}_{\mathrm{drift}} \cdot \mathcal{M}_{\mathrm{HQF}}$$

$$\mathcal{M}_{\mathrm{HQF}} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix} , \quad \mathcal{M}_{\mathrm{drift}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} , \quad \mathcal{M}_{\mathrm{QD}} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

$$\boxed{\mathcal{M}_{\mathrm{FODO}} = \begin{pmatrix} 1 - \frac{L^2}{2f^2} & 2L(1 + \frac{L}{2f}) \\ -\frac{L}{2f^2}(1 - \frac{L}{2f}) & 1 - \frac{L^2}{2f^2} \end{pmatrix}}$$

Betatron motion

- General solution of Hill's eq. for on-momentum linear motion of a particle:

$$\boxed{\begin{aligned} x(s) &= \sqrt{\epsilon\beta(s)}\cos(\psi(s)+\phi_0) \\ x'(s) &= \sqrt{\frac{\epsilon}{\beta(s)}}(\sin(\psi(s)+\phi_0)+\alpha(s)\cos(\psi(s)+\phi_0)) \end{aligned}}$$

Twiss parameters: $\beta(s), \quad \alpha(s) = -\frac{\beta'(s)}{2}, \quad \gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}$

Betatron phase: $\psi(s) = \int \frac{ds}{\beta(s)}$

Tune (in a ring): number of betatron oscillations per turn, or phase advance per turn in units of 2π

$$Q = \frac{1}{2\pi}\oint \frac{ds}{\beta(s)}$$

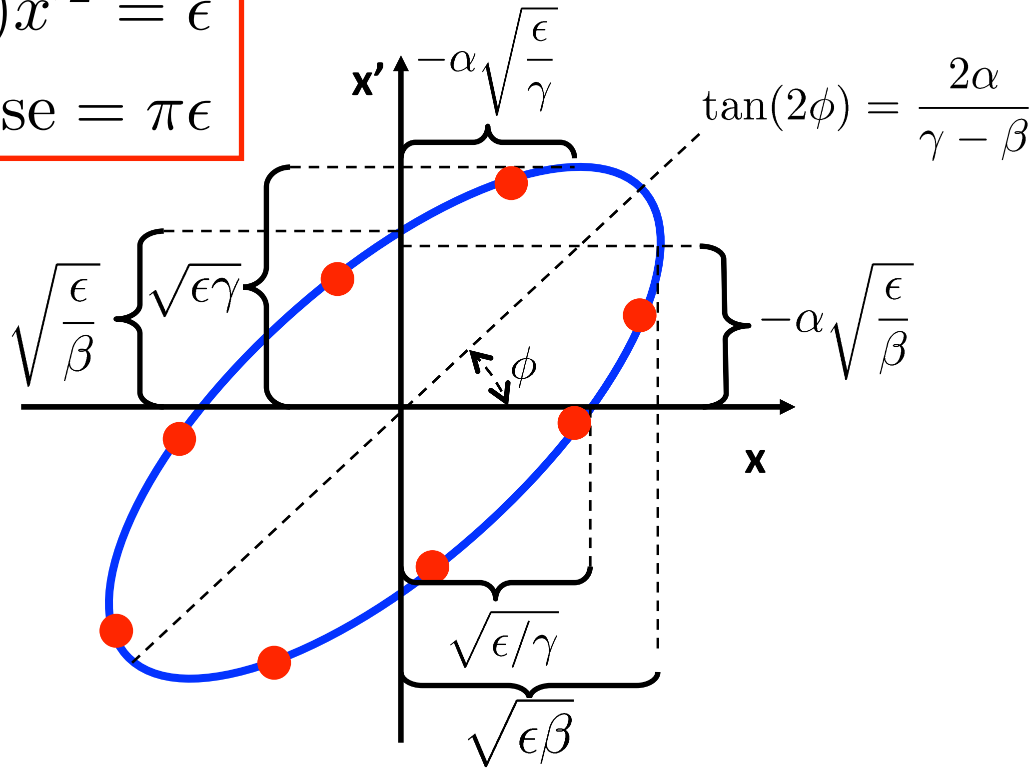
Phase space

- **The twiss parameters *α*, *β*, *γ* have a geometric meaning**

Phase space ellipse

$$\boxed{\begin{aligned}\gamma(s)x^2 + 2\alpha(s)xx' + \beta(s)x'^2 &= \epsilon \\ \text{Area of ellipse} &= \pi\epsilon\end{aligned}}$$

Courant-Snyder invariant: ϵ



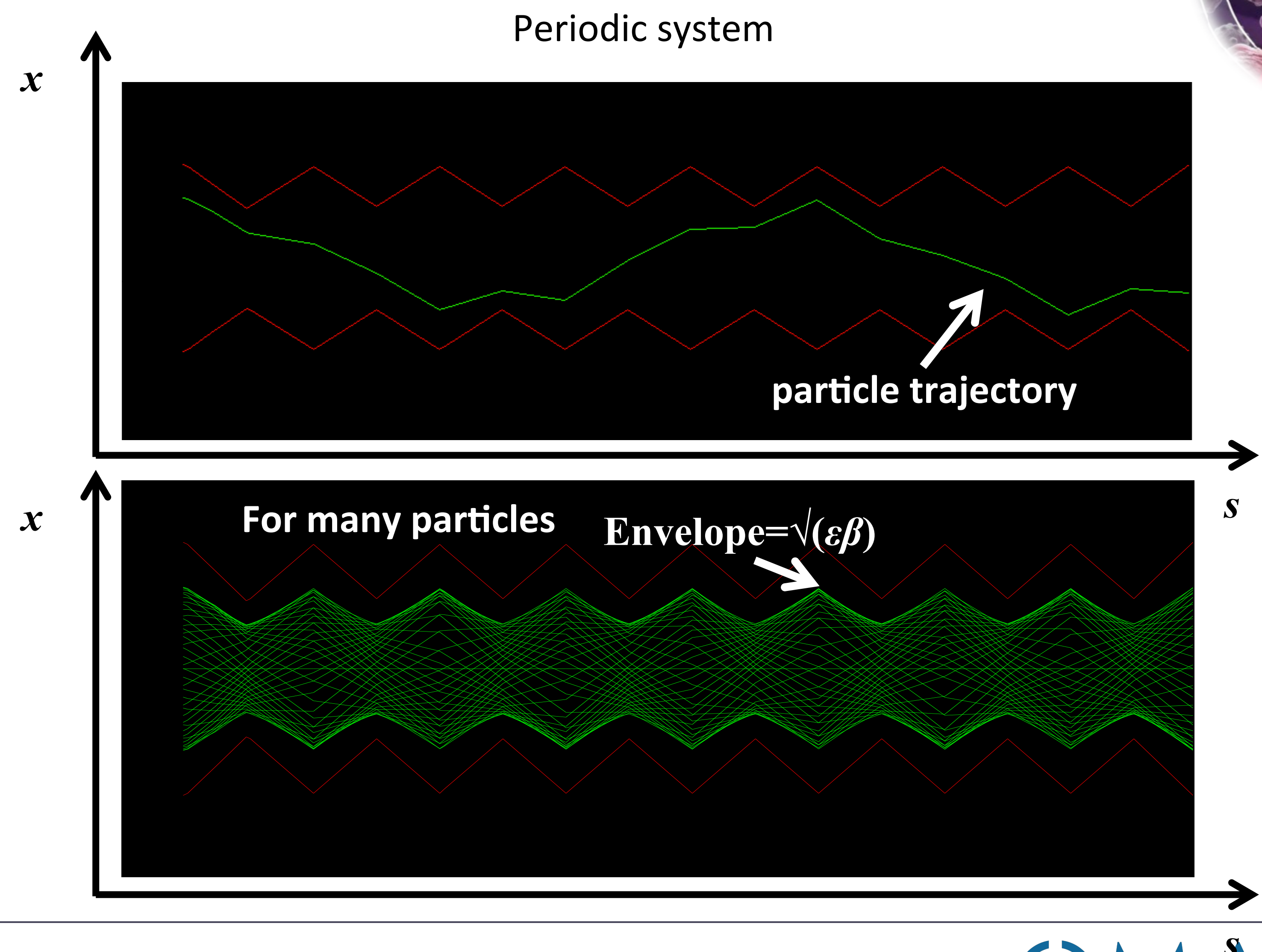
Beam envelope

$$E(s) = \sqrt{\epsilon\beta(s)}$$

Beam divergence

$$A(s) = \sqrt{\epsilon\gamma(s)}$$

Beam


Periodic system
x
particle trajectory
s
x
For many particles
Envelope=√(εβ)
s

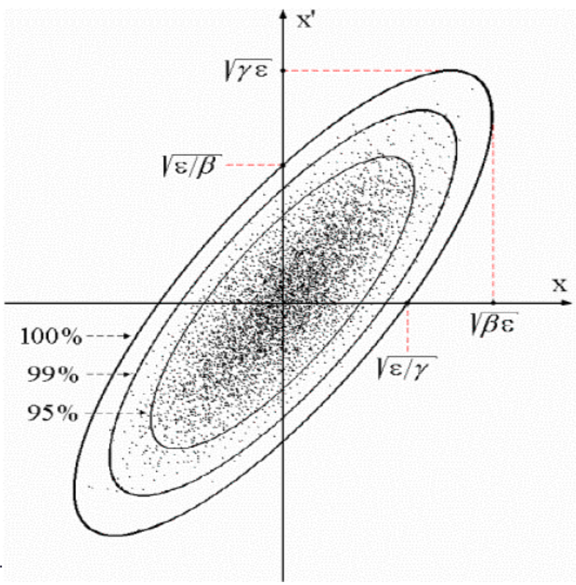
Beam

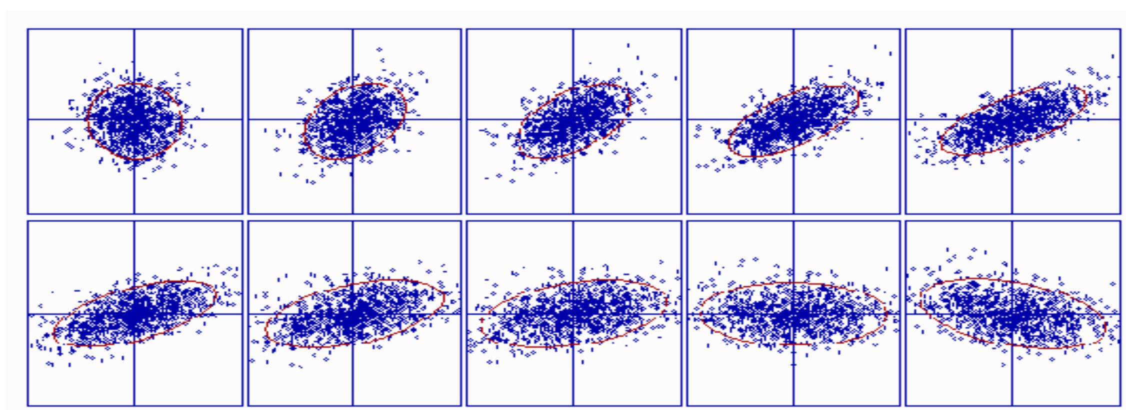
❑ Ensemble of particles

Beam is a set of millions/billions of particles (N)

For example, a Gaussian transverse distribution has a Gaussian density profile in phase space

$$f(x,x',y,y') = \frac{N}{A} exp\left(-\frac{\gamma_x x^2 + 2\alpha_x x x' + \beta_x x'^2}{2\epsilon_{x,\mathrm{rms}}} + \frac{\gamma_y y^2 + 2\alpha_y y y' + \beta_y y'^2}{2\epsilon_{y,\mathrm{rms}}}\right)$$





Emittance

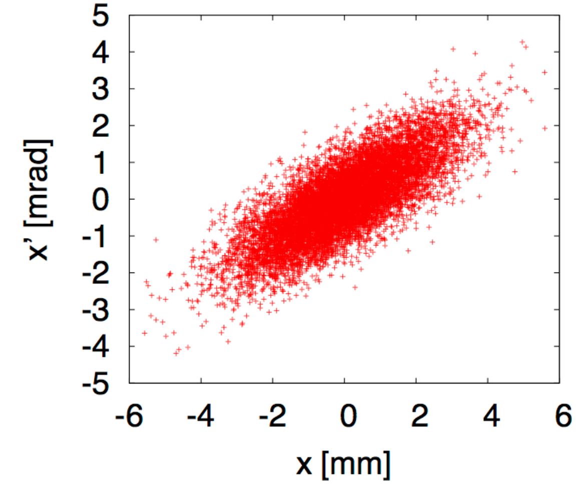
❑ Statistical definition

Let's consider first a 2D $x - x'$ beam phase space for an ensemble of particles. We need to characterise the spread of particles in phase space

Emittance is a measure of the phase space area occupied by a beam

$$\boxed{\epsilon_{x,rms} = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2}}$$

$$\begin{cases} \langle x^2 \rangle = \frac{1}{N} \sum_{i=1}^{N} (x_i - \langle x \rangle)^2 \\ \langle x'^2 \rangle = \frac{1}{N} \sum_{i=1}^{N} (x'_i - \langle x' \rangle)^2 \\ \langle x x' \rangle = \frac{1}{N} \sum_{i \neq j}^{N} \sum_{j=1}^{N} (x_i - \langle x \rangle)(x'_j - \langle x' \rangle) \end{cases}$$



Emittance

❑ Emittance and Courant-Snyder Invariant

Particle orbit in terms of lattice Twiss parameters

$$(i = 1 \text{ to } N)\begin{cases} x_i(s) = \sqrt{\epsilon_i \beta(s)}\cos(\psi(s)+\phi_i) \\ x'_i(s) = -\sqrt{\epsilon_i/\beta(s)}[\alpha(s)\cos(\psi(s)+\phi_i)+\sin(\psi(s)+\phi_i)] \end{cases}$$

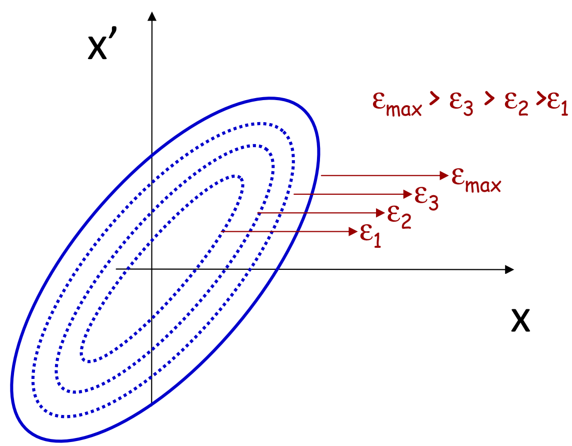
For a matched beam, for each ε_i, particles are uniformly distributed around the ellipse

$$\langle x^2\rangle = \frac{1}{N}\sum_{i=1}^{N}\epsilon_i\beta(s)\cos^2(\psi(s)+\phi_i) = \beta(s)\langle\epsilon\rangle$$

$$\langle x'^2\rangle = \gamma(s)\langle\epsilon\rangle$$

$$\langle xx'\rangle = \alpha(s)\langle\epsilon\rangle$$

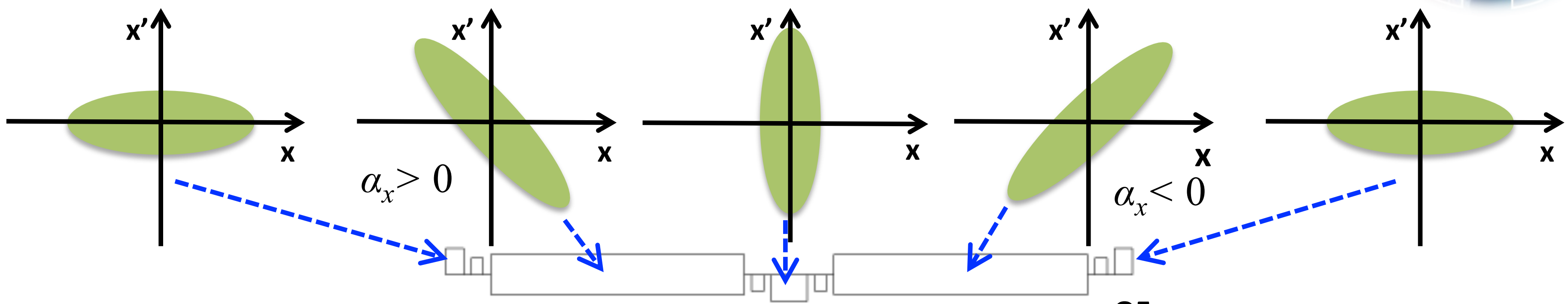
$$\boxed{\epsilon_{rms} = \frac{1}{N}\sum_{i=1}^{N}\epsilon_i = \langle\epsilon\rangle}$$



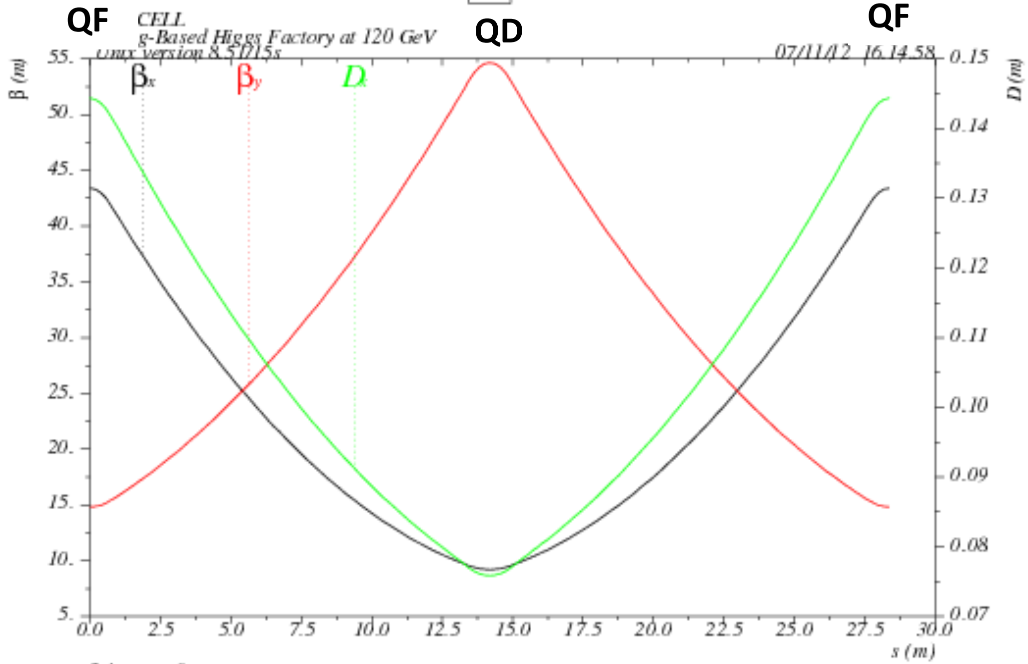
Courant-Snyder Invariant can be seen as a single particle emittance

Evolution of phase space

- A large β-function corresponds to a large beam size and a small beam divergence



In the middle of quad. QF , β_x is maximum (β_y minimum), and $\alpha_{x,y}$=0



In the middle of quad. QD, β_y is maximum (β_x minimum), and $\alpha_{x,y}$=0

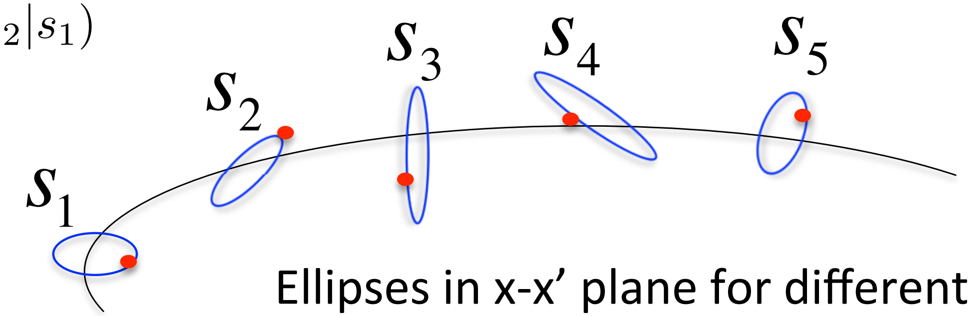
Liouville's Theorem

In a Hamiltonian system, i.e. in the absence of collisions or dissipative processes, the density in phase space along the trajectory is invariant

As particle moves along the orbit the shape and position of the phase space ellipse change according to $\beta(s)$, but the area remains constant.

$$M(s_5|s_1) = M(s_5|s_4)\cdots M(s_3|s_2)M(s_2|s_1)$$

$$\det(M) = 1$$



Ellipses in x-x' plane for different s

Poincare invariant:

$$\text{Area enclosed} = \iint dx dx' = \frac{1}{p}\iint dx dp_x = \pi\epsilon$$

Exceptions to Liouville

- Liouville Theorem not applicable when:
  - Collective effects: impedances (e.g. interaction with vacuum chamber), space-charge forces within the beam
  - Scattering and dissipation mechanisms
  - Damping effects, quantum diffusion due to emission of radiation (velocity-dependent effect)

Emittance (other definitions)

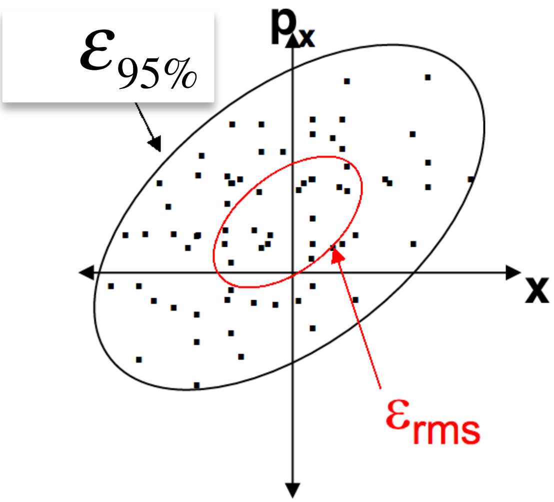
- **Normalised emittance:**

The geometric emittance, as defined before, decreases as $1/p$ or $1/(\beta\gamma)$, so when a beam is accelerated, it is not ϵ that is conserved, but the quantity

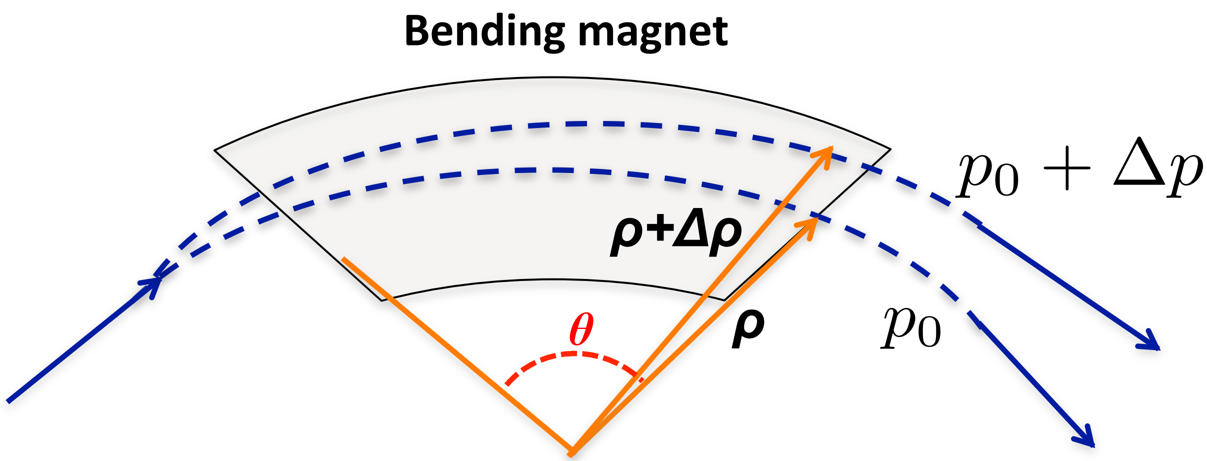
$$\epsilon_n = \beta\gamma\epsilon$$

- **Percentile emittance:**

e.g. 95% emittance, which defines the area of ellipse that contains 95% of the beam. It is a convenient definition when we have to deal with irregular beam distributions



Off-momentum particles



- Off-momentum particles get different deflection (different orbit)

$$\boxed{\Delta\theta = -\theta\frac{\Delta p}{p_0}}$$

Dispersion

❑ Inhomogeneous Hill's equation

$$x'' + K(s)x = \frac{1}{\rho}\frac{\Delta p}{p_0}$$

The solution is a sum of the **homogeneous** equation (on-momentum) and the **inhomogeneous** (off-momentum):

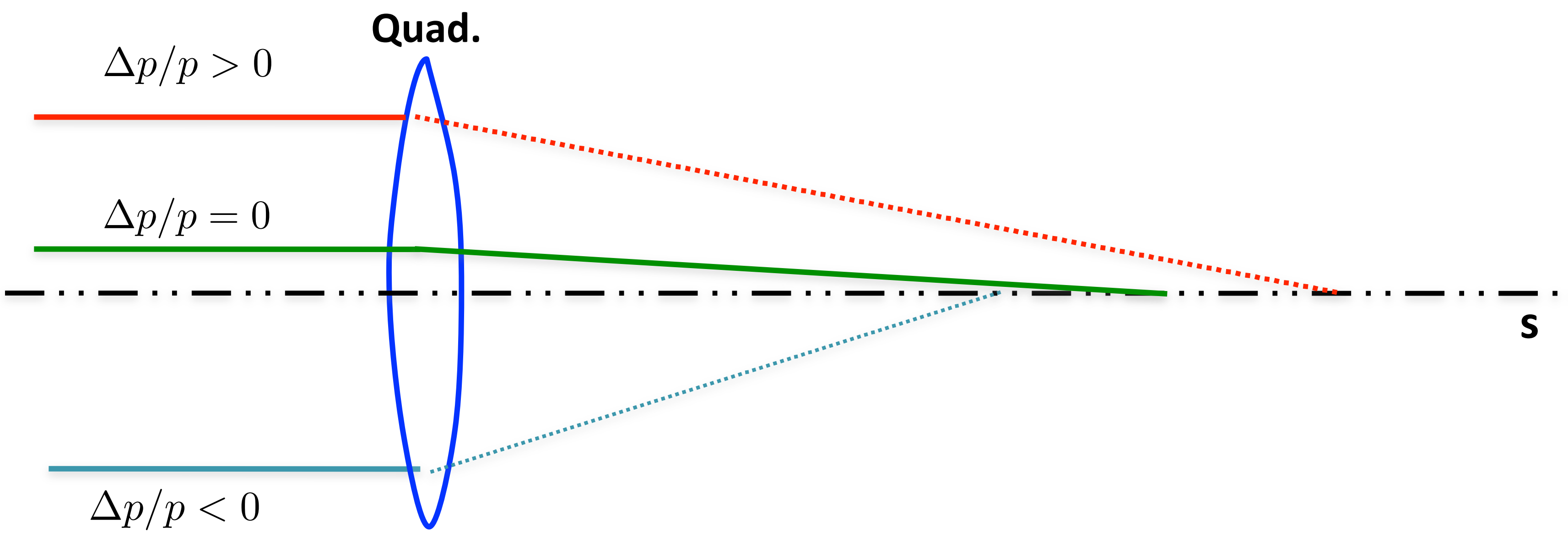
$$x(s) = x_\beta(s) + D(s)\frac{\Delta p}{p_0}$$

Dispersion function: $D(s)$

Dispersion equation: $D''(s) + K(s)D(s) = \frac{1}{\rho}$

Chromaticity

- Off-momentum particle gets different focusing

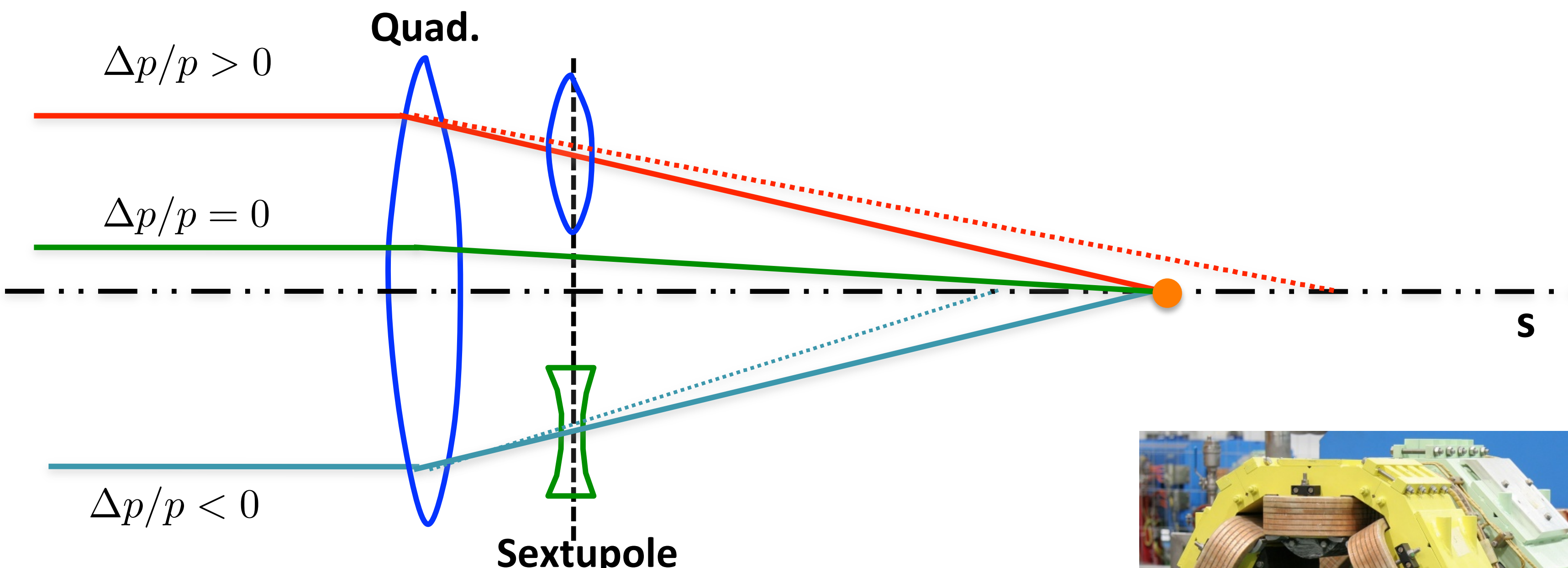


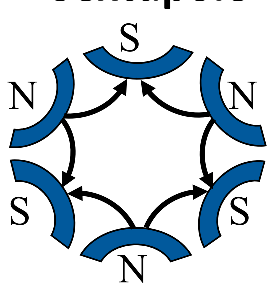
- Chromaticity acts like a quadrupole error (optical aberration), $\Delta K = -K\dfrac{\Delta p}{p}$ and leads to a tune spread:

$$\boxed{\xi = \frac{\Delta Q}{\frac{\Delta p}{p}}, \ \text{first order chromaticity}}$$

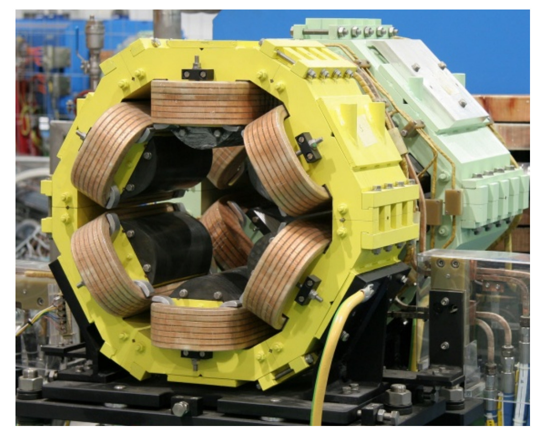
Chromaticity

- How to correct chromaticity?





CNAO sextupole



Nonlinearities

In reality, some lattice have significant higher order terms

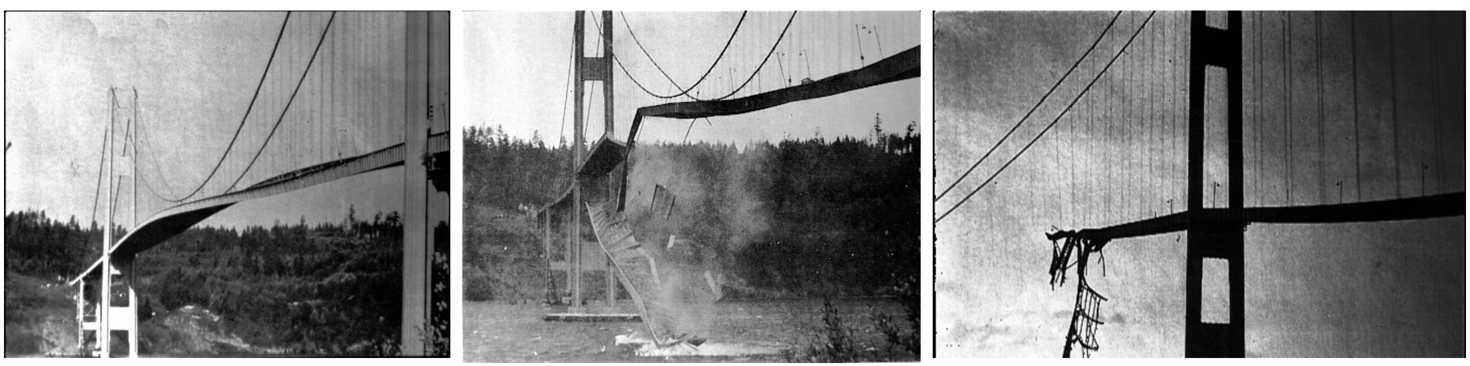
$$x'' + K(s)x = O(x^2) + \ldots$$

and magnetic imperfections, e.g. dipole errors

$$x'' + K(s)x = \delta(s - s_0)\theta_{\mathrm{error}}$$

which can drive **resonances**

Tacoma Narrow bridge 1940



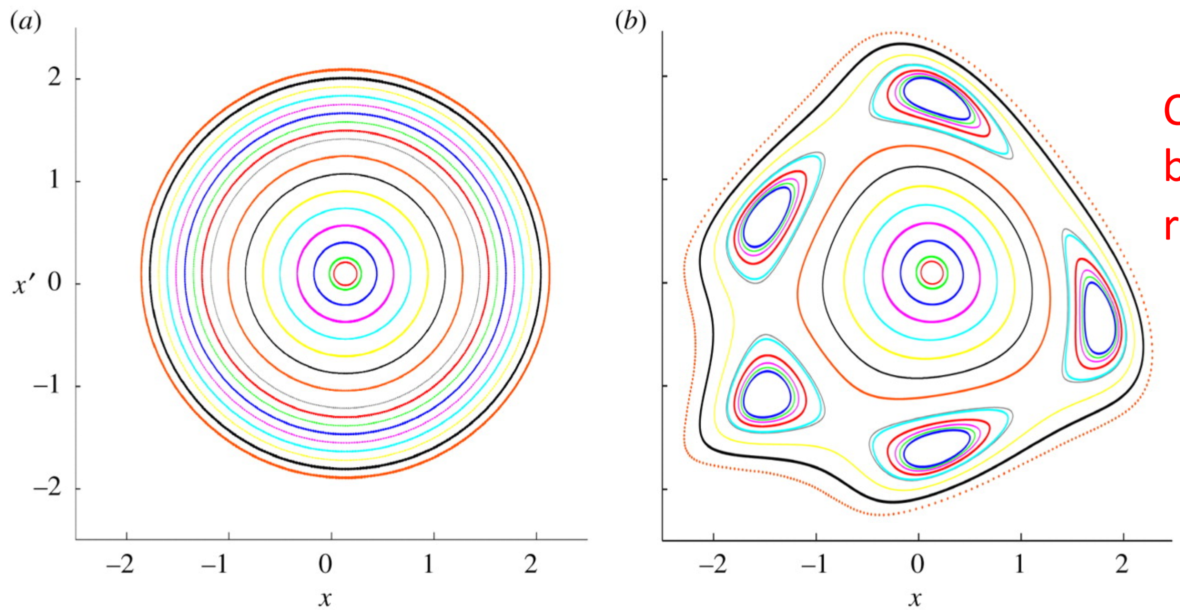
(Excitation by strong wind on the eigenfrequencies)

Nonlinearities

- Example:

Sextupoles are the most common magnet nonlinearities in accelerators

Phase space plot of particle motion close to a fifth-order resonance



Chaotic regimen between the resonant islands!

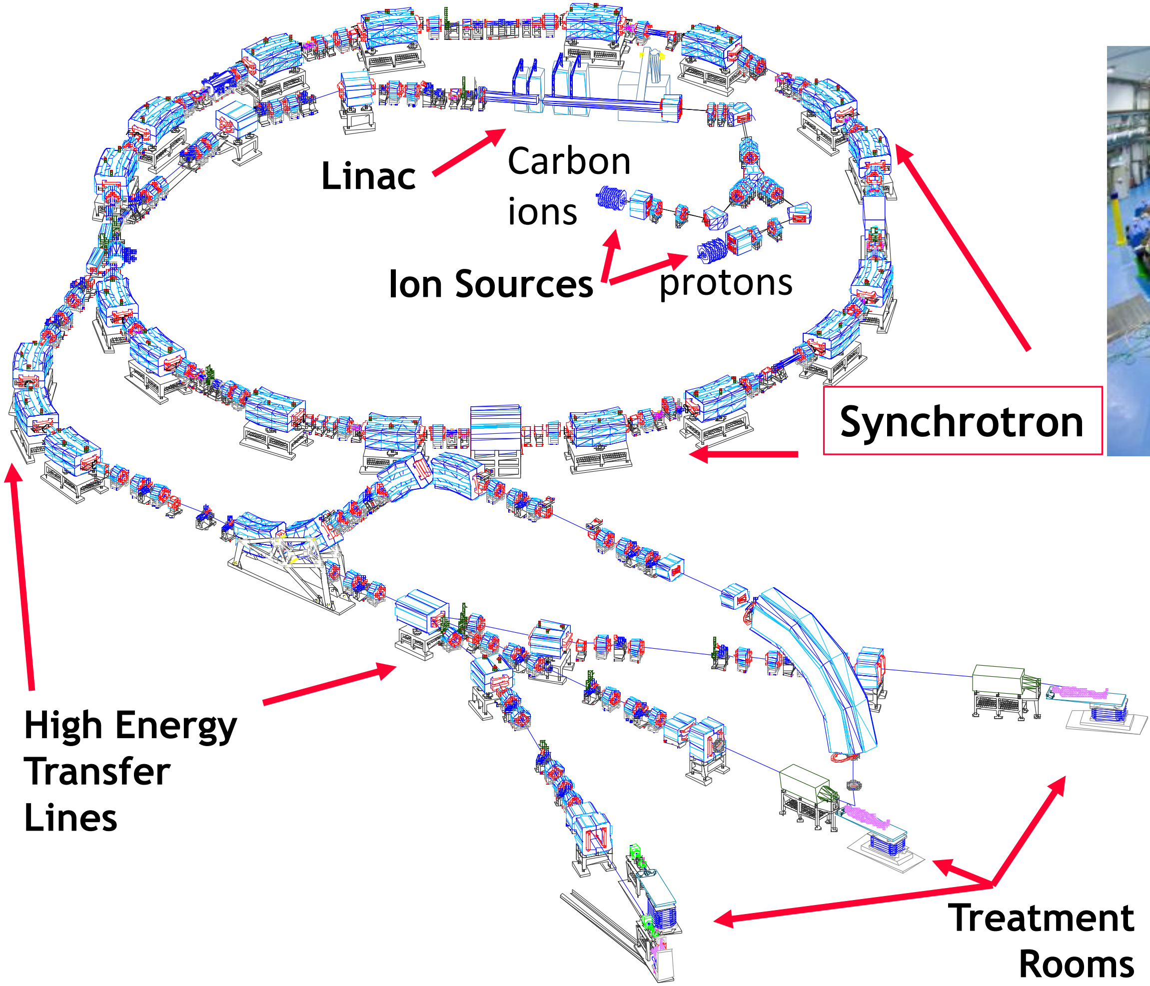
Linear machine

With a sextupole element

(Courtesy of W. Herr and E. Forest)

Real machine

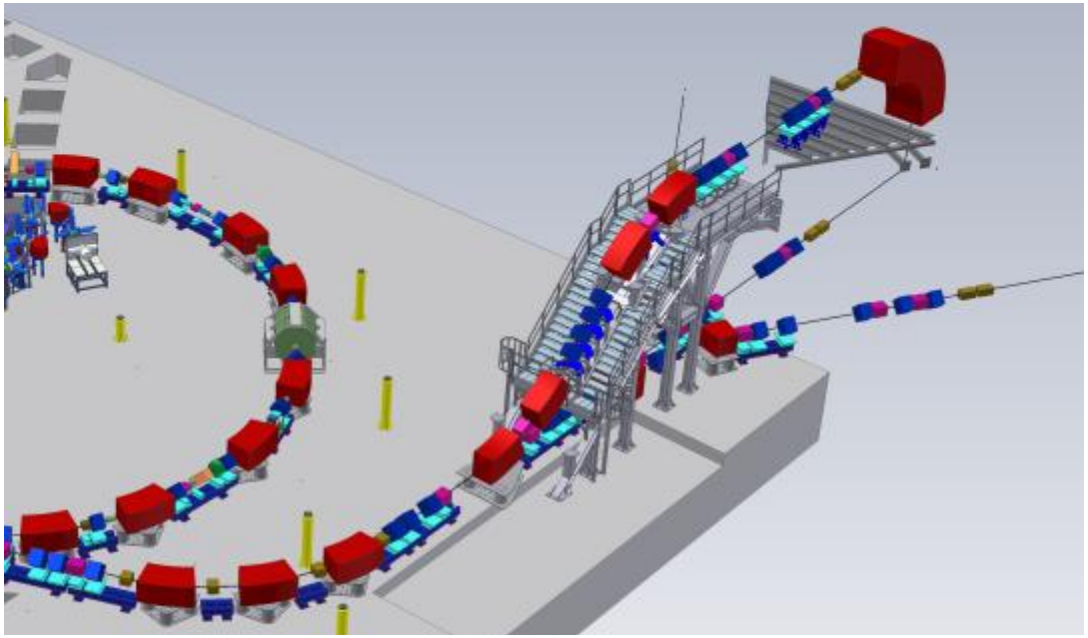
CNAO



Synchrotron
P : 60 – 250 MeV
C6+ :120 - 400 MeV

Transfer line examples

CNAO lines: 3 treatment rooms: 2 with horizontal line and 1 with horizontal and vertical one. The beginning of the line has 4 fast magnets (100 microsec) to dump the beam for patient security.

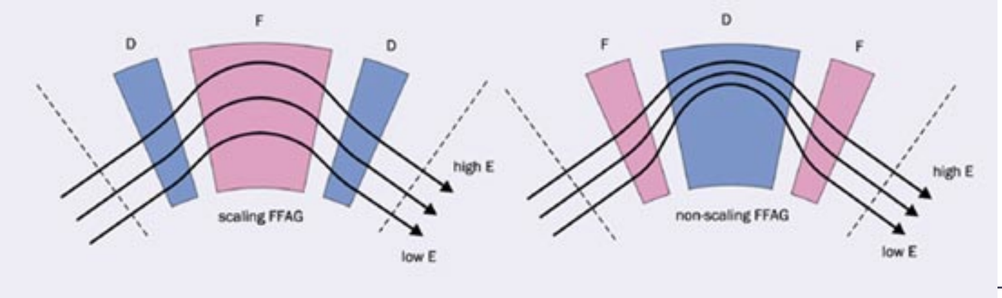


In a transfer line, the starting Twiss parameters determine the initial conditions from which to calculate the subsequent values

Looking at the future

❑ Worldwide R&D for more compact and/or advanced accelerators:

- **FFAG (Fixed Field Alternating Gradient)**
  - Magnetic field that does not change with time
  - Magnetic field does contain field gradients along the radial direction
  - Owing to the fixed magnetic field (as in a cyclotron). the FFAG accelerator can accelerate particles very rapidly
  - The magnetic gradient (as used in synchrotrons) allows a variable energy output
  - A fast energy change could be a good solution in treating moving organs



Looking at the future

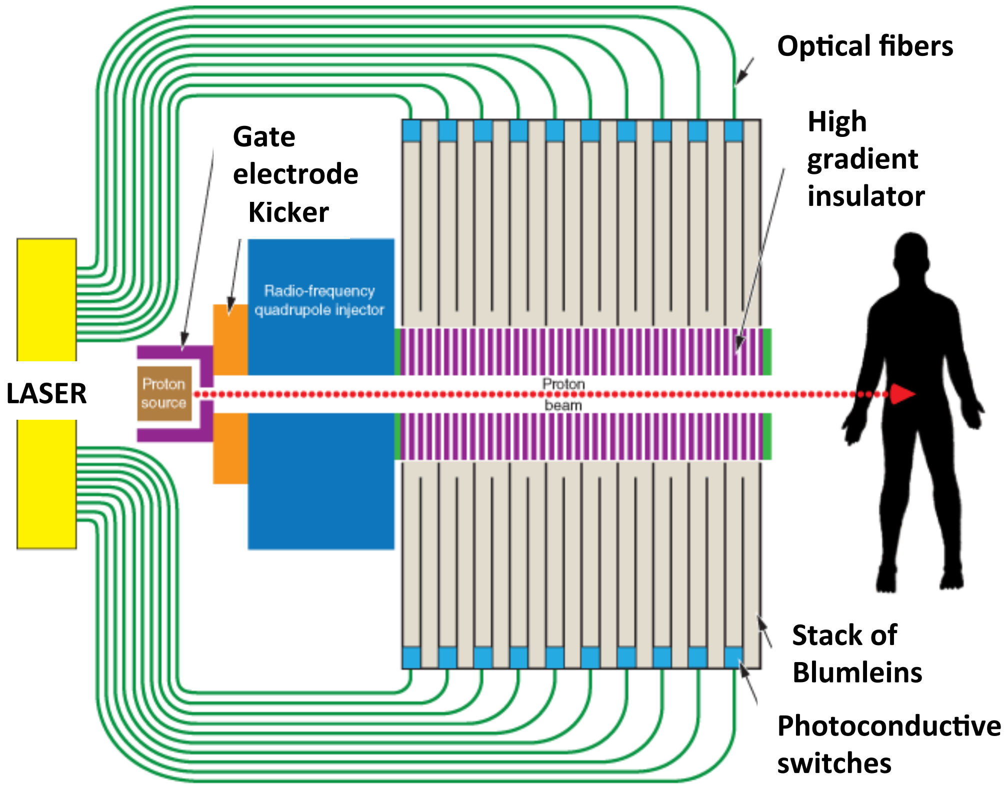
- **FFAG: EMMA at Daresbury**

Proof-of-principle
Electron acceleration: 10 to 20 MeV

Looking at the future

▪ DWA: Dielectric Wall Accelerator

- Acc. gradients ~ 100 MV/m
- Very high current in short pulses (< 1 ns)
- Both the proton energy and beam intensity can be varied simultaneously



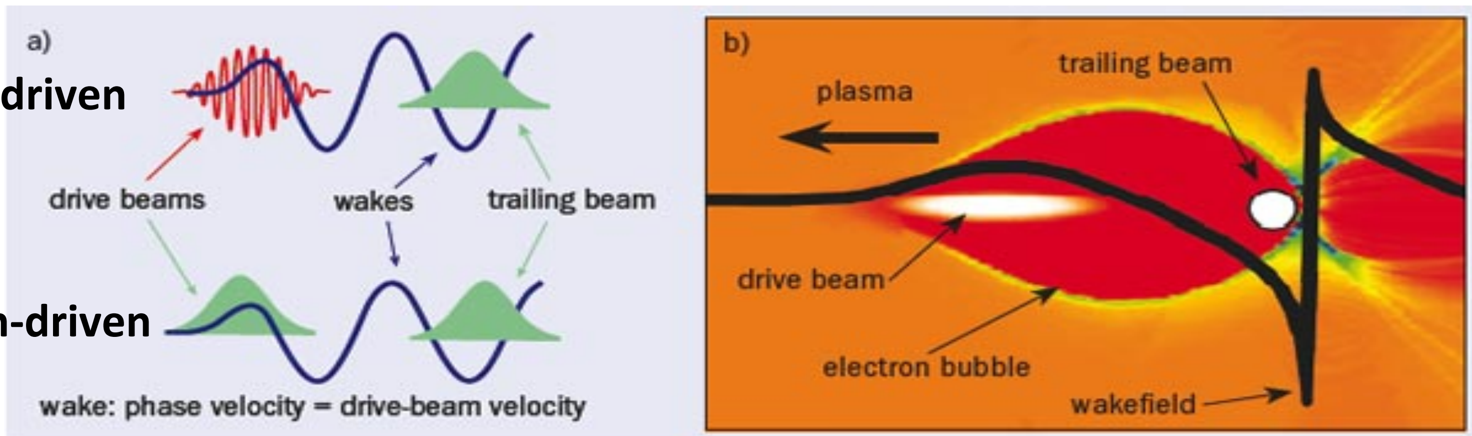
Lawrence Livermore National Laboratory, Livermore CA, USA

Looking at the future

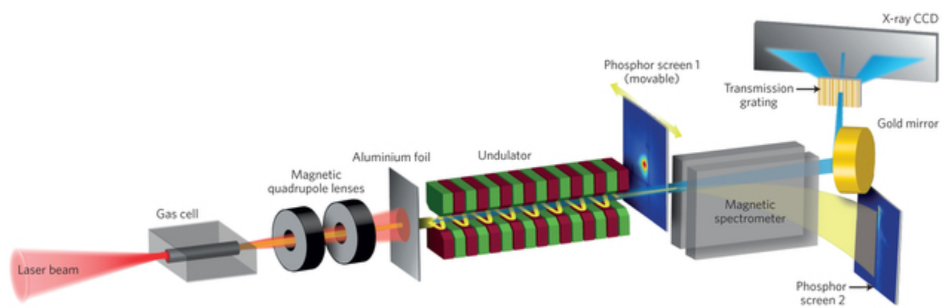
- **PWFA and LWFA**

Laser-driven

Beam-driven

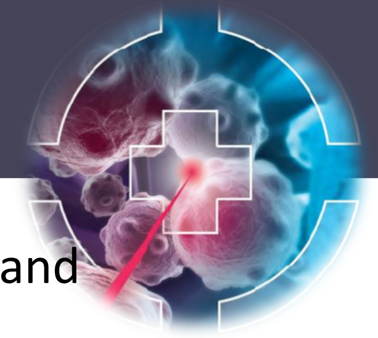


It would allow table-top machines for X-ray diagnostics and therapy



S. M. Hooker, Nature Photonics 7 (2013) 775

Summary



- Basic definitions (briefly) reviewed for both linear longitudinal and transverse beam dynamics

  - Starting from Lorentz equation
  - Techniques to accelerate particles (linacs, circular accelerators)
  - Magnets to guide the particle trajectory (dipoles, quadrupoles)
  - Single-particle beam dynamics
  - Hill's equation and betatron motion
  - Matrix description (maps) and basic blocks to design an accelerator: FODO lattice
  - Multiparticle-beam
  - Transverse phase space: C-S invariant, emittance, Liouville's Theorem
  - Off momentum particles: Dispersion function, Chromaticity

- However, the more advanced the problem … the more advanced the model you need!

- How to measure the beam properties… coming next!

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Maxwell's equations (in vacuum)

- In vacuum

$$\vec{D} = \varepsilon_0 \vec{E}, \quad \vec{B} = \mu_0 \vec{H}, \quad \varepsilon_0 \mu_0 = \frac{1}{c^2}$$

- Source-free equations:

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

- Source equations

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \times \vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{j}$$

- Equivalent integral forms (sometimes useful for simple geometries)

$$\oiint \vec{E} \cdot d\vec{S} = \frac{1}{\varepsilon_0} \iiint \rho \, dV$$

$$\oiint \vec{B} \cdot d\vec{S} = 0$$

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \iint \vec{B} \cdot d\vec{S} = -\frac{d\Phi}{dt}$$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \iint \vec{j} \cdot d\vec{S} + \frac{1}{c^2}\frac{d}{dt} \iint \vec{E} \cdot d\vec{S}$$

Relativity glossary

Relativistic beta: $\beta = v/c$ (not to confuse with the betatron function!)

Lorentz factor: $\gamma = \dfrac{1}{\sqrt{1-\beta^2}}$

Total energy: $E = \sqrt{m_0^2 c^4 + p^2 c^2}$ $\quad E = \gamma m_0 c^2$

Momentum: $p = \gamma m_0 v$

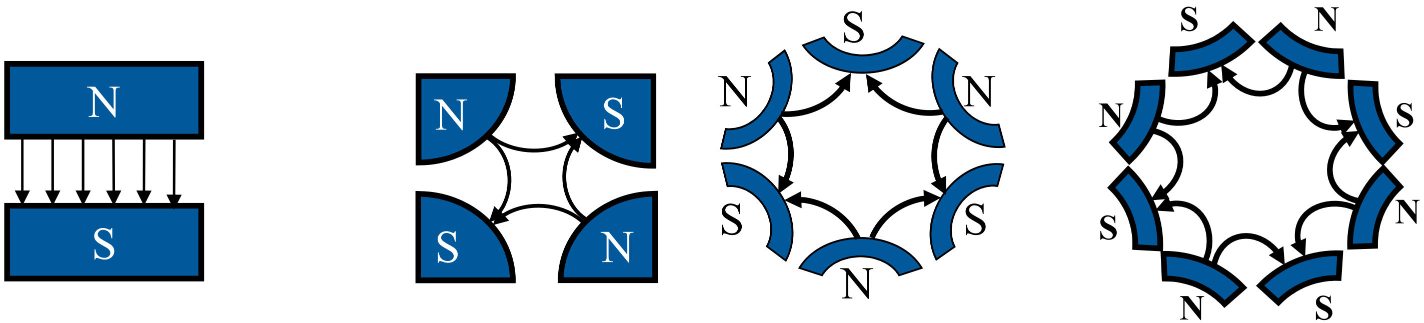
Kinetic energy: $W = (\gamma - 1) m_0 c^2$

Useful relations: $E = \dfrac{c}{\beta} p$ $\quad \dfrac{dp}{p} = \gamma^2 \dfrac{dv}{v}$ $\quad \dfrac{dE}{E} = \beta^2 \dfrac{dp}{p}$

Definition of magnets

- **2n-pole**:

dipole quadrupole sextupole octupole …

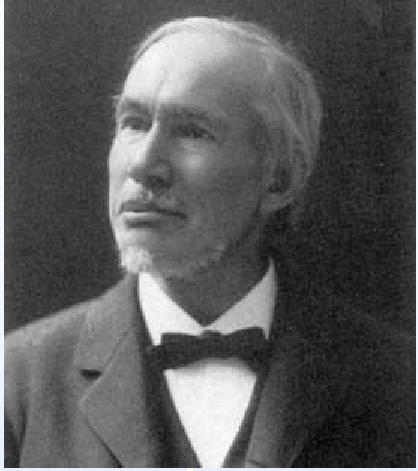


n: 1 2 3 4 …

- Normal: gap appears at the horizontal plane
- Skew: rotate around beam axis by **π/2n** angle
- Symmetry: rotating around beam axis by **π/n** angle, the field is reversed (polarity flipped)

D. Robin, MSU

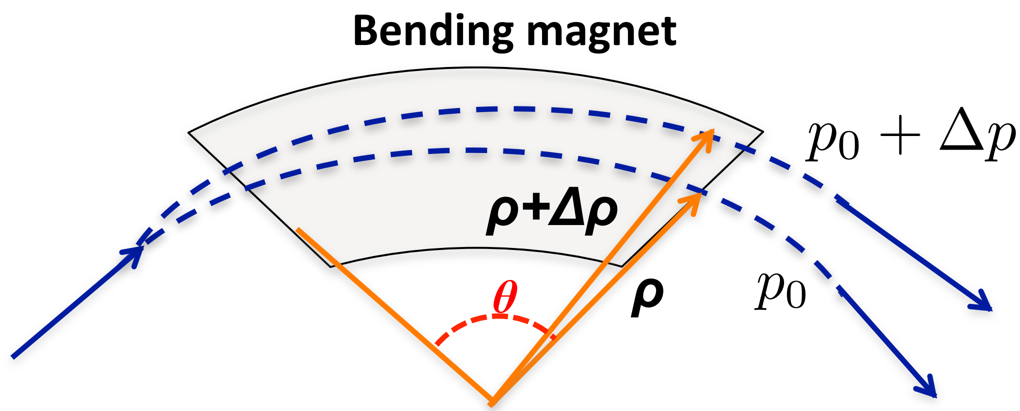
Hill’s Equation



George Hill

- Equation of transverse motion
  - Drift: $x'' = 0, \quad y'' = 0$
  - Solenoid: $x'' + 2k\,y' + k'\,y = 0, \quad y'' - 2k\,x' - k'\,x = 0$
  - Quadrupole: $x'' + k\,x = 0, \quad y'' - k\,y = 0$
  - Dipole: $x'' + \dfrac{1}{\rho^2}x = 0, \quad y'' = 0$
  - Sextupole: $x'' + k\left(x^2 - y^2\right) = 0, \quad y'' - 2kxy = 0$
- Hill’s Equation: $x'' + k_x(s)x = 0, \quad y'' + k_y(s)y = 0$

Off-momentum particles



- Recall that the magnetic rigidity $B\rho = \frac{p_0}{q}$, and for off-momentum particles

$$B(\rho + \Delta\rho) = \frac{p_0 + \Delta p}{q} \rightarrow \frac{\Delta\rho}{\rho} = \frac{\Delta p}{p_0}$$

- Considering the effective length of the dipole unchanged:

$$\theta\rho \simeq l_{eff} = \text{const.} \rightarrow \rho\Delta\theta + \theta\Delta\rho = 0 \rightarrow \frac{\Delta\theta}{\theta} = -\frac{\Delta\rho}{\rho} = -\frac{\Delta p}{p_0}$$

- Off-momentum particles get different deflection (different orbit)

$$\Delta\theta = -\theta\frac{\Delta p}{p_0}$$

Normalised emittance

- Apply some acceleration along *z* to all particles in the bunch
  - P_x is constant
  - P_z increases
  - $x' = P_x / P_z$ decreases!
- So the bunch emittance decreases
  - This is an example of something called *Liouville's Theorem*
  - ~"Emittance is conserved in (x,P_x) space"
- Define normalised emittance

$$\varepsilon_n = \frac{<p_z>}{m}\varepsilon$$

Types of accelerators

Three accelerators can provide clinical beam: LINAC, Cyclotrons, Synchrotrons. The energy and the species of hadrontherapy make LINAC up to now not very practical and feasible

Nowadays Hadrontherapy centers are Cyclotrons and Synchrotrons

Cyclotrons	Synchrotrons
Compact (4 m diameter)	More complicated
cheaper	More expensive
DC beam	Pulsed beam
High current (hundreds nA)	Lower currents (tens nA)

BUT…

Types of accelerators

…BUT

Cyclotrons are easy for protons; only one CHALLENGING PROPOSAL exists for carbon
Cyclotron compactness is partially offset by the place required by the medical structure
Passive scanning is needed with cyclotrons because the energy from accelerator is fixed

while

Synchrotrons can accelerate both protons and carbons.
A synchrotron designed for 300mm C^{6+} can accelerate 1<=Z<=6 and O up to 19 cm.
Synchrotron can perform active scanning.

Nowadays the best technological layout for a hadrontherapy center is a **Carbon Synchrotron equipped with active scanning**.

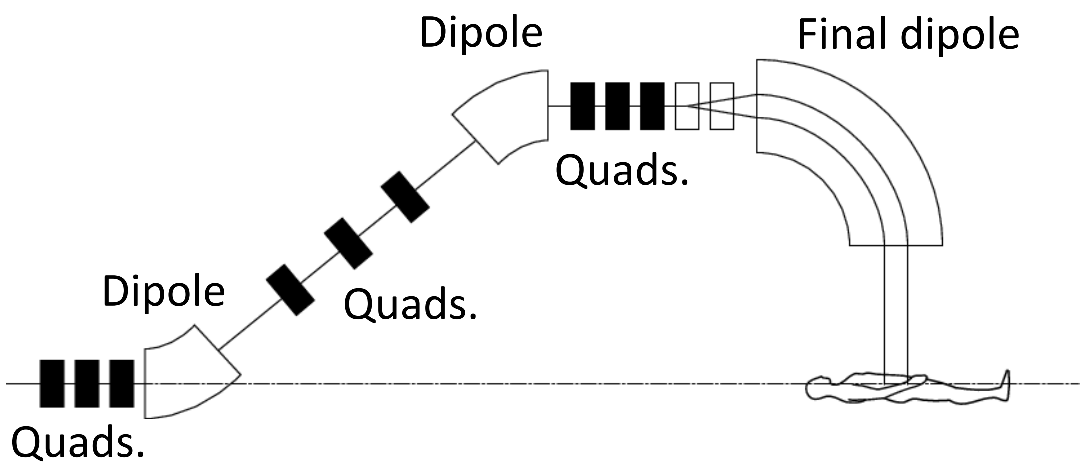
A carbon synchrotron facility is made up of:

1. A low energy injector
2. A ring
3. The extraction lines

Transfer line examples

Beam delivery System

Schematic of an isocentric 3 dipole gantry



Final dipole of Gantry 2 at PSI.
Large 45-tonne final dipole

H. Owen et al., Int. J. Mod. Phys. A29 (2014) 1441002

Resonances

Hill's equation is quasiharmonic, and whenever we have a harmonic system, the danger of exciting a resonance exists. Multiple sources of resonant driving terms exist in accelerators:

- Linear magnet imperfections
- Time varying fields
- Nonlinear magnets
- Collective effects
- etc., etc.

$$lQ_x + mQ_y = r$$

where (l, m, r) are integers

Tune diagram