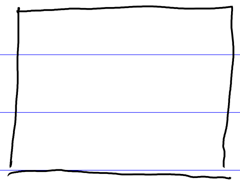
SSHEP – SARAJEVO 2017

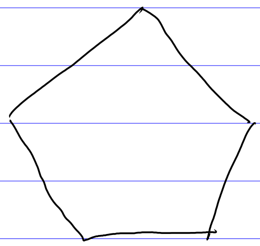
Symmetries in Physics

(K. Kumerički, Zagreb)

Which is more symmetric:



or

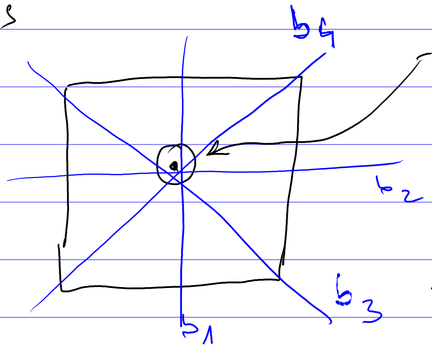


?

Intuition can be misleading. We need formal definition to be able to quantify symmetry.

Symmetry is a transformation that leaves the object unchanged. (Not very formal, but we can work with that.)

So, let's count symmetries, restricting ourselves to rotations



4 rotations about this axis:
$R(90^\circ) \equiv c,\ c^2,\ c^3,\ c^4 = R(0^\circ) \equiv e$

$b_1 \ldots b_4$ – rotations for 180°

TOTAL: 8 rotations

While pentangle has 10 and so is more symmetric

To really formalize the idea of symmetry we use a branch of mathematics known as group theory.

M. A. Armstrong: "Numbers measure size, groups measure symmetry"

Note that the set

$$G = \{g_1, g_2, \ldots\}$$

of all symmetry transformations of a given object has following properties:

1. $g_1, g_2 \in G \implies g_1 \circ g_2 \in G$

   (composition of two symmetry transformation is also a symmetry)

2. $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$

3. Transformation that does nothing (identity $\equiv e$) is always member of G.

4. $\forall g \in G \quad \exists g^{-1} \in G \mid g \circ g^{-1} = g^{-1} \circ g = e$

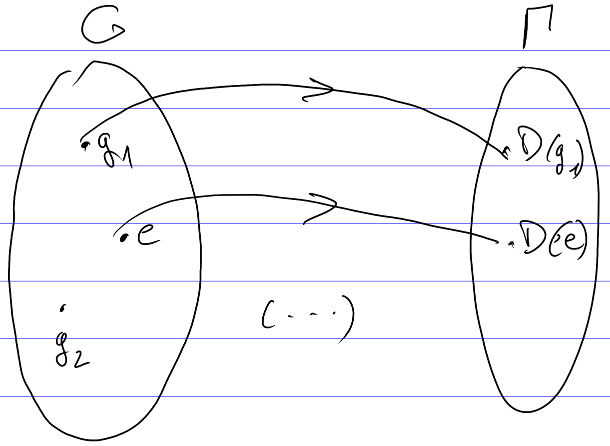
   (For each symmetry there is inverse transformation that is also a symmetry.)

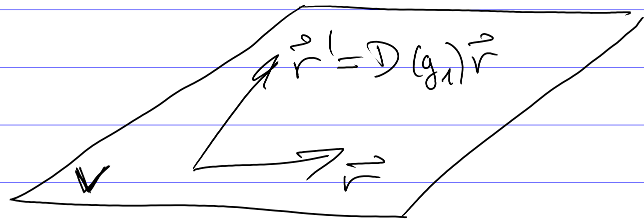
Abstract set of objects (not necesserly transformations with binary operation ($\circ$, not necesserily composition) satisfying the axioms above is called a group.

Each abstract group may have many different representations which are realizations of a group in terms of set of operators

$$\Gamma = \{D(g_1), D(g_2), \ldots\}$$

which operate on some vector space V.





$$D(g_1) \cdot D(g_2) = D(g_1 \cdot g_2)$$

Multiplication of operators has to have some structure as multiplication on the group. ⇒ homomorphism

In physics, V can often be Hilbert space and vectors are quantum states of some system, while $D(g)$ will be symmetry transformations on this system like e.g.

- rotation R
- Lorentz transformation Λ
- charge conjugation C
- change of particle type $|e^-\rangle \leftrightarrow |\nu\rangle$
- etc.

Note that most of the actual states are not symmetric under transformations listed above.

(Take e.g. state of electron at rest and consider what these transformations do to it.)

BTW, vacuum state $|0\rangle$ is quite symmetric, which is a non-trivial statement in quantum field theory (QFT) where vacuum has rich structure.

Also, vacuum is not symmetric under standard model (SM) symmetry group $SU(3)_C \times SU(2)_L \times U(1)_Y$.

Interactions (Lagrangian, action) are often symmetric, which has important consequences:

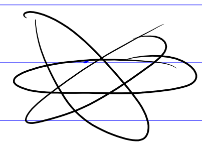
1. Set of all in nature possible states is symmetric. If you observe some state, transformed state should also occur in nature.

2. time evolution of a given state is constrained → Noether theorem → conservation laws

Example 1: Earth – Sun

– although governed by rotationally invariant Newton laws, actual orbit is not rotationally symmetric (because initial conditions aren't).

– but all orbits are possible.

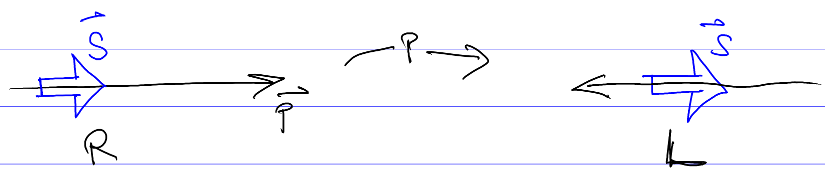


– $\vec{L}$ is conserved, trajectory is planar.

Example 2 parity $P: \vec{r} \to -\vec{r}$

$\vec{p} \to -\vec{p}$

$\vec{L} = \vec{r} \times \vec{p} \longrightarrow \vec{L}$ (axial vector)



i.e. Parity transforms left-hand (L) particles into right-hand ones (R).

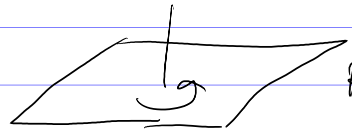
But in β^+ decay, neutrino is always ν_L → P violation

(For simplicity, we ignore chirality vs. helicity distinction.)

Example 3 CPT symmetry of QFT ⇒ for each particle there is antiparticle of exactly the same mass.

From now on, we concentrate on groups which have infinite number of elements, which can be mapped to points in a space known as group space or group manifold.

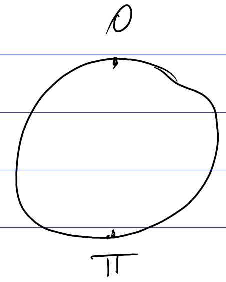
Example 1 – rotations in 2D plane



$R(\vartheta)$

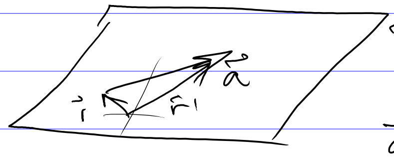
$\vartheta \in [0, 2\pi) \in \mathbb{R}$

group parameter



group space is circle

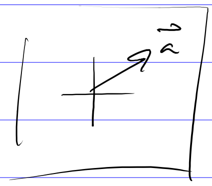
Example 2 – translations in 2D



$T(\vec{a}): \vec{r} \to \vec{r} + \vec{a}$

$\vec{a} = (a_x, a_y)$

two parameters



group space is 2D plane

Additional structure is brought by the fact that group elements (and their composition) are smooth/continuous/analytic function of parameters.

→ Lie groups

We can then use infinitesimal calculus, and do Taylor expansion for small value of parameters. E.g. for 2D rotations: $(\varepsilon \ll 1)$

$$R(\varepsilon) = R(0) + \varepsilon \underbrace{\left.\frac{\partial R(\vartheta)}{\partial \vartheta}\right|_{\vartheta=0}}_{\equiv X} + O(\varepsilon^2)$$

group generator: $\equiv X$ (independent of ϑ)

$$R(\vartheta) = \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix}$$

$$X = \frac{\partial R(\vartheta)}{\partial \vartheta}\bigg|_{\vartheta=0} = \begin{pmatrix} -\sin\vartheta & -\cos\vartheta \\ \cos\vartheta & -\sin\vartheta \end{pmatrix}_{\vartheta=0} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Finite transformations can be constructed as infinite composition of infinitesimal ones:

$$R(\vartheta) = \underbrace{R(\varepsilon)\cdot R(\varepsilon)\cdot \ldots R(\varepsilon)}_{N\times} = R(\varepsilon)^N$$

$$= (1+\varepsilon X)^N = \left\{ \begin{array}{l} \vartheta = N\varepsilon \to \text{const} \\ N \to \infty \\ \varepsilon \to 0 \end{array} \right\}$$

$$= \left(1+\frac{\vartheta}{N}X\right)^N \longrightarrow e^{\vartheta X}$$

Group elements are obtained by exponentiation of generators.

<u>Exercise</u>

Show: $e^{\vartheta X} = \sum_{k=0}^{\infty} \frac{\vartheta^k}{k!} X^k = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\vartheta + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\sin\vartheta$

as it must be.

So, infinitesimal generators (X) determine the whole group.

For other groups this is also largely true.

(Some subtleties involve topology.)

Going from 2D to 3D rotations:

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \longrightarrow X_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$R(\hat{n}, \vartheta) = \exp(\vartheta \hat{n} \cdot \vec{X})$$

axis of rotation ($\hat{n}$), angle of rotation (ϑ)

$$[X_1, X_2] \equiv X_1 X_2 - X_2 X_1 = \underbrace{(\ldots)}_{\text{exercise}} = X_3$$

generally:
$$[X_i, X_j] = \varepsilon_{ijk} X_k \tag{1}$$

ε_{ijk}: Levi-Civita tensor

$\varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} = 1$

$\varepsilon_{321} = \varepsilon_{213} = \varepsilon_{132} = -1$

$\varepsilon_{ijk} = 0$ if any two of (i,j,k) equal

Exercise:

Note that $(X_k)_{ij} = -\varepsilon_{ijk}$. Using identity

$$\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$

derive commutation relations $[X_i, X_j] = \varepsilon_{ijk} X_k$

Note that we do group theory and not quantum physics. So why are commutation relations interesting?

Consider composition of two group elements:

$$R(\hat{n}_1, \vartheta_1) \cdot R(\hat{n}_2, \vartheta_2) = e^{\underset{\equiv A}{\vartheta_1 \hat{n}_1 \cdot \vec{X}}} \, e^{\vartheta_2 \hat{n}_2 \cdot \vec{X}} \equiv B$$

and expand for small angles:

$$= \left(1 + A + \frac{1}{2}A^2 + \dots\right) \cdot \left(1 + B + \frac{1}{2}B^2 + \dots\right)$$

$$= 1 + A + B + \frac{1}{2}\left(A^2 + B^2 + 2AB\right) + \dots$$

$$+ BA - BA$$

$$= 1 + A + B + \underbrace{\frac{1}{2}\left(A^2 + B^2 + AB + BA\right)}_{(A+B)^2} + \frac{1}{2}\underbrace{\left(AB - BA\right)}_{[A,B]} + \dots$$

$$= e^{A + B + \frac{1}{2}[A,B]}$$ to considered order

It turns out (Baker-Campbell-Hausdorff formula) that higher orders also include only commutators $[A,B]$, $[A,[A,B]]$, ... etc.

so group operations are specified by commutators of generators.

$$[X, Y] = (\text{linear combination of generators})$$

$\downarrow$ Lie algebra

In QM we extract $-i$ from the $\vec{X} = -i\vec{J}$ and call $\vec{J}$ also a generator. For unitary transformations (such as rotations) $\vec{J}$ is Hermitean and is observable.
(In case of rotations it is operator of angular momentum.)

Using <u>only</u> commutation relations (1) from p.7 one can show (see any book from the list at the end of these lectures) that possible irreducible*) representations of rotation group are on $(2j+1)$–dimensional vector spaces spanned by eigenvectors of J_z.

$$\underbrace{|j, m=j\rangle, |j, m=j-1\rangle, \ldots, |j, m=-j\rangle}_{2j+1}$$

$$J_z|j\,m\rangle = m|j\,m\rangle$$

where $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ are labels of these representations called „spins".

Note that no knowledge of quantum physics is needed for this construction, only Lie algebra of rotation group!

Also, explicit matrix representations can be constructed

$$j=0 \;:\; J_j = (1)$$

$$j=\frac{1}{2} \;:\; J_j = \frac{1}{2}\sigma_j$$

σ_j – Pauli matrices

$j=1$: J_j – combination of 3D matrices on previous pages

$(\ldots)$

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*) irreducible = cannot be decomposed into a direct sum of lower dimensional representations

Lorentz group

$c = 1$

Lorentz boost: $r^\mu \to \Lambda(\vec{v})^\mu{}_\nu \, r^\nu$

$v_i \in [0, c) \qquad i = 1,2,3$

(three parameters)

Boost along z-axis: $\vec{v} = v\hat{z}$

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -v\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -v\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \qquad \gamma = \frac{1}{\sqrt{1-v^2}}$$

generator:

$$X_3 = \frac{\partial}{\partial v}\Big(\quad\Big)_{v=0} = \begin{pmatrix} \frac{-v}{(1-v^2)^{3/2}} & 0 & 0 & \left(\frac{v^2}{(1-v^2)^{3/2}} - \frac{1}{\sqrt{1-v^2}}\right) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\text{''}- & 0 & 0 & -\text{''}- \end{pmatrix}_{v=0}$$

$$= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \equiv -i K_3$$

$$K_2 = i \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & & & \\ -1 & & 0 & \\ 0 & & & \end{pmatrix} \qquad K_1 = i \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & & & \\ 0 & & 0 & \\ 0 & & & \end{pmatrix}$$

How does the Lie algebra looks like?

$$[K_1, K_2] = -i J_3 \qquad \text{(check!)} \qquad \underset{\substack{\uparrow \\ 4D}}{J_3} = \begin{pmatrix} 0 & 0\;0\;0 \\ \begin{matrix}0\\0\\0\end{matrix} & \underset{\substack{\uparrow \\ 3D}}{J_3} \end{pmatrix}$$

and, more generally,

$$[K_i, K_j] = -i \varepsilon_{ijk} J_k$$

so the algebra of just Lorentz boosts is not closed and we need to include rotations

Rest of the algebra is now closed:

$$[J_i, K_j] = i \varepsilon_{ijk} K_k$$

$$[J_i, J_j] = i \varepsilon_{ijk} J_k \qquad \text{(as we know)}$$

So full Lorentz group has 6 generators and 6 parameters, and rotations are subgroup.

To see what are representations of Lorentz group, we define

$$\vec{A} \equiv \frac{1}{2}(\vec{J} + i\vec{K}) \quad , \quad \vec{B} \equiv \frac{1}{2}(\vec{J} - i\vec{K})$$

In terms of $\vec{A}$ and $\vec{B}$ Lorentz algebra becomes (check):

$$[A_i, A_j] = i \varepsilon_{ijk} A_k$$

$$[B_i, B_j] = i \varepsilon_{ijk} B_k$$

$$[A_i, B_j] = 0$$

so it decomposes into two <u>separate</u> rotation algebras.

So, representations can be labeled by pairs (j_A, j_B) and are $(2j_A+1)(2j_B+1)$-dimensional.

$(0,0)$ - scalar rep. $J_i, K_i = (1)$ 1D

$(0,\frac{1}{2})$ - left Weyl rep. $J_i = \frac{\sigma_i}{2}$, $K_i = \frac{i\sigma_i}{2}$ 2D

$(\frac{1}{2},0)$ - right Weyl rep $J_i = \frac{\sigma_i}{2}$ $K_i = -\frac{i\sigma_i}{2}$ 2D

$(\frac{1}{2},\frac{1}{2})$ - 4-vector rep. J_i, K_i - see previous pages 4D

$(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ - Dirac rep. $J^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ 4D

$$\left(\begin{array}{l} J^{mn} = \varepsilon^{mni} J^i \\ J^{i0} = K^i \end{array}\right)$$

can be used to write also other representations in relativistically covariant form, where all three commutation relations of Lorentz group become one:

$$i[J^{\mu\nu}, J^{\rho\sigma}] = g^{\mu\rho} J^{\nu\sigma} - g^{\nu\rho} J^{\mu\sigma} + g^{\mu\sigma} J^{\rho\nu} - g^{\sigma\nu} J^{\rho\mu}$$

<u>Exercise</u> check that 4-vector 4D representation can be written in the form

$$(J^{\mu\nu})^\alpha{}_\beta = i\left(g^{\mu\alpha} g^\nu{}_\beta - g^\nu{}_\beta\, g^{\nu\alpha}\right)$$

Include space-time translation symmetries:

$$x^\mu \to x^\mu + a^\mu$$
↑ 4 parameters

4 generators: P^μ $\quad P^0 = H, \vec{P}$

$$[P^\mu, P^\nu] = 0$$
$$i[P^\mu, J^{\rho\sigma}] = g^{\mu\sigma} P^\rho - g^{\mu\rho} P^\sigma$$

→ 10 parameters Poincare group

Can we go beyond that?

Consider dilatations: $x^\mu \to \lambda x^\mu$
↑ 1 parameter

1 generator, call it D

This is not a symmetry of our Nature, because it would imply continuous spectrum of particle masses:
$$p^\mu \to \lambda p^\mu \Rightarrow m^2 = p^2 \to \lambda^2 m^2 \quad \forall \lambda \ \lightning$$

But this is a symmetry of some theories of theoretical interest, like (classical) massless chromodynamics.

$$[P^\mu, D] = i P^\mu$$
$$[J^{\mu\nu}, D] = 0$$

So algebra closes. However, in QFT dilatation symmetry often implies symmetry under:

$x^\mu \to x'^\mu$ where

$$\frac{x'^\mu}{x'^2} = \frac{x^\mu}{x^2} + b^\mu \quad \text{– special conformal transf.}$$

b^μ – 4 parameters

$\Rightarrow$ 4 generators $\equiv K^\mu$

$\Rightarrow$ 15 parameters conformal group

And this is the end of the story.

Some theories can be symmetric under transformation generated by „fermionic" generators satisfying „graded" Lie algebra, involving anti-commutators:

$$\{Q_\alpha, \overline{Q}_\beta\} = 2(\sigma^\mu)_{\alpha\beta} P_\mu$$

$$Q|\text{fermion}\rangle = |\text{boson}\rangle \quad \to \text{supersymmetry}$$

Literature

1. H.F. Jones, Groups, Representations and Physics

2. J.J. Sakurai, Modern Quantum Mechanics

3. K. Kumerički, Grupe, simetrije i tenzori u fizici (in Croatian)

4. J. Schwichtenberg, Physics from Symmetry