Two-loop Feynman calculations

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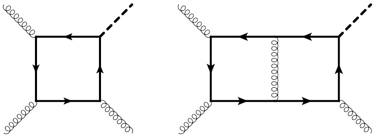
Introduction

The ϕ^* project: Higgs+jet production.



Two-loop: ½ a page in the main text, 20 pages of appendix.

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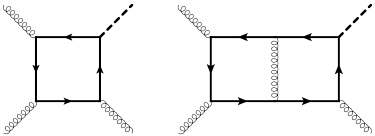

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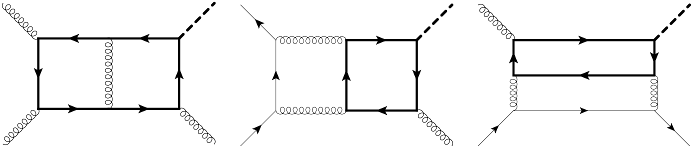
Appendix A: Two-loop Feynman integrals

- A.1: Extracting the numerator
- A.2: Reduction to scalar integrals
- A.3: Integrand reduction
- A.4: Integration-By-Parts identities
- A.5: The differential equation method
- A.6: Generalized polylogarithms
- A.7: Symbols
- A.8: Elliptic integrals

A1: Extracting the numerator

Three processes contribute



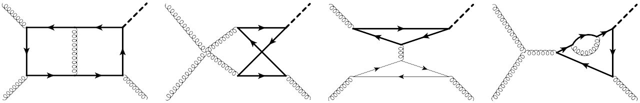
$gg \to gH$ $\qquad$ $q\bar{q} \to gH$ $\qquad$ $qg \to qH$

The gluon channel contains 286 Feynman diagrams,
the quark channels each contain 61.

In the following we will mostly discuss $gg \to gH$.

A1: Extracting the numerator

The Feynman diagrams may be generated with FeynArts.
Four of the 286 diagrams contributing to $gg \to gH$



gluon propagator:	1 term
quark propagator:	2 terms
ggg vertex:	6 terms
$gggg$ vertex:	6 terms
$q\bar{q}g$ vertex:	1 term
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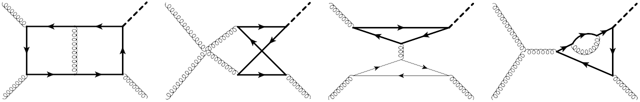

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$$\mathrm{tr}\left(\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_n}\right) = (n-1)!! \text{ terms}$$

In total: approx. $100\,000$ terms.

$$M = \sum_{i \in \text{diagrams}} \int \frac{\prod_l^L \mathrm{d}^d k_l}{(i\pi^{d/2})^L} \frac{N_i(\{k\})}{\prod_{j \in i} D_j(\{k\})}$$

$$D_1 = (k_1)^2 \,, \quad D_2 = (k_1 + p_1)^2 \,, \quad D_3 = (k_2 + p_1)^2 - m^2 \,, \ \ldots$$

Integrand reduction is cancellations between numerator and denominator.

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Example:

$$C \frac{k \cdot p}{(k^2)\,((k-p)^2 - m^2)} = \frac{C}{2} \left(\frac{p^2 - m^2}{(k^2)\,((k-p)^2 - m^2)} - \frac{1}{k^2} + \frac{1}{(k-p)^2 - m^2} \right)$$

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Doing this systematically (perhaps using algebraic geometry) gives

$$M = \sum_{i \in \text{topologies}} \int \frac{\prod_l^L \mathrm{d}^d k_l}{(i\pi^{d/2})^L} \frac{\Delta_i(\{k\})}{\prod_{j \in i} D_j(\{k\})}$$

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More reduction is desired...

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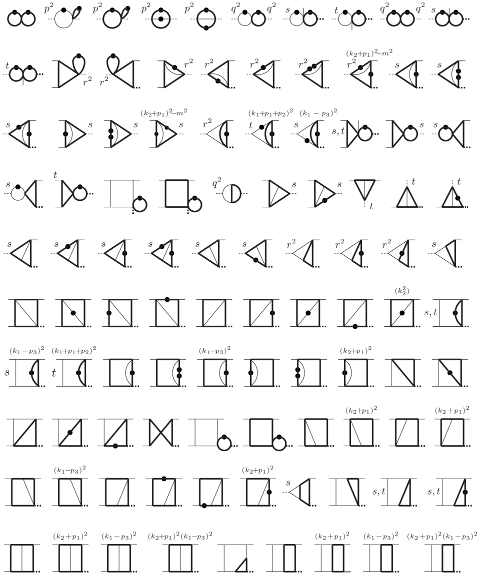
Different choices of v^μ, k, $F(k)$ imples a lot of identities.
Reduction to a minimal set called “master integrals”.

Mass-less $2 \to 2$ at one-loop has 3 master integrals:
two 'bubbles' and a 'box'.

4-scale two-loop systems have $\approx$ 100

A4: Integration-By-Parts identities



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$$\mathrm{d}\mathbf{I} = \epsilon(\mathrm{d}\tilde{A})\mathbf{I} \;, \qquad \tilde{A}_{ij} = \sum_l c_{ijl} \log(f_l(s))$$

Canonical form

A5: The differential equation method

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$$I_i = \sum_{k=0}^{\infty} I_i^{(k)} \epsilon^k$$

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The generalized polylogarithm (GPL)

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$$G(a_1,\ldots,a_n;x) = G(za_1,\ldots,za_n;zx) \quad \text{if} \quad z \neq 0,\ a_n \neq 0$$

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$$G(\bar{0}_{m_1-1},a_1,\ldots,\bar{0}_{m_n-1},a_n;x) = (-1)^n \mathrm{Li}_{m_1,\ldots,m_n}\left(\tfrac{x}{a_1},\tfrac{a_1}{a_2},\ldots,\tfrac{a_{n-1}}{a_n}\right)$$

$$\mathrm{Li}_{m_1,\ldots,m_n}(x_1,\ldots,x_n) = \sum_{i_1>\cdots>i_n>0} \frac{x_1^{i_1}}{i_1^{m_1}} \cdots \frac{x_n^{i_n}}{i_n^{m_n}}$$

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$$\mathcal{S}\Big(G(a_1,\ldots,a_n;a_0)\Big) = \sum_{i=1}^{n-1}\Bigg(\mathcal{S}\Big(G(a_1,\ldots,\hat{a}_i,\ldots,a_n;a_0)\Big)\otimes(a_i-a_{i-1}) - \mathcal{S}\Big(G(a_1,\ldots,\hat{a}_i,\ldots,a_n;a_0)\Big)\otimes(a_i-a_{i+1})\Bigg)$$

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$$\mathrm{Li}_2\big(\tfrac{1}{x}\big) = -\mathrm{Li}_2(x) - \frac{1}{2}\log(-x)^2 - \frac{\pi^2}{6}$$

$$\begin{aligned}
\mathcal{S}\big(\mathrm{Li}_2\big(\tfrac{1}{x}\big)\big) &= \big((1-x)\otimes x\big) - \big(x\otimes x\big)\\
\mathcal{S}\big(\mathrm{Li}_2(x)\big) &= -\big((1-x)\otimes x\big)\\
\mathcal{S}\big(\log(-x)^2\big) &= 2\big(x\otimes x\big)\\
\mathcal{S}\big(\pi^2/6\big) &= 0
\end{aligned}$$

The DGR (Duhr, Gangl, Rhodes) algorithm:

1) Find the symbol of your expression
2) Find a basis of function w. matching symbol
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The symbol can be derived directly from the canonical diff.eq:

$$\mathrm{d}f^{(n)} = c(\mathrm{d}\log(y))f^{(n-1)} \Leftrightarrow \mathcal{S}(f^{(n)}) = c\mathcal{S}(f^{(n-1)}) \otimes y$$

A basis can usually be found in terms of $\log$, Li_n, and $\mathrm{Li}_{2,2}$.

Finding arguments and inversion is where most computer time is spent.

A more sophisticated version of the symbol denoted the 'co-product' can help fix many of the remaining terms.

Interlude: Implementations of GPLs and symbols

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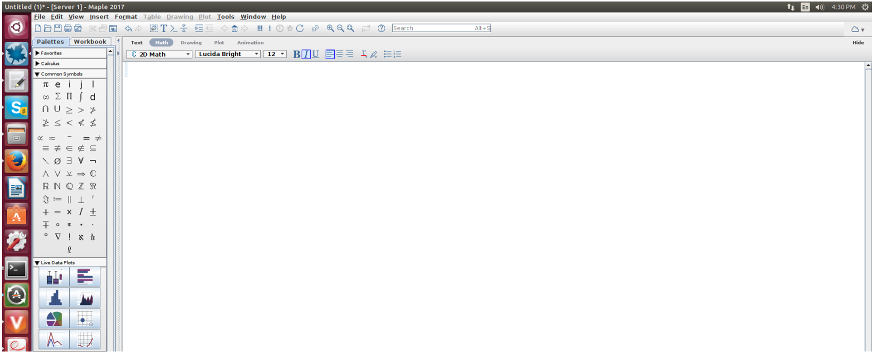

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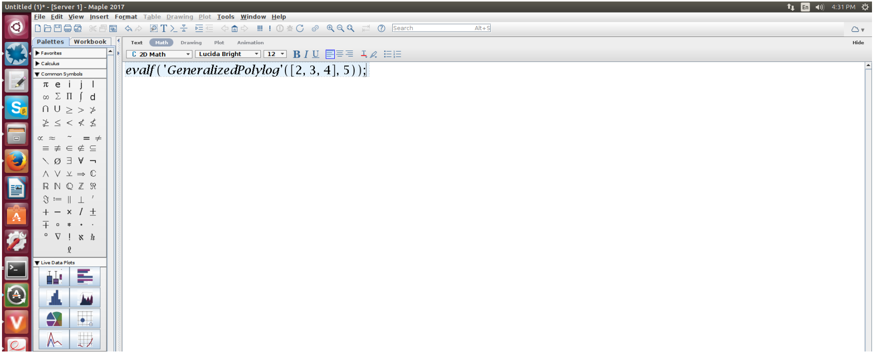

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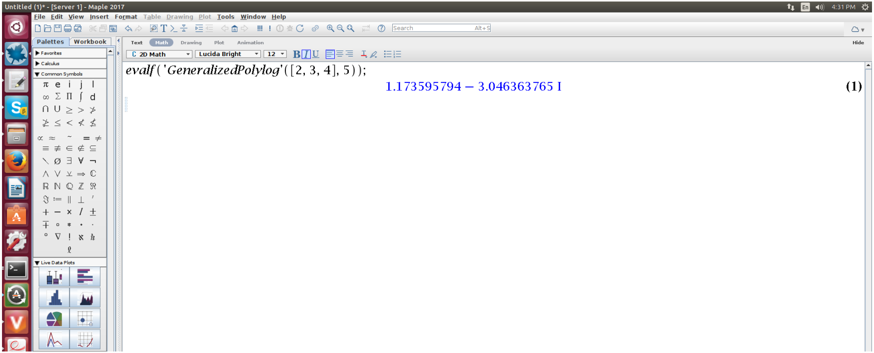

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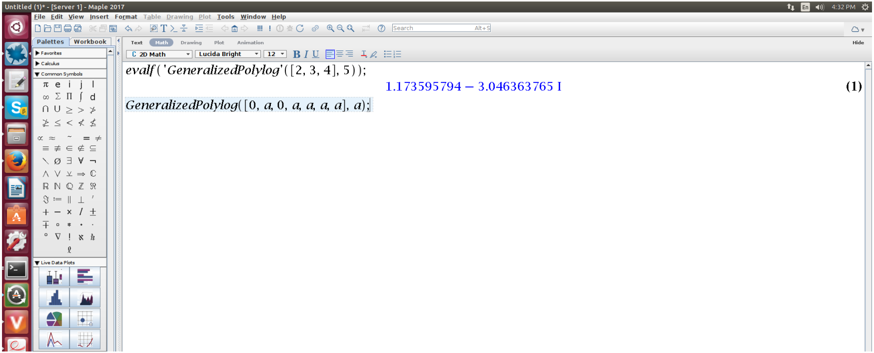

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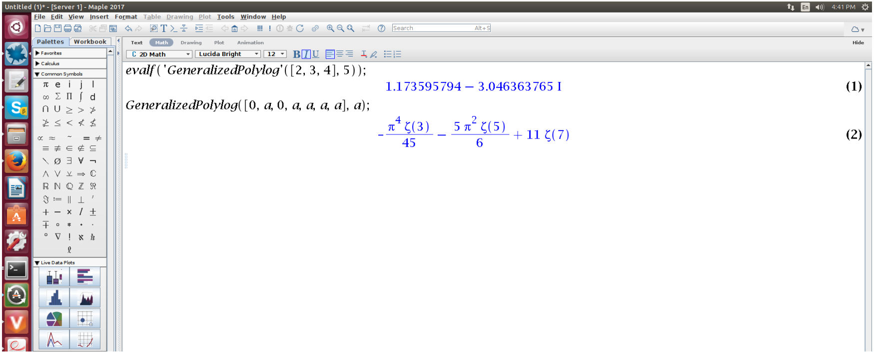

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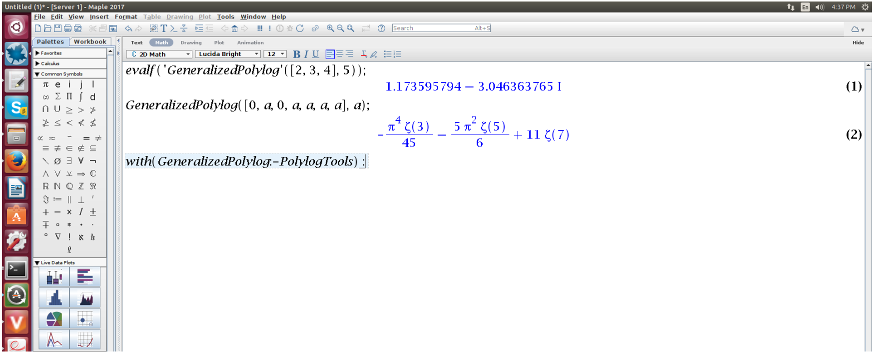

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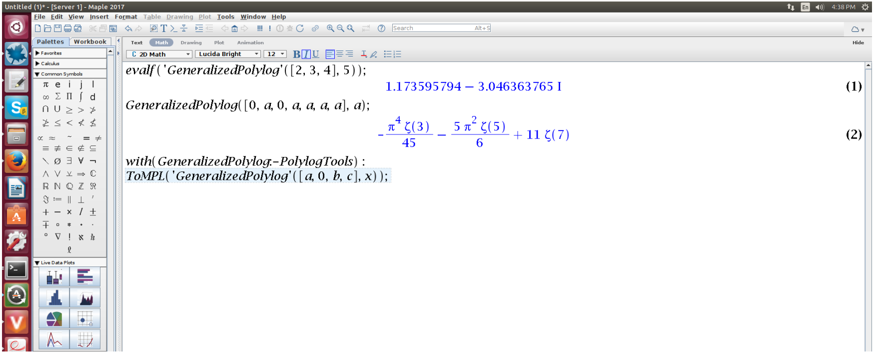

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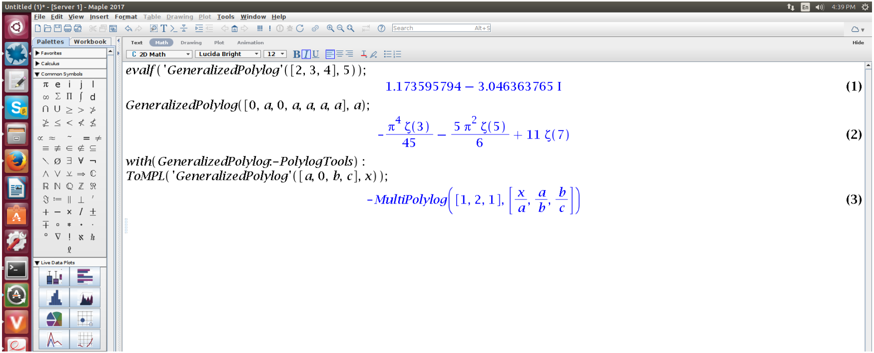

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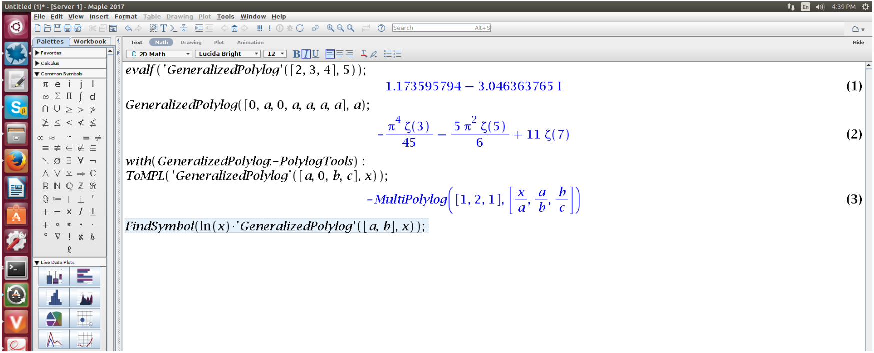

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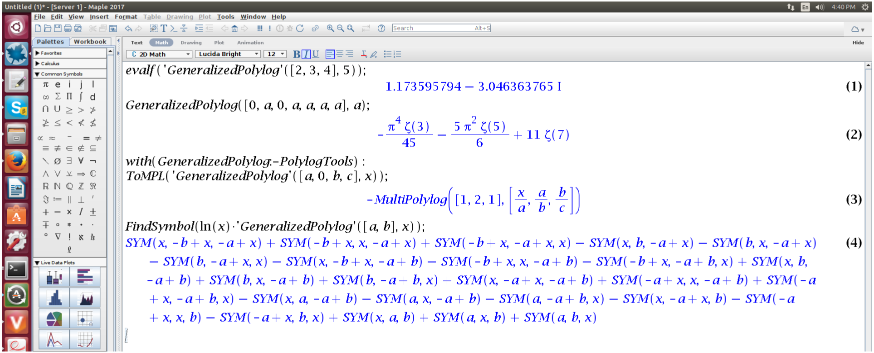

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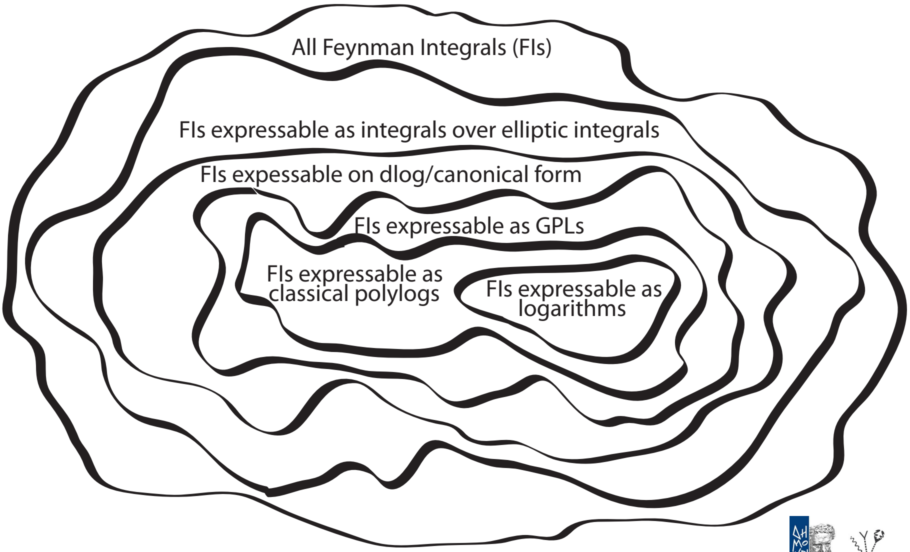

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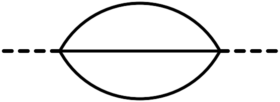

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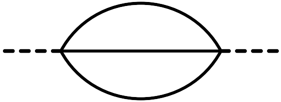

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$$K(k) \equiv \int_0^1 \frac{\mathrm{d}x}{\sqrt{(1-x^2)(1-k^2x^2)}} \qquad E(k) \equiv \int_0^1 \sqrt{\frac{1-k^2x^2}{1-x^2}}\, \mathrm{d}x$$

Complete elliptic integrals of first and second kind

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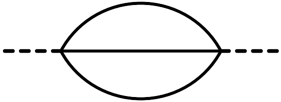

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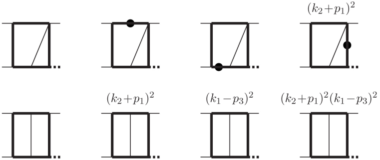

Summary

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- A.2: Reduction to scalar integrals
- A.3: Integrand reduction
- A.4: Integration-By-Parts identities
- A.5: The differential equation method
- A.6: Generalized polylogarithms
- A.7: Symbols
- A.8: Elliptic integrals

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Thank you for listening.