Tutorial 1. Solutions

OMA School, Pavia,

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1 Acceleration

1. Why does not a magnetic field contribute to the longitudinal acceleration of a charged particle with initial speed v?

Solution:

In presence of an electromagnetic field, the force on a charged particle with a charge e is given by the Lorentz force:

$$\vec{F} = \frac{d\vec{p}}{dt} = e(\vec{E} + \vec{v} \times \vec{B})$$

The second term goes with the cross product and, therefore, is always perpendicular to the direction of motion, so it does not give any longitudinal acceleration and it does not increase the energy of particle.

2. If we assume an electric field and the acceleration to be along the z direction, calculate the kinetic energy gained from the field.

Solution:

According to the Lorentz force we have

$$\frac{dp}{dt} = eE_z.$$

The total energy E of a particle is the sum of the rest energy E_0 and the kinetic energy W:

$$E = E_0 + W.$$

In relativistic dynamics, the total energy E and the momentum p are linked by

$$E^2 = E_0^2 + p^2c^2,$$

from which it follows that $dE = vdp$.
The rate of energy gain per unit length of acceleration (along the z direction) is then given by

$$\frac{dE}{dz} = v\frac{dp}{dz} = \frac{dp}{dt} = eE_z$$

and the kinetic energy gained from the field along the z path follows from $dW = dE = eE_z dz$:

$$W = e\int E_z dz = eV,$$

where V is just an electric potential.

2 Local radius, rigidity

We wish to design an electron ring with a radius of R=200 m. Let us assume that only 50% of the circumference is occupied by bending magnets:

- What will be the local radius of bend ρ in these magnets if they all have the same strength? **Solution:**

$$2\pi\rho = 50\% \cdot 2\pi R \longrightarrow \rho = 100 \text{ m}$$

- If the momentum of the electrons is 12 GeV/c, calculate the rigidity $B\rho$ and the field in the dipoles. **Solution:**
  Using the rigidity definition:

$$B\rho = 3.3356 \cdot p[\text{GeV}/c] = 40.03 \text{ T·m}$$

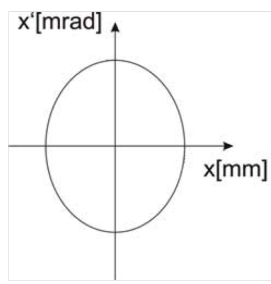
  and therefore $B = 0.4$ T.

3 Phase space

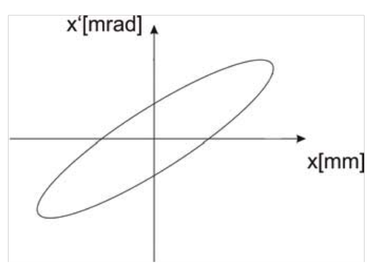
1. Sketch the emittance ellipse of a particle beam in horizontal x-x' phase space (I) at the position of a transverse waist, (II) when the beam is divergent and (III) when the beam is convergent.

Solution:

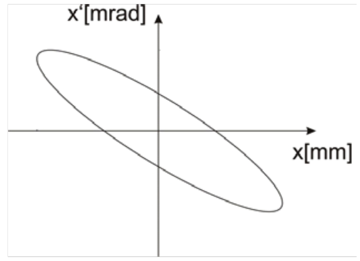
(I) Beam at the position of a transverse (x) waist – upright/round ellipse:



(II) Divergent beam – positive slope:



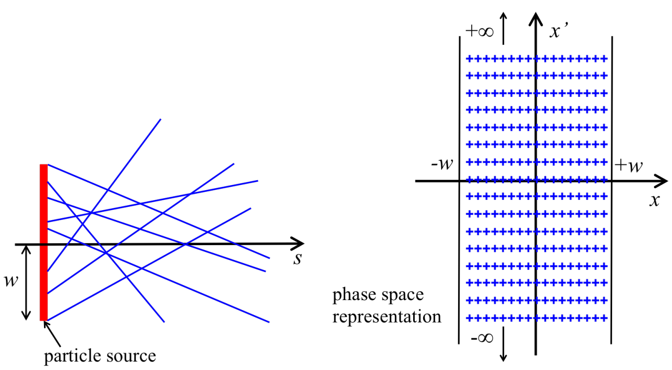
(III) Convergent beam – negative slope:



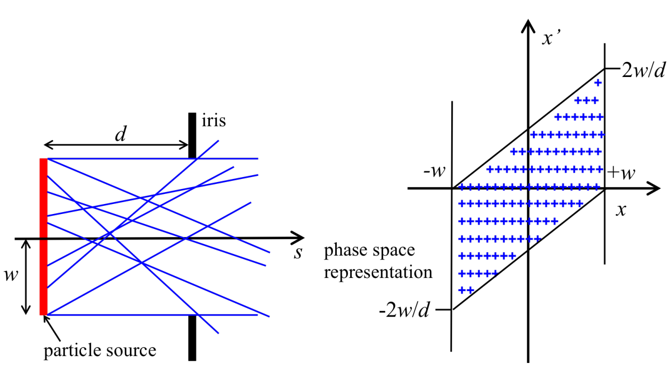
2. Phase Space Representation of a Particle Source:

Solution:

- Consider a source at position s_0 with radius w emitting particles. Make a drawing of this setup in configuration space and in phase space. Which part of phase space can be occupied by the emitted particles?
  Particles are emitted from the entire source surface $x \in [-w, +w]$ with a trajectory slope $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, i.e. the particles can have any $x' \in \mathbb{R}$. The occupied phase space area is infinite.



- Any real beam emerging from a source like the one above will be clipped by aperture limitations of the vacuum chamber. This can be modeled by assuming that a distance d away from the source there is an iris with an opening with radius $R = w$. Make a drawing of this setup in configuration and phase space. Which part of phase space is occupied by the beam at a location after the iris?
  Particles with angle $x' = 0$ are emitted from the entire source surface $x \in [-w, +w]$ and arrive behind the iris opening. For $x = \pm w$ there is a maximum angle $x' = \pm 2w/d$ that will still be accepted by the iris. This leads to a parallelogram in phase space. Such a beam has a specific emittance given by the occupied phase space area.

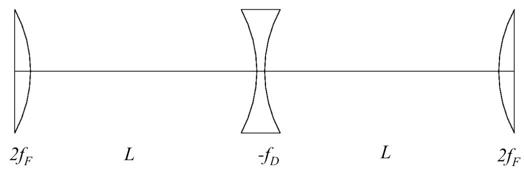


4 FODO lattice

A quadrupole doublet consists of two lenses of focal length f_1 and f_2 separated by a drift length L. Assume that the lenses are thin and show that the transport matrix of this system is

$$M = \begin{pmatrix} 1 - L/f_1 & L \\ -1/f^* & 1 - L/f_2 \end{pmatrix} \text{ where } \frac{1}{f^*} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{L}{f_1 f_2}$$

A FODO cell can be considered as the simplest block of the magnetic structure of modern accelerators and storage rings. It consists of a magnet structure of focusing (F) and defocusing (D) quadrupole lenses in alternating order (see schematic below). Its transfer matrix can be calculated using the matrix of the quadrupole doublet (above) with $f_1 = +2f$ and $f_2 = -2f$ followed (and multiplied) by another quadrupole doublet matrix with $f_1 = -2f$ and $f_2 = +2f$.



Show that the transfer matrix of a FODO system in thin lens approximation is as follows:

$$M_{FODO} = \begin{pmatrix} 1 - \frac{L^2}{2f^2} & 2L + \frac{L^2}{f} \\ -\frac{L}{2f^2}(1 - \frac{L}{2f}) & 1 - \frac{L^2}{2f^2} \end{pmatrix}$$

Solution:

Taking into account the previous result:

$$M_{FODO} = \begin{pmatrix} 1 + \frac{L}{2f} & L \\ -\frac{L}{4f^2} & 1 - \frac{L}{2f} \end{pmatrix} \begin{pmatrix} 1 - \frac{L}{2f} & L \\ -\frac{L}{4f^2} & 1 + \frac{L}{2f} \end{pmatrix} = \begin{pmatrix} 1 - \frac{L^2}{2f^2} & 2L + \frac{L^2}{f} \\ -\frac{L}{2f^2}(1 - \frac{L}{2f}) & 1 - \frac{L^2}{2f^2} \end{pmatrix}$$

5 Cyclotron

Let us to consider a cyclotron with a magnetic field strength of 1 T. If a proton beam is injected:

- What is the maximum velocity of the protons for a maximum radius of 50 cm? **Solution:**

$$v = \frac{eBR}{m} = \frac{(1.6 \times 10^{-19}\,\mathrm{C})(1\,\mathrm{T})(0.5\,\mathrm{m})}{(1.7 \times 10^{-27}\,\mathrm{kg})} = 4.7 \times 10^7\,\mathrm{m/s}$$

- What is the corresponding maximum kinetic energy? **Solution:**

$$W_{max} = \frac{1}{2}mv^2 = 1.9 \times 10^{-12}\,\mathrm{J} \; = 12\,\mathrm{MeV}$$

- If the maximum voltage across the gap is 50 kV, how many revolutions does a proton make before it reaches its maximum energy? **Solution:**

  The proton crosses the gap twice in a full revolution and gains kinetic energy

$$\Delta W = 2eV = (2e)(50\,\mathrm{kV}) = 100\,\mathrm{keV}$$

  The number of revolutions it needs to gain 12 MeV is $n = (12 \times 10^6\,\mathrm{eV})/(100 \times 10^3\,\mathrm{eV}) = 120$.

- How much time does a proton spend in this accelerator? **Solution:**

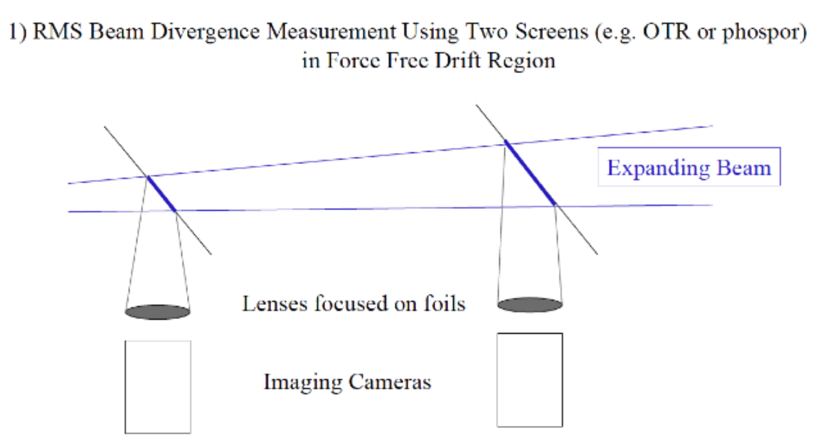
$$t = nT = \frac{n}{f} = \frac{120}{1.5 \times 10^7\,\mathrm{s}^{-1}} = 8\,\mu\mathrm{s}$$

6 OTR

1. Optical Transition Radiation (OTR) is a workhorse for diagnosing a beam's spatial and angular properties. OTR is generated as the beam intercepts a metallic foil. Imaging the beam with OTR provides a high resolution measurement of the spatial distribution of the beam; imaging the beam on two screens separated in a free drift region allows a measure of the beam's rms divergence. Under the proper conditions OTR can thus be used to measure the beam's rms emittance.

   - Assume that the beam is axially symmetric and has equal rms horizontal (x) and vertical (y) divergences. Show how the divergence can be measured by writing a simple equation for divergence in terms of the rms size of the beam as it travels in a free drift space between two screens separated by a distance L. Provide a diagram to illustrate your calculations.

   ### Solution:

   If A and B are the measured rms sizes of the beam at the first and second foils respectively, the divergence is just: $D = (B - A)/L$.

2. The angular distribution of OTR produced from a single foil can also be used to measure the beam divergence. This is due to the fact that the angular distribution of OTR for a single charged particle in the beam has the form:

$$I(\theta) \sim \frac{\theta^2}{(\gamma^{-2} + \theta^2)^2}$$

   where $I(\theta)$ is the intensity of the radiation, θ is the angle measured between the velocity vector of the particle and the direction of observation and γ is the Lorentz factor of the beam. In the absence of beam divergence this pattern has a null in the center. Beam divergence increases the observed minimum value and widens the distribution.

   - Can you qualitatively explain why divergence has this effect? **Solution:**
     If one has two different electrons with different trajectory angles, the OTR angular distribution of the first electron will be shifted in angle space from the second, i.e. the position of the null for the first will be displaced from that of the second. The total radiation pattern from the two electrons will not have a zero value in the center and the total pattern will be widened because the peak angle of emission will be shifted from that of the first as well even if both electrons have the same energy. For an ensemble of electrons with divergence (i.e. a random ensemble of trajectory angles), the total OTR pattern will be filled in the center (where zero angle is now defined by the average particle velocity) and the pattern will be widened as well.
   - How would you go about computing the effect of divergence if e.g. the beam particles had a Gaussian distribution of trajectory angles? **Solution:**
     One can numerically compute the effect of divergence by first modelling the distribution function representing the angular distribution of particle trajectories as a Gaussian characterized by an rms width equal to the rms beam divergence, and then convolving this distribution with the OTR angular distribution. The resulting function will represent the OTR angular distribution for a beam with divergence. One could then compare or fit this calculated distribution to the measured OTR angular distribution to find the divergence.

3. Practically speaking, single foil OTR AD pattern can be used to extract the beam divergence down to a value of about 10% of the angle of peak intensity. Once the rms (x, y) beam sizes and divergences are known, the rms (x, y) geometrical emittances can be calculated; e.g., for x, the x rms geometric emittance is given by

$$\epsilon_x = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle}$$

If the beam has an energy $E = 100$ MeV:

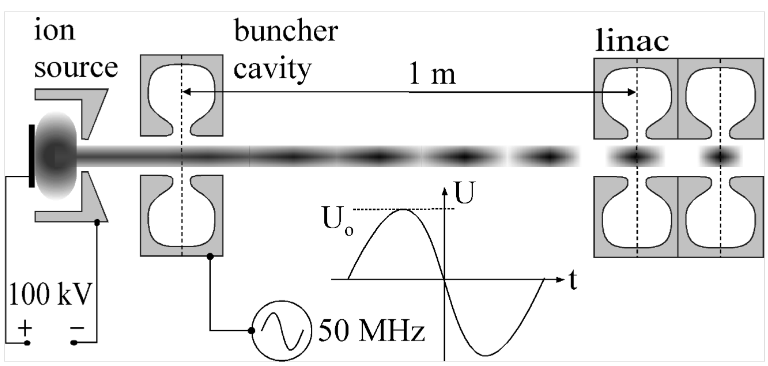
- What is minimum value of the rms x divergence that can be measured with single foil OTR? **Solution:**
First, calculate the peak angle of OTR intensity. This can conveniently be done by writing $I(\theta)$ in the form $I(x) \sim \frac{x^2}{(1+x^2)^2}$, where $x = \gamma\theta$, and setting the derivative of $I(x)$ to 0 to find the maximum value $x = 1$, which gives the angle of peak emission $\theta_p = 1/\gamma$. Single foil OTR can measure divergences down to 10% of this angle. If $E = 100$ MeV, then $\gamma \sim 200$ and $0.1/\gamma \sim 0.5$ mrad.
- If the beam is focussed to an x waist condition and a value of rms $x = 0.1$ mm, what is the corresponding normalized rms x emittance of the beam? **Solution:**
When the beam is at a waist, the correlation term $\langle xx' \rangle = 0$, so that the x geometrical emittance just equal to the product $(x_{rms} \cdot x'_{rms})$. Then the normalized rms x emittance for the divergence $x' = 0.5$ mrad, given above in a), when rms $x = 0.1$ mm is just

$$\epsilon = \beta\gamma x_{rms} \cdot x'_{rms} \sim 10\,\text{mm mrad}$$

Bonus!

7 Ion source and bunching

Assume a continuous proton beam from a 100 kV ion source. A buncher cavity operating at a frequency of 50 MHz modulates the energy of the protons. The corresponding modulation of velocities leads to different times of flight to a location further downstream, and thus to a formation of beam bunches from the initially continuous current. This process is called velocity bunching or ballistic bunching. At a distance $L = 1$ m after the buncher the bunching should be optimum in order to inject the protons into a linac there.



Question: What's the required voltage amplitude U_0 of the buncher cavity?
Hints:

- What's the proton beam velocity?
- Is it allowed to do a non-relativistic calculation?
- Assume that the buncher is "short", i.e. any variation of fields inside the buncher may be neglected.
- Calculate time of flight for a proton at time δt with respect to a reference particle.
- Introduce approximations:

  a) linearize the time dependance ($\sin \omega t$) of the electric field in the buncher.

  b) Assume the energy change due to the buncher is small compared to the initial energy.

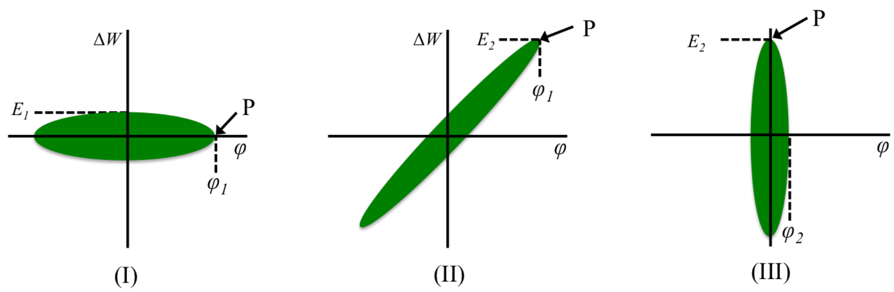
Solution:

The kinetic energy given to the particle generated by this ion source (proton) with potential difference of 100 kV is (using relativistic expressions):

$$W = (\gamma - 1)m_0c^2 = 100\,\text{keV}$$

where $\gamma = 1/\sqrt{1-\beta^2}$ is the Lorentz factor and $\beta = v/c$ the relativistic ratio of the particle speed v to the speed of light c. Then, we obtain $\gamma = 1.0001$ and $\beta \simeq 0.014$. Therefore, since $\gamma \approx 1$ and $\beta \ll 1$, in this case the non-relativistic approximation is valid.

The initial bunch arrives at the buncher cavity, whose peak effective voltage is U_0. Here, for simplicity we will assume a thin cavity located at the gap center of the real cavity. The ballistic bunching and compression process can be illustrated through a simple model, represented schematically in the figure below:



(I) We approximate the initial beam, which may in reality have a uniform continuous distribution, as a sequence of bunches each with upright ellipses in longitudinal phase space, whose total phase width for the ellipse that we will follow may be chosen equal to 2π. The initial bunch arrives at the buncher cavity, whose peak effective voltage is U_0. Here, for simplicity we will assume a thin cavity located at the gap center of the real cavity.

Assume that the semi-axes of the phase-energy ellipse are φ_1 (phase) and E_1 (energy). The reference particle is at the origin $\varphi = 0$ and $\Delta W = 0$. For a cavity to perform as a buncher, the reference particle arrives at the gap when voltage is rising in time and is zero, so there will be essentially zero average energy gain. We will follow the point P that has initial phase-space coordinates $(\varphi_{1,}\Delta W = 0)$.

(II) The buncher cavity delivers a phase-dependent kick which changes the upright ellipse to a tilted ellipse. Now point P has coordinates $(\varphi_{1,}E_2)$ with respect to the reference particle.

Before the kick the phase-space coordinates of particle P are $(\varphi_{1,}0)$. After the kick its coordinates are $(\varphi_{1,}E_2)$, where:

$$E_2 = qU_0 \sin(\omega t) \simeq qU_0\varphi_1$$

with $\varphi_1 = \omega t_1$.

(III) After the buncher cavity, the beam propagates in a drift space of length L and arrives at the phase focus (defined downstream, at the middle of the first Linac cavity), where the ellipse is upright again with a compressed width. The particle P has coordinates $(0, E_2)$. Since we have assumed a thin buncher cavity, where any variation of fields can be neglected and practically no energy variation with respect to the initial energy, the phase-space area occupied by the beam is conserved, so $E_1 \cdot \varphi_1 = E_2 \cdot \varphi_2$. Therefore, assuming the initial semi-axes φ_1 and E_1 are known, the phase width of the bunch at the phase focus is given by this phase-space area conservation:

$$\varphi_2 = \frac{E_1 \cdot \varphi_1}{E_2} \simeq \frac{E_1}{qU_0}$$

The phase change of particle P relative to the reference particle at the origin determines the location L of the phase focus relative to the cavity. The time difference for two particles with velocity difference $\delta\beta$ to travel a distance L between the cavity and the phase focus (located at the middle of the first Linac cavity) is:

$$\delta t = -\frac{L}{c}\frac{\delta\beta}{\beta^2} = \frac{L\delta W}{m_0c^3\beta^3\gamma^3}$$

where we have taken into account the derivative of the energy $\delta W = \delta(\gamma)m_0c^2 = -\beta m_0c^2\gamma^3\delta\beta$. For particle P, the variation of energy $\delta W = E_2 - E_1 = E_2$, and we can write

$$\delta\varphi = -\varphi_1 = -\omega\delta t = -2\pi f_{RF}L\frac{E_2}{m_0c^3\beta^3\gamma^3} = -2\pi f_{RF}L\frac{qU_0\varphi_1}{m_0c^3\beta^3\gamma^3}$$

Solving for U_0,

$$U_0 = \frac{m_0c^3\beta^3\gamma^3}{2\pi qLf_{RF}}$$

As indicated above, we can apply the non-relativistic calculation

$$U_0 \simeq \frac{m_0v^3}{2\pi qLf_{RF}}$$

Using the parameters given above, we obtain $U_0 \simeq 2.5$ kV.