

Kaluza-Klein spectrum
and altered dispersion relations of
fermions
in asymmetrically warped
five dimensional spacetimes

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Motivation

- Theoretically compelling hints towards sterile neutrinos
(e.g. seesaw mechanism)
- SM-gauge-singlets can enter the extra dimension in RS-like models
- LSND hints towards sterile neutrinos
→ Can it be resolved with asymmetric warping?
(see e.g. [S.Hollenberg, O.Micu, H.Päs, *Prog.Part.Nucl.Phys.* 64 (2010) 193-195])

Main goal:

- Asymmetric warping scenario → shape of the Kaluza-Klein tower ?
- Asymmetric warping → dispersion relation on the 3-brane?

General Set-Up

- General warped 5D geometry, including compactified extra dimension
- x^0 and x^i coordinates are warped differently \rightarrow 'asymmetric warping'

5D metric $G_{MN} = \begin{pmatrix} -A^2 & & & & \\ & B^2 & & & \\ & & B^2 & & \\ & & & B^2 & \\ & & & & (rC)^2 \end{pmatrix}$

- Warp factors are functions of ϕ (e.g. $A = A(\phi)$)
- S^1/Z_2 -orbifolding ($\phi = \phi + 2\pi$, $\phi = -\phi$)

- 5D action of Dirac-fermion field Ψ in the bulk

[M. Neubert, Y. Grossman, *Phys. Lett. B* 474 (2000), S. 361–371]

$$S = \int d^4x \int d\phi \sqrt{\det G} \left\{ E_a^A \left[\frac{i}{2} \bar{\Psi} \gamma^a \left(\partial_A - \overset{\leftrightarrow}{\partial}_A \right) \Psi \right] - m \operatorname{sgn}(\phi) \bar{\Psi} \Psi \right\} + \dots$$

- $\Psi_{1/2} = \frac{1}{2}(1 \mp \gamma^5)\Psi \Rightarrow$

$$S = \int d^4x \int d\phi \sqrt{\det G} \left\{ \bar{\Psi}_{1/2} i \left[\frac{\gamma^0}{A} \partial_0 + \frac{\gamma^k}{B} \partial_k \right] \Psi_{1/2} \right\} + \bar{\Psi}_{1/2} \mathcal{O}(\partial_\phi, m, C) \Psi_{2/1}$$

- Expand 5D spinors into series of 4D brane spinors $\psi_n^{1/2}(x^\mu) \times$ eigenfunctions $\hat{f}_n^{1/2}(\phi)$ of hermitian operator yet to determine

$$\Psi_{1/2} = \sum_n \psi_n^{1/2}(x^\mu) \frac{1}{\sqrt{2r\xi}} \hat{f}_n^{1/2}(\phi) \quad \text{with} \quad \xi = \xi(\phi) = \frac{\sqrt{\det G}}{2rC}$$

Kaluza-Klein decomposition

- Integrating out 5'th dimension: Compare 4D projection with $S_{\text{Dirac}} \Rightarrow$

$$S_{\text{DiracCorr}} = \sum_n \int d^4x \left\{ \overline{\psi_n}(x) \left(i\not{\partial} - M_n + \hat{\Omega} \right) \psi_n(x) \right\}$$

with conditions

$$\int d\phi \hat{f}_n^{1/2} \frac{C}{A} \hat{f}_m^{1/2\dagger} = \delta_{nm} \quad \wedge \quad \left(\mp \frac{\partial_\phi}{r} - mC \right) \hat{f}_m^{2/1} = -M_m \frac{C}{A} \hat{f}_m^{1/2}$$

Result:

- No ' B ' \rightarrow asymmetry irrelevant for KK spectrum
- Solving differential equations + Z_2 -symmetry \rightarrow shape of the KK spectrum
- Asymmetric warping induces alteration of the dispersion relation

Extracting the Altered Dispersion Relation (ADR)

- Closer look at S_{kin} after KK decomposition

$$S = \int d^4x \sum_n \left\{ \mathcal{L}_{\text{Dirac}_n} + \sum_m \sum_{j=1}^2 \left[\underbrace{\overline{\psi_n^j} \int d\phi \hat{f}_n^{j\dagger} \left(\frac{C}{B} - \frac{C}{A} \right) \hat{f}_m^j}_{{= \tilde{I}_{nm}^j \triangleq \text{correction integral}}} i \gamma^k \partial_k \psi_m^j \right] \right\}$$

- New dispersion relation on the brane

$$\Rightarrow E^2 = \left(1 + \tilde{I}_{nn} \right) \vec{p}^2 + M_n^2$$

- \tilde{I}_{00} calculated analytically

Oscillations between Bulk- and Brane-Neutrino

- Expand dispersion relation in $\tilde{I}_{nn} \ll 1$

$$\begin{aligned}\tilde{E} &= \sqrt{\vec{p}^2 + M_n^2 + \tilde{I}_{nn} \vec{p}^2} \approx \sqrt{\vec{p}^2 + M_n^2} + \frac{\vec{p}^2 \tilde{I}_{nn}}{2\sqrt{\vec{p}^2 + M_n^2}} \\ &:= E + \frac{\chi}{2E} \quad \text{with } \chi = \vec{p}^2 \tilde{I}_{nn}\end{aligned}$$

- χ treated as effective potential
 (as in [H.Päs, S.Pakvasa, T.J.Weiler, *Phys. Rev. D*72(2005), S. 095017])
- New effective $\sin^2 2\theta$ and Δm^2 in $P_{\alpha \rightarrow \beta}$

$$\sin^2 2\tilde{\theta} = \frac{\sin^2(2\theta)}{\sin^2(2\theta) + \cos^2(2\theta) \left(\frac{\chi}{\Delta m^2 \cos(2\theta)} - 1 \right)^2}$$

$$\Delta \tilde{m}^2 = \Delta m^2 \sqrt{\left(\frac{\chi}{\Delta m^2} - \cos(2\theta) \right)^2 + \sin^2(2\theta)}$$

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Back Slides

4D projection

- Before decomposition

$$S_{kin} = \int d^4x \int d\phi \sqrt{\det G} \left\{ \bar{\Psi}_1 i \left[\frac{\gamma^0}{A} \partial_0 + \frac{\gamma^k}{B} \partial_k \right] \Psi_1 + \bar{\Psi}_2 i \left[\frac{\gamma^0}{A} \partial_0 + \frac{\gamma^k}{B} \partial_k \right] \Psi_2 \right\}$$

$$\begin{aligned} S_{lower} = \int d^4x \int d\phi & \left\{ - \left[\bar{\Psi}_1 \left(\frac{\sqrt{\det G}}{2rC} \partial_\phi + \partial_\phi \frac{\sqrt{\det G}}{2rC} \right) \Psi_2 \right. \right. \\ & \left. \left. - \bar{\Psi}_2 \left(\frac{\sqrt{\det G}}{2rC} \partial_\phi + \partial_\phi \frac{\sqrt{\det G}}{2rC} \right) \Psi_1 \right] \right. \\ & \left. - \sqrt{\det G} m \operatorname{sgn}(\phi) [\bar{\Psi}_1 \Psi_2 + \bar{\Psi}_2 \Psi_1] \right\} \end{aligned}$$

■ After decomposition

$$\begin{aligned}
 S_{\text{kin}} = & \int d^4x \int d\phi \sum_n \sum_m \left\{ \frac{C}{A} \left[\overline{\psi_n^1} \hat{f}_n^{1\dagger} i \left(\not{\partial} + \frac{(A-B)\gamma^k}{B} \partial_k \right) \psi_m^1 \hat{f}_m^1 \right. \right. \\
 & \left. \left. + \overline{\psi_n^2} \hat{f}_n^{2\dagger} i \left(\not{\partial} + \frac{(A-B)\gamma^k}{B} \partial_k \right) \psi_m^2 \hat{f}_m^2 \right] \right\} \\
 S_{\text{tower}} = & \int d^4x \int d\phi \sum_n \sum_m \left\{ \overline{\psi_n^1} \hat{f}_n^{1\dagger} \left(-\frac{\partial_\phi}{r} - m \operatorname{sgn}(\phi) C \right) \psi_m^2 \hat{f}_m^2 \right. \\
 & \left. + \overline{\psi_n^2} \hat{f}_n^{2\dagger} \left(+\frac{\partial_\phi}{r} - m \operatorname{sgn}(\phi) C \right) \psi_m^1 \hat{f}_m^1 \right\}
 \end{aligned}$$

Backup Slides

calculation of the spectrum

- Assuming a set of well-defined functions

$$A(\phi) = e^{-\frac{1}{2}kr\phi}$$

$$B(\phi) = e^{-\frac{1}{2}\tilde{k}r\phi}$$

$$C(\phi) = 1$$

- New dimensionless variables ([3],[8])

$$t := e^{kr(\phi - \pi)} := \epsilon e^{kr\phi} \in [\epsilon, 1]$$

$$q := \frac{m}{k}$$

$$X_n := \frac{M_n}{\epsilon k}$$

- Rescale $\hat{f}_n^{1/2} \rightarrow \sqrt{\epsilon kr} f_n^{1/2}$

■ Conditions in new variables

$$\int dt f^{1/2\dagger}(t) f_m^{1/2}(t) = \delta_{nm}$$

$$(\pm t\partial_t - q) f_n^{1/2}(t) = -X_n t f_n^{2/1}(t)$$

■ Zero mode decouples the system:

$$\Rightarrow f_0^{1/2} = \hat{K}_{1/2} t^{\pm q}$$

■ Diff. Eq. for Non-zero modes → maths → Bessel's Diff. Eq.

■ Solutions to the equations

$$f_n^1(t) = \sqrt{t} \left[\alpha_n^1 J_{\frac{1}{2}-q}(X_n t) + \alpha_n^2 J_{-\frac{1}{2}+q}(X_n t) \right]$$

$$f_n^2(t) = \sqrt{t} \left[\alpha_n^2 J_{\frac{1}{2}+q}(X_n t) - \alpha_n^1 J_{-\frac{1}{2}-q}(X_n t) \right]$$

Boundary Conditions and the Spectrum

- Boundaries of the orbifold
- Z_2 -behavior (set manually for Ψ_1 and Ψ_2)
- Two options:
 1. option $Z_2(\Psi_1) = \Psi_1, Z_2(\Psi_2) = -\Psi_2$
 2. option $Z_2(\Psi_2) = \Psi_2, Z_2(\Psi_1) = -\Psi_1$

$$\Rightarrow f_n^{1/2}(t=1) = f_n^{1/2}(t=\epsilon) = 0, \quad \text{with} \quad Z_2(\Psi_{2/1}) = +\Psi_{2/1}$$

- Example: Option 1 (Option 2 then is analogue)

$$\left[\alpha_n^2 J_{\frac{1}{2}+q}(X_n) - \alpha_n^1 J_{-\frac{1}{2}-q}(X_n) \right] = 0$$

$$\sqrt{\epsilon} \left[\alpha_n^2 J_{\frac{1}{2}+q}(X_n \epsilon) - \alpha_n^1 J_{-\frac{1}{2}-q}(X_n \epsilon) \right] = 0$$

- Approximation: $X_n \epsilon \ll 1$

$$J_\mu(X_n \epsilon) \sim (X_n \epsilon)^\mu$$

- \Rightarrow Coefficient $\alpha_n^1 = 0$
- \Rightarrow KK spectrum: $J_{\frac{1}{2}+q}(X_n) = 0$

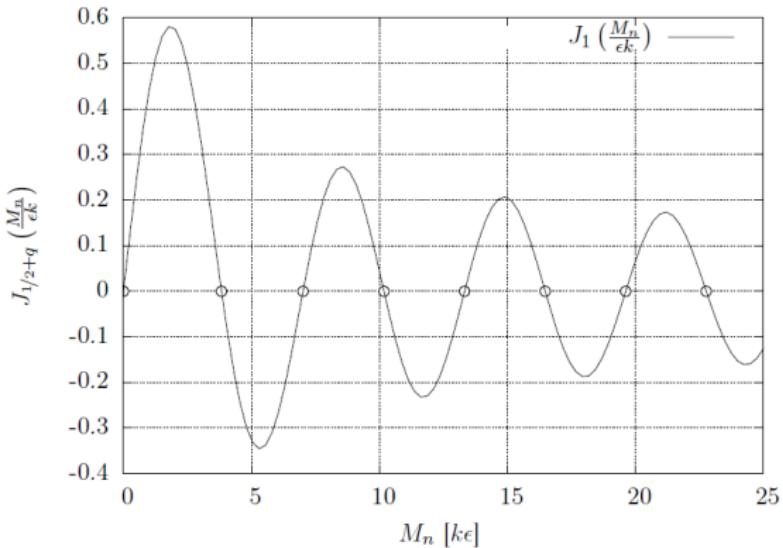


Figure: The Kaluza-Klein spectrum M_n displayed as the roots of the Bessel function $J_{\frac{1}{2}+q}\left(\frac{M_n}{k\epsilon}\right)$ in units of $k\epsilon$, exemplarily for $q = 0.5$.

Backup slides

Calculation of the ADR

- Asymmetric warping affects the alteration of the dispersion relation
- Responsible term is the $\hat{\Omega}$ -term
- Expectation: $\hat{\Omega}(A, B)$
- Closer look at S_{kin} after KK decomposition

$$S = \int d^4x \sum_n \left\{ \overline{\psi_n} (i\not{\partial} - M_n) \psi_n + \underbrace{\int d\phi \sum_m \sum_{j=1}^2 \left[\overline{\psi_n^j} \hat{f}_n^{j\dagger} \left(\frac{C}{B} - \frac{C}{A} \right) i \gamma^k \partial_k \psi_m^j \hat{f}_m^j \right]}_{=\text{correction term } S_{\text{corr}}} \right\}$$

- The correction term with $\int d\phi \hat{f}_n^{1/2} \frac{C}{A} \hat{f}_m^{1/2\dagger} := \delta_{nm}$ reads

$$S_{corr} = \int d^4x \sum_{n,m} \sum_{j=1}^2 \left[\overline{\psi_n^j} \underbrace{\int d\phi \hat{f}_n^{j\dagger} \frac{C}{B} \hat{f}_m^j}_{l_{nm}^j(x)} i \gamma^k \partial_k \psi_m^j - \overline{\psi_n} \delta_{nm} i \gamma^k \partial_k \psi_m \right]$$

- Deviation from unity

$$l_{nm}^j = \delta_{nm} + \tilde{l}_{nm}^j$$

- Handy expression

$$S_{corr} = \int d^4x \sum_{n,m} \sum_{j=1}^2 \left[\overline{\psi_n^j} \tilde{l}_{nm}^j i \gamma^k \partial_k \psi_m^j \right]$$

- The whole action reads

$$S = \int d^4x \sum_n \left[\bar{\psi}_n \left(i\partial - M_n + i\tilde{L}_{nn} \gamma^k \partial_k \right) \psi_n \right]$$

- The dispersion-relation can be calculated out of the whole action S via variation $\delta S = 0$.

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}_n} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \bar{\psi}_n)} = 0 \quad \text{with } S = \int d^4x \mathcal{L}$$

- It can written as

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}_n} = \left(i\partial - M_n + i\tilde{L}_{nn} \gamma^k \partial_k \right) \psi_n = 0$$

- Expressing this equation in terms of momenta (Fourier transformation from position- to momentum space $\Rightarrow i\partial_\mu = p_\mu$) it is

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}_n} = \left(\not{p} - M_n + \tilde{I}_{nn} \gamma^k p_k \right) \psi_n(p) = 0$$

- Using the dirac representations for the γ -matrices

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}.$$

- It is obvious, that an eigenvalue equation with eigenvalue M_n arises, that reads

$$\underbrace{\begin{pmatrix} p_0 \mathbb{1} & (\tilde{I}_{nn} p_k + p_k) \sigma^k \\ -(\tilde{I}_{nn} p_k + p_k) \sigma^k & -p_0 \mathbb{1} \end{pmatrix}}_{:= \mathbb{F}} \psi_n(p) = M_n \psi_n(p).$$

- To solve this problem, the equation $\det(\mathbb{F} - M_n \mathbb{1}) = 0$ has to be solved. Since $\mathbb{F} - M_n \mathbb{1}$ is in block matrix form :

$$\begin{aligned}\det \begin{pmatrix} \mathbb{V} & \mathbb{W} \\ \mathbb{X} & \mathbb{Y} \end{pmatrix} &= \det \left[\underbrace{\begin{pmatrix} \mathbb{V} & 0 \\ \mathbb{X} & \mathbb{1} \end{pmatrix}}_{:=\mathbb{A}} \underbrace{\begin{pmatrix} \mathbb{1} & \mathbb{V}^{-1}\mathbb{W} \\ 0 & \mathbb{Y} - \mathbb{X}\mathbb{V}^{-1}\mathbb{W} \end{pmatrix}}_{:=\mathbb{B}} \right] && \text{, if } \mathbb{V} \text{ is invertible} \\ &= \det(\mathbb{A}) \det(\mathbb{B}) \\ &= \det(\mathbb{V}) \det(\mathbb{Y} - \mathbb{X}\mathbb{V}^{-1}\mathbb{W}).\end{aligned}$$

- Applied on our problem \Rightarrow

$$\begin{aligned}\det(\mathbb{F} - M_n \mathbb{1}) &= \det(\mathbb{V}) \det(\mathbb{Y} - \mathbb{X}\mathbb{V}^{-1}\mathbb{W}) \\ &= \det[(p_0 - M_n) \mathbb{1}] \cdot \det \left[-(p_0 + M_n) \mathbb{1} - \frac{-(\tilde{l}_{nn} + 1)^2 (p_k \sigma^k)^2}{p_0 - M_n} \right] \\ &= \det[(p_0 - M_n) \mathbb{1}] \cdot \det \left[-(p_0 + M_n) \mathbb{1} + \frac{(\tilde{l}_{nn} + 1)^2 \vec{p}^2}{p_0 - M_n} \mathbb{1} \right] = 0\end{aligned}$$

- Since $(p_k \sigma^k)^2 = (p_1^2 + p_2^2 + p_3^2) \mathbb{1} = \vec{p}^2 \mathbb{1}$ and $\mathbb{V} \sim \mathbb{1}$ is indeed invertible, this way of calculating the determinant is valid. With $p_0 = E$:

$$\begin{aligned}(E - M_n)(E + M_n) &= (\tilde{I}_{nn} + 1)\vec{p}^2 \\ \Leftrightarrow E^2 &= (\tilde{I}_{nn} + 1)\vec{p}^2 + M_n^2.\end{aligned}$$