Elliptic Integrals in Higher Loop Calculations

- from IBPs to η-weighted elliptic polylogarithms -

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Introduction

One of the main and difficult issues in high energy physics is the calculation of involved multi-dimensional integrals.
In the following our attitude will be their analytic integration.
For quite some classes of integrals, particularly at lower order in the coupling constant, quite a series of analytic computational methods exist.
cf. e.g. [arXiv:1509.08324] for the algorithm.

- Hypergeometric functions.
- Summation methods based on difference fields, implemented in the `Mathematica` program `Sigma` [C. Schneider, 2005–].
  - Reduction of the sums to a small number of key sums.
  - Expansion of the summands in ε.
  - Simplification by symbolic summation algorithms based on $\Pi\Sigma$-fields [Karr 1981 J. ACM, Schneider 2005–].
  - Harmonic sums, polylogarithms and their various generalizations are algebraically reduced using the package HarmonicSums [Ablinger 2010, 2013, Ablinger, Blümlein, Schneider 2011,2013].

Introduction

- Mellin-Barnes representations.
- In the case of convergent massive 3-loop Feynman integrals, they can be performed in terms of Hyperlogarithms [Generalization of a method by F. Brown, 2008, to non-vanishing masses and local operators].
- Systems of Differential Equations.
- Almkvist-Zeilberger Theorem as Integration Method.

In the following we will concentrate on the method of Differential Equations since these are automatically obtained from the integration-by-parts identities representing all integrals by the so-called master integrals.
These may either be considered directly or in terms of difference equations obtained through a formal power-series ansatz or a Mellin transform.

Function Spaces

Sums

Harmonic Sums

$$\sum_{k=1}^{N} \frac{1}{k} \sum_{l=1}^{k} \frac{(-1)^l}{l^3}$$

gen. Harmonic Sums

$$\sum_{k=1}^{N} \frac{(1/2)^k}{k} \sum_{l=1}^{k} \frac{(-1)^l}{l^3}$$

Cycl. Harmonic Sums

$$\sum_{k=1}^{N} \frac{1}{(2k+1)} \sum_{l=1}^{k} \frac{(-1)^l}{l^3}$$

Binomial Sums

$$\sum_{k=1}^{N} \frac{1}{k^2} \binom{2k}{k} (-1)^k$$

Integrals

Harmonic Polylogarithms

$$\int_0^x \frac{dy}{y} \int_0^y \frac{dz}{1+z}$$

gen. Harmonic Polylogarithms

$$\int_0^x \frac{dy}{y} \int_0^y \frac{dz}{z-3}$$

Cycl. Harmonic Polylogarithms

$$\int_0^x \frac{dy}{1+y^2} \int_0^y \frac{dz}{1-z+z^2}$$

root-valued iterated integrals

$$\int_0^x \frac{dy}{y} \int_0^y \frac{dz}{z\sqrt{1+z}}$$

iterated integrals on CIS fct.

$$\int_0^z dx \frac{\ln(x)}{1+x} {}_2F_1\left[\begin{matrix}\frac{4}{3}, \frac{5}{3} \\ 2\end{matrix}; \frac{x^2(x^2-9)^2}{(x^2+3)^3}\right]$$

Special Numbers

multiple zeta values

$$\int_0^1 dx \frac{\mathrm{Li}_3(x)}{1+x} = -2\mathrm{Li}_4(1/2) + \ldots$$

gen. multiple zeta values

$$\int_0^1 dx \frac{\ln(x+2)}{x-3/2} = \mathrm{Li}_2(1/3) + \ldots$$

cycl. multiple zeta values

$$\mathbf{C} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$$

associated numbers

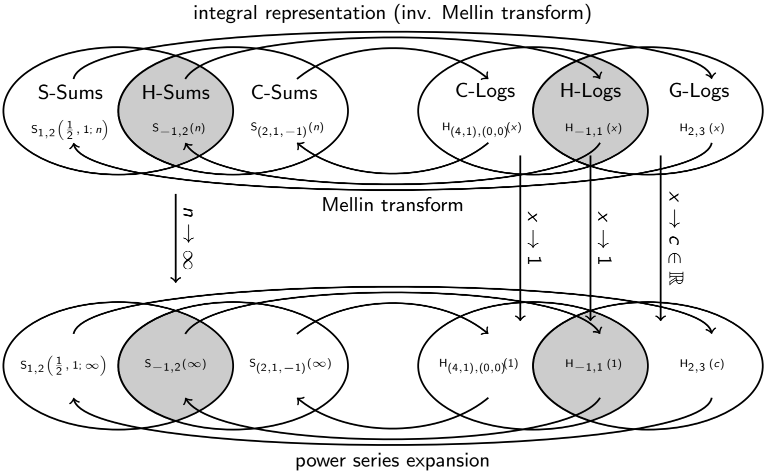
$$\mathrm{H}_{8,w_3} = 2\mathrm{arccot}(\sqrt{7})^2$$

associated numbers

$$\int_0^1 dx \; {}_2F_1\left[\begin{matrix}\frac{4}{3}, \frac{5}{3} \\ 2\end{matrix}; \frac{x^2(x^2-9)^2}{(x^2+3)^3}\right]$$

shuffle, stuffle, and various structural relations $\Longrightarrow$ algebras

Except the last line integrals, all other ones stem from 1st order factorizable equations.



square-root valued letters $\Longleftrightarrow$ nested binomial sums $\binom{2i}{i}$

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Decoupling of Systems

- We consider linear systems of N inhomogeneous differential equations and decouple them into a single scalar equation $+$ $(N-1)$ other determining equations.
- Usually one may use a series ansatz ($+\ \ln^k(x)$ modulation)

$$f(x) = \sum_{k=1}^{\infty} a(k) x^k$$

and obtain

$$\sum_{k=0}^{m} p_k(N) F(N+k) = G(N)$$

.

- The latter equation is now tried to be solved using difference-field techniques.
- If the equation has successive 1st order solutions one ends up with a nested sums solution. All these cases have been algorithmized. [arXiv:1509.08324 [hep-ph]].
- This even applies for some cases ending up elliptic in x-space [arXiv:1310.5645 [math-ph]].

An Example: Master integrals for the ρ-parameter @ $O(\alpha_s^3)$

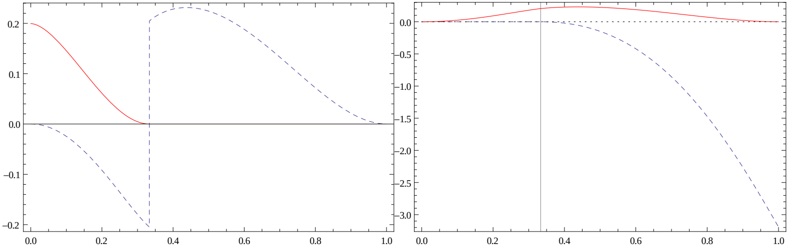
$$\frac{d^2}{dx^2} f_{8a}(x) + \frac{9-30x^2+5x^4}{x(x^2-1)(9-x^2)} \frac{d}{dx} f_{8a}(x) - \frac{8(-3+x^2)}{(9-x^2)(x^2-1)} f_{8a}(x) = I_{8a}(x)$$

Homogeneous solutions:

$$\begin{aligned}
\psi_3^{(0)}(x) &= -\frac{\sqrt{1-3x}\sqrt{x+1}}{2\sqrt{2\pi}} \left[(x+1)\left(3x^2+1\right) \mathbf{E}(z) - (x-1)^2(3x+1)\mathbf{K}(z) \right] \\
\psi_4^{(0)}(x) &= -\frac{\sqrt{1-3x}\sqrt{x+1}}{2\sqrt{2\pi}} \left[8x^2\mathbf{K}(1-z) - (x+1)\left(3x^2+1\right)\mathbf{E}(1-z) \right], \\
z &= \frac{16x^3}{(x+1)^3(3x-1)} .
\end{aligned}$$

K, E are the complete elliptic integrals of the 1st and 2nd kind.
I_{8a} contains rational functions of x and HPLs.

Solutions with a Singularity



Inhomogeneous Solution

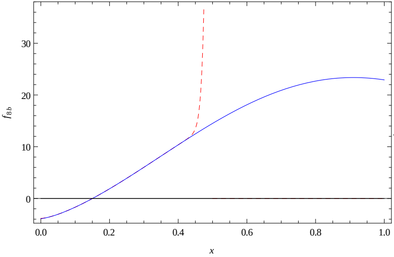
$$\psi(x) = \psi_3^{(0)}(x)\left[C_1 - \int dx\,\psi_4^{(0)}(x)\frac{N(x)}{W(x)}\right] + \psi_4^{(0)}(x)\left[C_2 - \int dx\,\psi_3^{(0)}(x)\frac{N(x)}{W(x)}\right]$$

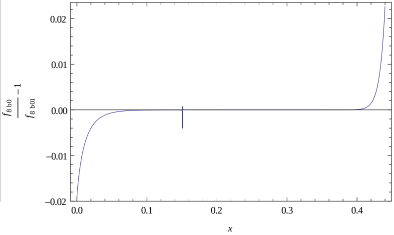
Series Solution

$$
\begin{aligned}
f_{8a}(x) \quad = \quad & -\sqrt{3}\Bigg[\pi^3\left(\frac{35x^2}{108}-\frac{35x^4}{486}-\frac{35x^6}{4374}-\frac{35x^8}{13122}-\frac{70x^{10}}{59049}-\frac{665x^{12}}{1062882}\right)+\Bigg(12x^2-\frac{8x^4}{3} \\
& -\frac{8x^6}{27}-\frac{8x^8}{81}-\frac{32x^{10}}{729}-\frac{152x^{12}}{6561}\Bigg)\mathrm{Im}\left[\mathrm{Li}_3\left(\frac{e^{-\frac{i\pi}{6}}}{\sqrt{3}}\right)\right]\Bigg]-\pi^2\Bigg(1+\frac{x^4}{9}-\frac{4x^6}{243}-\frac{46x^8}{6561} \\
& -\frac{214x^{10}}{59049}-\frac{5546x^{12}}{2657205}\Bigg)-\left(-\frac{3}{2}-\frac{x^4}{6}+\frac{2x^6}{81}+\frac{23x^8}{2187}+\frac{107x^{10}}{19683}+\frac{2773x^{12}}{885735}\right)\psi^{(1)}\left(\frac{1}{3}\right) \\
& -\sqrt{3}\pi\left(\frac{x^2}{4}-\frac{x^4}{18}-\frac{x^6}{162}-\frac{x^8}{486}-\frac{2x^{10}}{2187}-\frac{19x^{12}}{39366}\right)\ln^2(3)-\Bigg[33x^2-\frac{5x^4}{4}-\frac{11x^6}{54} \\
& -\frac{19x^8}{324}-\frac{751x^{10}}{29160}-\frac{2227x^{12}}{164025}+\pi^2\left(\frac{4x^2}{3}-\frac{8x^4}{27}-\frac{8x^6}{243}-\frac{8x^8}{729}-\frac{32x^{10}}{6561}-\frac{152x^{12}}{59049}\right) \\
& +\left(-2x^2+\frac{4x^4}{9}+\frac{4x^6}{81}+\frac{4x^8}{243}+\frac{16x^{10}}{2187}+\frac{76x^{12}}{19683}\right)\psi^{(1)}\left(\frac{1}{3}\right)\Bigg]\ln(x)+\frac{135}{16}+19x^2 \\
& -\frac{43x^4}{48}-\frac{89x^6}{324}-\frac{1493x^8}{23328}-\frac{132503x^{10}}{5248800}-\frac{2924131x^{12}}{236196000}-\left(\frac{x^4}{2}-12x^2\right)\ln^2(x) \\
& -2x^2\ln^3(x)+O\left(x^{14}\ln(x)\right)
\end{aligned}
$$

The solution can be easily extended to accuracies of $O(10^{-30})$ using Mathematica or Maple.

Solutions with a Singularity





Non-iterative Iterative Integrals

A New Class of Integrals in QFT:

$$\begin{aligned}\mathbb{H}_{a_1,\dots,a_{m-1};\{a_m;F_m(r(y_m))\},a_{m+1},\dots,a_q}(x) &= \int_0^x dy_1 f_{a_1}(y_1)\int_0^{y_1} dy_2 \dots \int_0^{y_{m-1}} dy_m f_{a_m}(y_m)\\ &\quad \times F_m[r(y_m)] H_{a_{m+1},\dots,a_q}(y_{m+1}),\\ F[r(y)] &= \int_0^1 dz g(z, r(y)), \quad r(y) \in \mathbb{Q}[y],\end{aligned}$$

In general, this spans all solutions and the story would end here.
May be, most of the practical physicists, would led it end here anyway.
This type of solution applies to many more cases beyond ${}_2F_1$-solutions (if being properly generalized).

Elliptic Solutions and Analytic q-Series

Map:

$$x \to q : q = \exp[-\pi \mathbf{K}(1 - z(x))/\mathbf{K}(z(x)], \quad |q| < 1$$

- One attempts to calculate the integrals of the inhomogeneous solution in terms of q-series analytically.
- It is expected to write it in terms of products (and integrals over) elliptic polylogarithms [and possibly other functions].
- Note that the corresponding results are rather deep multi-series!
- Inspiration from algebraic geometry.

Elliptic polylogarithm (as a partly suitable frame):

$$\mathrm{ELi}_{n,m}(x, y, q) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{x^j}{j^n} \frac{y^k}{k^m} q^{jk}$$

Is it (and its generalizations) a modular form ?
$\Longrightarrow$ The central functions turn out to be more special ones.

The Individual Steps: from IBPs to Closed Form q-Series

- Generate the master integrals, determine their hierarchy, and look whether you have only 1st order factorization or also 2nd order terms
- The latter can be trivial in case; check whether they persist in Mellin space
- If yes, analyze the 2nd order differential equation
- One usually finds a ${}_2F_1$-solution with rational argument $r(z)$, where $r(z)$ has additional singularities, i.e. the problem is of 2nd order, but has more than 3 singularities.
- Triangle group relations may be used to map the ${}_2F_1$ depending on the rational parameters a,b,c to the complete elliptic integrals or not.
- In the latter case return to the formalism on slide 11 and stop.
- If yes, one may walk along the q-series avenue.
- Different Levels of Complexity:
  - 1st order factorization in Mellin space:

$$\mathbf{M}[\mathbf{K}(1-z)](N) = \frac{2^{4N+1}}{(1+2N)^2\binom{2N}{N}^2}$$

$$\mathbf{M}[\mathbf{E}(1-z)](N) = \frac{2^{4N+2}}{(1+2N)^2(3+2N)\binom{2N}{N}^2}$$

The Individual Steps: from IBPs to Closed Form q-Series

- Criteria by Herfurtner (1991), Movasati et al. (2009) are obeyed.
  $\Longrightarrow$ 2-loop sunrise and kite diagrams, cf. Weinzierl et al. 2014-17.
  Only $\mathbf{K}(r(z))$ and $\mathbf{K}'(r(z))$ contribute as elliptic integrals.
- Also $\mathbf{E}(r(z))$ and $\mathbf{E}'(r(z))$, square roots of quadratic forms etc. contribute (present case)

- Transform now: $x \to q$.
  - The kinematic variable x:

$$
\begin{aligned}
k^2 &= \frac{-x^3}{(1+x)^3(1-3x)} = \frac{\vartheta_2^4(q)}{\vartheta_3^4(q)} \\
x &= \frac{\vartheta_2^2(q)}{3\vartheta_2^2(q^3)}, \quad \text{i.e.} \quad x \in [1, +\infty[
\end{aligned}
$$

by a cubic transformation (Legendre-Jacobi).

$$
x = \frac{1}{3}\frac{\eta^2(2\tau)\eta^2(3\tau)}{\eta^2(\tau)\eta^4(6\tau)}, \quad \text{singular}, \propto \frac{1}{q}
$$

The Individual Steps: from IBPs to Closed Form q-Series

- Map to a Modular Form, which can be represented by Lambert Series
  - How to find the η-ratio ? $\Longrightarrow$ Many are listed as sequences in Sloan's OEIS.
  - To find a modular form, situated in a corresponding finite-dimensional vector space M_k one has to meet a series of conditions and usually split off a factor $1/\eta^k(\tau), k > 0$.
  - The remainder modular form is now a polynomial over $\mathbb{Q}$ of Lambert-Eisenstein series

$$\sum_{n=0}^{\infty} \frac{m^n q^{an+b}}{1 - q^{an+b}} .$$

<u>Example:</u>

$$\mathbf{K}(z(x)) = \frac{\pi}{2} \sum_{k=1}^{\infty} \frac{q^k}{1 + q^{2k}}$$

  - In this case, two q series are equal, if both are modular forms, and agree in a series of k first terms, where k is predicted for each congruence sub-group of $\Gamma(N)$.

The Individual Steps: from IBPs to Closed Form q-Series

- Map Lambert-Eisenstein Series into the frame of Elliptic Polylogarithms
- Examples:

$$\begin{aligned}
\mathbf{K}(z) &= \frac{\pi}{2}\sum_{k=1}^{\infty}\frac{q^k}{1+q^{2k}} = \frac{\pi}{i}\sum_{k=1}^{\infty}\left[\mathrm{Li}_0\left(iq^k\right)-\mathrm{Li}_0\left(-iq^k\right)\right] \\
&= \frac{\pi}{4}\overline{E}_{0,0}(i,1,q), \\
q\frac{\vartheta_4'(q)}{\vartheta_4(q)} &= -\frac{1}{2}\left[\mathrm{ELi}_{-1;0}(1;1;q)+\mathrm{ELi}_{-1;0}(-1;1;q)\right] \\
&\quad +\left[\mathrm{ELi}_{0;0}(1;q^{-1};q)+\mathrm{ELi}_{0;0}(-1;q^{-1};q)\right] \\
&\quad -\left[\mathrm{ELi}_{-1;0}(1;q^{-1};q)+\mathrm{ELi}_{-1;0}(-1;q^{-1};q)\right].
\end{aligned}$$

- New type of elliptic polylogarithm, e.g.:
  $\mathrm{ELi}_{-1;0}(-1;q^{-1};q)$, $\quad y = y(q)$!
- Argument synchronization necessary: $-q \to q, \quad q^k \to q$ (cyclotomic).

Elliptic Solutions and Analytic q-Series

- Terms to be translated:
  - rational functions in x
  - $\mathbf{K}, \mathbf{E}$
  - $\sqrt{(1-3x)(1+x)}$
  - $H_{\vec{a}}(x)$

Examples:

$$\begin{aligned} H_{-1}(x) = \ln(1+x) &= -\ln(3q) - \overline{E}_{0;-1;2}(-1;-1;q) + \overline{E}_{0;-1;2}(\rho_6;-1;q) \\ &\quad -\overline{E}_{0;-1;2}(\rho_3;-i;q) - \overline{E}_{0;-1;2}(\rho_3;i;q) \\ H_1(x) &= -\left. H_{-1}(x)\right|_{q\to -q} + 2\pi i, etc.; \quad \rho_m = exp(2\pi i/m) \end{aligned}$$

$$I(q) = \frac{1}{\eta^k(\tau)} \cdot \mathbf{P}\left[\ln(q), \mathrm{Li}_0(q^m), \mathrm{ELi}_{k,l}(x,y,q), \mathrm{ELi}_{k',l'}(x,q^{-1},q)\right]$$

$$\int \frac{dq}{q} I(q)$$

is usually not an elliptic polylogarithm, due to the η-factor, but a higher transcendental function in q.

We are still in the unphysical region and have to map back to $x \in [0,1]$.

Conclusions

- We have automated the chain from IBPs to 2nd order solutions within the theory of differential equations [Before we had solved the 1st order factorizing cases for whatsoever basis of MIs.]
- General solution in the case not 1st order factorizing: Non-iterative iterative integrals $\mathbb{H}$.
- These solutions might be sufficient and are very precise numerically and the result has a compact representation.
- In the elliptic cases we were enforced to generalize to structures not yet appearing in the case of the sunrise/kite integrals.
- Our tools are close to those applied for number theoretic problems. Modular forms need to become a manifest part of knowledge for particle physicist working on fundamental QFTs [String 'theory' needs it as well, but in a simpler way so far.]
- We can solve any η ratio.

Conclusions

- The general solution is given in terms of polynomials of elliptic polylogarithms, more precisely: Lambert-Eisenstein series and a few simpler functions in q-space
- Singularity treatment ?
- How to map back to the different physical regions ?
- What are the minimal bases ?
  $\Longrightarrow$ An important mathematical research topic.
- Interesting observation: q-series for equal mass sunrise appeared in 1987 in a similar form in Beukers' 2nd proof of the irrationality of ζ_3 in form of an Eichler-integral [Zagier].
- What comes next ? Abel integrals ? K3 surfaces (Kummer, Kähler, Kodaira), Calabi-Yau structures...?
- Again a new and exciting territory for theoretical physics ...