Cuts of Feynman Integrals in Baikov representation

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INPP, NCSR "Demokritos"

3rd HiggsTools annual meeting, Torino, May 15, 2017

Outline

1. Introduction
2. The HOC frontier – NNLO
3. Baikov representation, cuts and DE
4. Summary - Discussion

Feynman

PHYSICAL REVIEW VOLUME 76, NUMBER 6 SEPTEMBER 15, 1949

Space-Time Approach to Quantum Electrodynamics

R. P. Feynman
Department of Physics, Cornell University, Ithaca, New York
(Received May 9, 1949)

In this paper two things are done. (1) It is shown that a considerable simplification can be attained in writing down matrix elements for complex processes in electrodynamics. Further, a physical point of view is available which permits them to be written down directly for any specific problem. Being simply a and presumably consistent, method is therefore available for the calculation of all processes involving electrons and photons.
The simplification in writing the expressions results from an emphasis on the over-all space-time view resulting from a study of the solution of the equations of electrodynamics. The relation

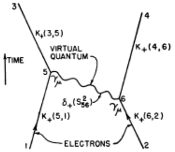
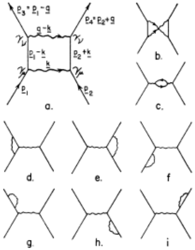


Fig. 1. The fundamental interaction Eq. (4). Exchange of one quantum between two electrons.

D. More Complex Problems

Matrix elements for complex problems can be set up in a manner analogous to that used for the simpler cases. We give three illustrations; higher order corrections to the Møller scatter-



C. G. Papadopoulos, D. Tommasini and C. Wever, JHEP **1604**, 078 (2016) [arXiv:1511.09404 [hep-ph]].

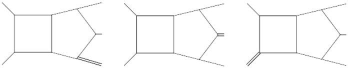


Figure 1. The three planar pentaboxes of the families P_1 (left), P_2 (middle) and P_3 (right) with one external massive leg.

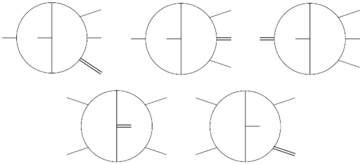


Figure 2. The five non-planar families with one external massive leg.

$$
\begin{aligned}
\mathbf{G} = {} & \varepsilon^{-2}\mathbf{b}_0^{(-2)} + \varepsilon^{-1}\left(\sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(-2)} + \mathbf{b}_0^{(-1)}\right) \\
& + \varepsilon^0 \left(\sum \mathcal{G}_{ab}\mathbf{M}_a\mathbf{M}_b\mathbf{b}_0^{(-2)} + \sum \mathcal{G}_a\mathbf{M}_a\mathbf{b}_0^{(-1)} + \mathbf{b}_0^{(0)}\right) \\
& + \varepsilon \left(\sum \mathcal{G}_{abc}\mathbf{M}_a\mathbf{M}_b\mathbf{M}_c\mathbf{b}_0^{(-2)} + \sum \mathcal{G}_{ab}\mathbf{M}_a\mathbf{M}_b\mathbf{b}_0^{(-1)} + \sum \mathcal{G}_a\mathbf{M}_a\mathbf{b}_0^{(0)} + \mathbf{b}_0^{(1)}\right) \\
& + \varepsilon^2 \left(\sum \mathcal{G}_{abcd}\mathbf{M}_a\mathbf{M}_b\mathbf{M}_c\mathbf{M}_d\mathbf{b}_0^{(-2)} + \sum \mathcal{G}_{abc}\mathbf{M}_a\mathbf{M}_b\mathbf{M}_c\mathbf{b}_0^{(-1)} \right. \\
& \qquad \left. + \sum \mathcal{G}_{ab}\mathbf{M}_a\mathbf{M}_b\mathbf{b}_0^{(0)} + \sum \mathcal{G}_a\mathbf{M}_a\mathbf{b}_0^{(1)} + \mathbf{b}_0^{(2)}\right) \qquad (3.6)
\end{aligned}
$$

LO - Dyson-Schwinger Recursive Equations

From Feynman Diagrams to recursive equations: taming the *n*!

- 1999 `HELAC`: The first code to calculate recursively tree-order amplitudes for (practically) arbitrary number of particles

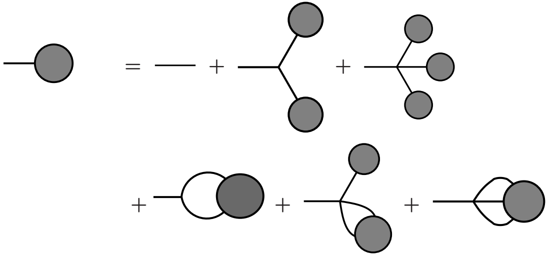
From Feynman Diagrams to recursive equations: taming the n!

- 1999 `HELAC`: The first code to calculate recursively tree-order amplitudes for (practically) arbitrary number of particles

A. Kanaki and C. G. Papadopoulos, Comput. Phys. Commun. **132** (2000) 306 [arXiv:hep-ph/0002082].

F. A. Berends and W. T. Giele, Nucl. Phys. B **306** (1988) 759.

F. Caravaglios and M. Moretti, Phys. Lett. B **358** (1995) 332.



Unfortunately not so much on the second line !

From Feynman graphs ...

$gg \to ng$	2	3	4	5	6	7	8	9
# FG	4	25	220	2,485	34,300	559,405	10,525,900	224,449,225

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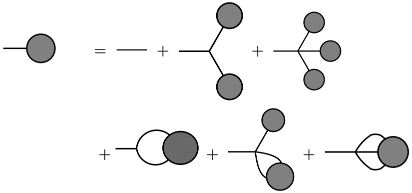
to Dyson-Schwinger recursion! `Helac-Phegas`



$gg \to ng$	2	3	4	5	6	7	8	9
#	5	15	35	70	126	210	330	495

What do we need for an NLO calculation ?

$$p_1, p_2 \to p_3, \ldots, p_{m+2}$$

$$\begin{aligned} \sigma_{NLO} &= \int_m d\Phi_m |M_m^{(0)}|^2 J_m(\Phi) \\ &+ \int_m d\Phi_m 2Re(M_m^{(0)*} M_m^{(1)}(\epsilon_{UV}, \epsilon_{IR})) J_m(\Phi) \\ &+ \int_{m+1} d\Phi_{m+1} |M_{m+1}^{(0)}|^2 J_{m+1}(\Phi) \end{aligned}$$

$J_m(\Phi)$ jet function: Infrared safeness $J_{m+1} \to J_m$

Perturbative QCD at NLO

What do we need for an NLO calculation ?

$$p_1, p_2 \to p_3, \ldots, p_{m+2}$$

$$\begin{aligned}
\sigma_{NLO} &= \int_m d\Phi_m^{D=4}\left(|M_m^{(0)}|^2 + 2Re(M_m^{(0)*} M_m^{(CT)}(\epsilon_{UV}))\right) J_m(\Phi) \\
&+ \int_m d\Phi_m^{D=4} 2Re(M_m^{(0)*} M_m^{(1)}(\epsilon_{UV}, \epsilon_{IR})) J_m(\Phi) \\
&+ \int_{m+1} d\Phi_{m+1}^{D=4-2\epsilon_{IR}} |M_{m+1}^{(0)}|^2 J_{m+1}(\Phi)
\end{aligned}$$

IR and UV divergencies, Four-Dimensional-Helicity scheme; scale dependence μ_R

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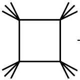
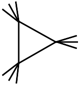
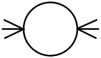
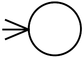
QCD factorization$-\mu_F$ Collinear counter-terms when PDF are involved basis of scalar integrals:

G. Passarino and M. J. G. Veltman, Nucl. Phys. B **160** (1979) 151.

Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, Nucl. Phys. B **425** (1994) 217 [arXiv:hep-ph/9403226].

$\mathcal{A} = \sum d_{i_1 i_2 i_3 i_4}$  $+ \sum c_{i_1 i_2 i_3}$  $+ \sum b_{i_1 i_2}$  $+ \sum a_{i_1}$  $+ R$

$a, b, c, d \rightarrow$ cut-constructible part $\qquad R \rightarrow$ rational terms

$$\mathcal{A} = \sum_{I \subset \{0,1,\cdots,m-1\}} \int \frac{\mu^{(4-d)} d^d q}{(2\pi)^d} \frac{\bar{N}_I(\bar{q})}{\prod_{i \in I} \bar{D}_i(\bar{q})}$$

OPP "MASTER" FORMULA - I

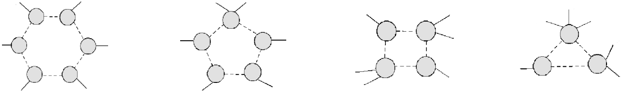
General expression for the 4-dim $N(q)$ at the integrand level in terms of D_i

$$
\begin{aligned}
N(q) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3)\right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i \\
&+ \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2)\right] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i \\
&+ \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1) + \tilde{b}(q; i_0 i_1)\right] \prod_{i \neq i_0, i_1}^{m-1} D_i \\
&+ \sum_{i_0}^{m-1} \left[a(i_0) + \tilde{a}(q; i_0)\right] \prod_{i \neq i_0}^{m-1} D_i
\end{aligned}
$$

The one-loop calculation in a nutshell

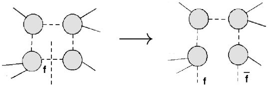
The computation of $pp(p\bar{p}) \to e^{+}\nu_e \mu^{-} \bar{\nu}_\mu b\bar{b}$ involves up to six-point functions.
The most generic integrand has therefore the form

$$\mathcal{A}(q) = \sum \underbrace{\frac{N_i^{(6)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\cdots\bar{D}_{i_5}}} + \underbrace{\frac{N_i^{(5)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\cdots\bar{D}_{i_4}}} + \underbrace{\frac{N_i^{(4)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\cdots\bar{D}_{i_3}}} + \underbrace{\frac{N_i^{(3)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\bar{D}_{i_2}}} + \cdots$$



In order to apply the OPP reduction, HELAC evaluates numerically the numerators $N_i^6(q), N_i^5(q), ...$ with the values of the loop momentum q provided by CutTools

- generates all inequivalent partitions of 6,5,4,3... blobs attached to the loop, and check all possible flavours (and colours) that can be consistently running inside
- hard-cuts the loop (q is fixed) to get a $n+2$ *tree-like* process



The R_2 contributions (rational terms) are calculated in the same way as the tree-order amplitude, taking into account *extra vertices*

→ MadGraph, RECOLA, OpenLoops

The one-loop calculation in a nutshell

Institute of Nuclear Physics "Demokritos" Bergische Universitat Wuppertal Institute of Nuclear Physics PAN RWTH Aachen University

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HELAC-NLO & Associated Tools

Projects

HELAC-PHEGAS - A generator for all parton level processes in the Standard Model

HELAC-DIPOLES - Dipole formalism for the arbitrary helicity eigenstates of the external partons

HELAC-1LOOP - A program for numerical evaluation of QCD virtual corrections to scattering amplitudes

ONELOOP - A program for the evaluation of one-loop scalar functions

CUTTOOLS - A program implementing the OPP reduction method to compute one-loop amplitudes

PARNI - A program for importance sampling and density estimation

KALEU - A general-purpose parton-level phase space generator

HELAC-ONIA - An automatic matrix element generator for heavy quarkonium physics

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People

Giuseppe Bevilacqua
Michal Czakon
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Andreas van Hameren
Adam Kardos
Yiannis Malamos
Costas G. Papadopoulos
Roberto Pittau
Malgorzata Worek
Hua-Sheng Shao

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Contact us

If you have a question, comment, suggestion or bug report, please e-mail us at:

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J.Malamos(at)science.ru.nl
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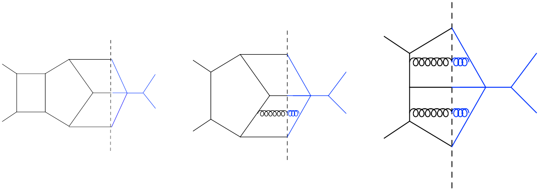
Last modified by Malgorzata Worek
Thursday, January 10th, 2013

Proof of concept: the first NLO public code

What do we need for an NNLO calculation ?

$$p_1, p_2 \to p_3, \dots, p_{m+2}$$



Perturbative QCD at NNLO

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$$\begin{aligned}
\sigma_{NNLO} \quad \to \quad & \int_m d\Phi_m \left(2Re(M_m^{(0)*} M_m^{(2)}) + \left| M_m^{(1)} \right|^2 \right) J_m(\Phi) & \quad VV \\
+ \quad & \int_{m+1} d\Phi_{m+1} \left(2Re \left(M_{m+1}^{(0)*} M_{m+1}^{(1)} \right) \right) J_{m+1}(\Phi) & \quad RV \\
+ \quad & \int_{m+2} d\Phi_{m+2} \left| M_{m+2}^{(0)} \right|^2 J_{m+2}(\Phi) & \quad RR
\end{aligned}$$

$RV + RR \to$ Antenna-S, Colorfull-S, STRIPPER, q_T, N-jetiness

A. Gehrmann-De Ridder, T. Gehrmann and M. Ritzmann, JHEP **1210** (2012) 047
P. Bolzoni, G. Somogyi and Z. Trocsanyi, JHEP **1101** (2011) 059
M. Czakon and D. Heymes, Nucl. Phys. B **890** (2014) 152
S. Catani and M. Grazzini, Phys. Rev. Lett. **98** (2007) 222002
R. Boughezal, C. Focke, X. Liu and F. Petriello, Phys. Rev. Lett. **115** (2015) no.6, 062002

OPP AT TWO LOOPS

coefficients of MI $\oplus$ spurious terms

$$\frac{N(q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}} = \sum_{i_0<i_1<i_2<i_3}^{m-1} \frac{d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3)}{\bar{D}_{i0} \bar{D}_{i1} \bar{D}_{i2} \bar{D}_{i2}}$$
$$+ \sum_{i_0<i_1<i_2}^{m-1} \frac{c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2)}{\bar{D}_{i0} \bar{D}_{i1} \bar{D}_{i2}}$$
$$+ \sum_{i_0<i_1}^{m-1} \frac{b(i_0 i_1) + \tilde{b}(q; i_0 i_1)}{\bar{D}_{i0} \bar{D}_{i1}}$$
$$+ \sum_{i_0}^{m-1} \frac{a(i_0) + \tilde{a}(q; i_0)}{\bar{D}_{i0}}$$
$$+ \text{ rational terms}$$

G. Ossola, C. G. Papadopoulos and R. Pittau, Nucl. Phys. B **763**, 147 (2007)

- Write the "OPP-type" equation at two loops

$$\frac{N\left(l_1, l_2; \{p_i\}\right)}{D_1 D_2 \ldots D_n} = \sum_{m=1}^{\min(n,8)} \sum_{S_{m;n}} \frac{\Delta_{i_1 i_2 \ldots i_m}\left(l_1, l_2; \{p_i\}\right)}{D_{i_1} D_{i_2} \ldots D_{i_m}}$$

$$\sum \frac{\Delta_{i_1 i_2 \ldots i_m}\left(l_1, l_2; \{p_i\}\right)}{D_{i_1} D_{i_2} \ldots D_{i_m}} \to \text{spurious} \oplus \text{ISP} - \text{irreducible integrals}$$

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ISP-irreducible integrals → use IBPI to Master Integrals

Libraries in the future: `QCD2LOOP`, `TwOLOop`

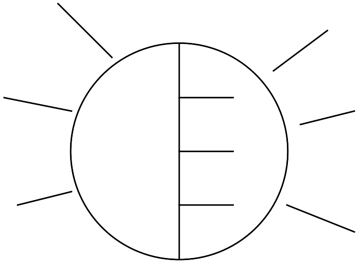
P. Mastrolia, T. Peraro and A. Primo, arXiv:1605.03157 [hep-ph].

J. Gluza, K. Kajda and D. A. Kosower, Phys. Rev. D **83** (2011) 045012

H. Ita, arXiv:1510.05626 [hep-th].

C. G. Papadopoulos, R. H. P. Kleiss and I. Malamos, PoS Corfu **2012** (2013) 019.

Two-loop graph



IBPI: The current approach

- m independent momenta l loops, $N = l(l+1)/2 + lm$ scalar products
- basis composed by $D_1 \ldots D_N$, allows to express all scalar products $D_i = (\{k, l\} + p_i)^2 - M_i^2$
-

$$F[a_1, \ldots, a_N] = \int d^d k d^d l \frac{1}{D_1^{a_1} \ldots D_N^{a_N}}$$

$$\int d^d k d^d l \;\; \frac{\partial}{\partial \{k^\mu, l^\mu\}} \left(\frac{\{k^\mu, l^\mu, v^\mu\}}{D_1^{a_1} \ldots D_N^{a_N}} \right) = 0$$

- IBP Laporta: `FIRE`, `AIR`, `Reduze` reduce these to MI
- MI computed, Feynman parameters, Mellin-Barnes, Differential Equations
- Or numerical: `SecDec`, Weinzierl

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F. V. Tkachov, Phys. Lett. B **100** (1981) 65.

K. G. Chetyrkin and F. V. Tkachov, Nucl. Phys. B **192** (1981) 159.

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S. Laporta, Int. J. Mod. Phys. A **15** (2000) 5087

C. Anastasiou and A. Lazopoulos, JHEP **0407** (2004) 046

C. Studerus, Comput. Phys. Commun. **181** (2010) 1293

A. V. Smirnov, Comput. Phys. Commun. **189** (2014) 182

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Z. Bern, L. J. Dixon and D. A. Kosower, Phys. Lett. B **302** (1993) 299.

V. A. Smirnov, Phys. Lett. B **460** (1999) 397

T. Gehrmann and E. Remiddi, Nucl. Phys. B **580** (2000) 485 [hep-ph/9912329].

J. M. Henn, Phys. Rev. Lett. **110** (2013) 25, 251601 [arXiv:1304.1806 [hep-th]].

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- MI computed, Feynman parameters, Mellin-Barnes, Differential Equations
- Or numerical: `SecDec`, Weinzierl

S. Borowka, G. Heinrich, S. P. Jones, M. Kerner, J. Schlenk and T. Zirke, Comput. Phys. Commun. **196** (2015) 470

S. Becker, C. Reuschle and S. Weinzierl, JHEP **1012** (2010) 013

- Find a better IBP algorithm ... Generating function technique, Baikov ?

P. A. Baikov, Nucl. Instrum. Meth. A **389** (1997) 347

V. A. Smirnov and M. Steinhauser, Nucl. Phys. B **672** (2003) 199

$$F_{a_1 \dots a_N} = \sum_{i=\text{masters}} c^{(i)}_{a_1 \dots a_N} G_i$$

- Baikov polynomial $\leftrightarrow$ LZ construction
- Sector $\leftrightarrow$ cut

$$\delta\left((k+p)^2 - m^2\right) \leftrightarrow \oint_{z=0} dz \frac{1}{z^{n=1}}$$

- Cut with higher powers in denominator

IBPI: The current approach

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K. J. Larsen and Y. Zhang, Phys. Rev. D **93** (2016) no.4, 041701

A. Georgoudis, K. J. Larsen and Y. Zhang, arXiv:1612.04252 [hep-th].

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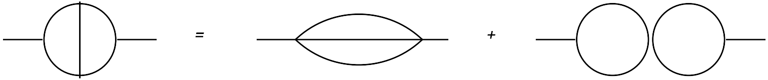

$$F_{11111} = \frac{(3d-10)(3d-8)}{(d-4)^2(p^2)^2} F_{10011} + \frac{(3d-10)(3d-8)}{(d-4)^2(p^2)^2} F_{01101} - 2\frac{(d-3)}{(d-4)\,p^2} F_{11110}$$

Differential Equations Approach

The integral is a function of external momenta, so one can set-up differential equations by differentiating and using IBP

$$p_j^\mu \frac{\partial}{\partial p_i^\mu} G[a_1, \ldots, a_n] \to \sum C_{b_1,\ldots,b_n} F[b_1, \ldots, b_n] \to \sum C_{a'_1,\ldots,a'_n} G[a'_1, \ldots, a'_n]$$

- Find the proper parametrization; Bring the system of equations in a form suitable to express the MI in terms of GPs

$$\partial_m f\left(\varepsilon, \{x_i\}\right) = \varepsilon A_m\left(\{x_i\}\right) f\left(\varepsilon, \{x_i\}\right)$$
$$\partial_m A_n - \partial_n A_m = 0 \quad [A_m, A_n] = 0$$

  ⋆ f not MI!

  J. M. Henn, Phys. Rev. Lett. **110** (2013) 25, 251601 [arXiv:1304.1806 [hep-th]].

- Boundary conditions: expansion by regions or regularity conditions.

  B. Jantzen, A. V. Smirnov and V. A. Smirnov, Eur. Phys. J. C **72** (2012) 2139 [arXiv:1206.0546 [hep-ph]].

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- Iterated Integrals — K. T. Chen, Iterated path integrals, Bull. Amer. Math. Soc. 83 (1977) 831
- Multiple Polylogarithms, Symbol algebra
- Goncharov Polylogarithms

$$\mathcal{G}\left(a_n,\ldots,a_1,x\right)=\int_0^x dt\,\frac{1}{t-a_n}\mathcal{G}\left(a_{n-1},\ldots,a_1,t\right)$$

with the special cases, $\mathcal{G}(x)=1$ and

$$\mathcal{G}\left(\underbrace{0,\ldots 0}_{n},x\right)=\frac{1}{n!}\log^n\left(x\right)$$

- Shuffle algebra

Differential Equations Approach

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- Multiple Polylogarithms, Symbol algebra

A. B. Goncharov, M. Spradlin, C. Vergu and A. Volovich, Phys. Rev. Lett. **105** (2010) 151605.

C. Duhr, H. Gangl and J. R. Rhodes, JHEP **1210** (2012) 075 [arXiv:1110.0458 [math-ph]].

C. Bogner and F. Brown

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$$\mathcal{G}\left(a_1, a_2; x\right) \mathcal{G}\left(b_1; x\right) = \mathcal{G}\left(a_1, a_2, b_1; x\right) + \mathcal{G}\left(a_1, b_1, a_2; x\right) + \mathcal{G}\left(b_1, a_1, a_2; x\right)$$

The Simplified Differential Equations Approach

C. G. Papadopoulos, JHEP **1407** (2014) 088

Making the whole procedure systematic (algorithmic) and straightforwardly expressible in terms of GPs.

- Introduce one parameter

$$G_{11\ldots1}(x) = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{(k^2)\,(k + x\,p_1)^2\,(k+p_1+p_2)^2 \ldots (k+p_1+p_2+\ldots+p_n)^2}$$

- Factorizing external momenta dependence:

$$x : (q_1 = xp_1, q_2 = p_{12} - xp_1, \ldots) \to x \otimes (q_1 = p_1, q_2 = p_2, \ldots)$$

- Now the integral as a function of x, allows to define a differential equation with respect to x, schematically given by

$$\frac{\partial}{\partial x} G_{11\ldots1}(x) = -\frac{1}{x} G_{11\ldots1}(x) + xp_1^2 G_{12\ldots1} + \frac{1}{x} G_{02\ldots1}$$

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5BOX - ONE LEG OFF-SHELL: ALL FAMILIES

C. G. Papadopoulos, D. Tommasini and C. Wever, arXiv:1511.09404 [hep-ph].

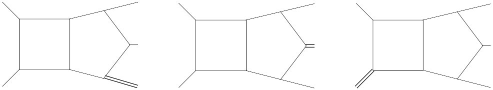

FIGURE : The three planar pentaboxes of the families P_1 (left), P_2 (middle) and P_3 (right) with one external massive leg.

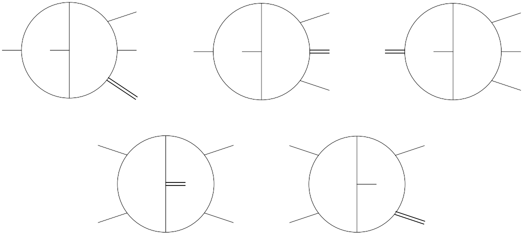

FIGURE : The five non-planar families with one external massive leg.

$$p(q_1)p'(q_2) \to V(q_3)j_1(q_4)j_2(q_5), \quad q_1^2 = q_2^2 = 0, \quad q_3^2 = M_3^2, \quad q_4^2 = q_5^2 = 0.$$

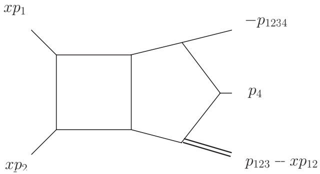

FIGURE : The parametrization of external momenta in terms of x for the planar pentabox of the family P_1. All external momenta are incoming.

$$s_{12} := p_{12}^2, \quad s_{23} := p_{23}^2, \quad s_{34} := p_{34}^2, \quad s_{45} := p_{45}^2 = p_{123}^2, \quad s_{51} := p_{15}^2 = p_{234}^2,$$

$$q_1^2 = q_2^2 = q_4^2 = q_5^2 = 0 \quad q_3^2 = (s_{45} - s_{12}x)(1-x)$$
$$q_{12}^2 = s_{12}x^2 \;\; q_{23}^2 = s_{45}(1-x) + s_{23}x \;\; q_{34}^2 = (s_{34} - s_{12}(1-x))\,x \;\; q_{45}^2 = s_{45} \;\; q_{51}^2 = s_{51}x$$

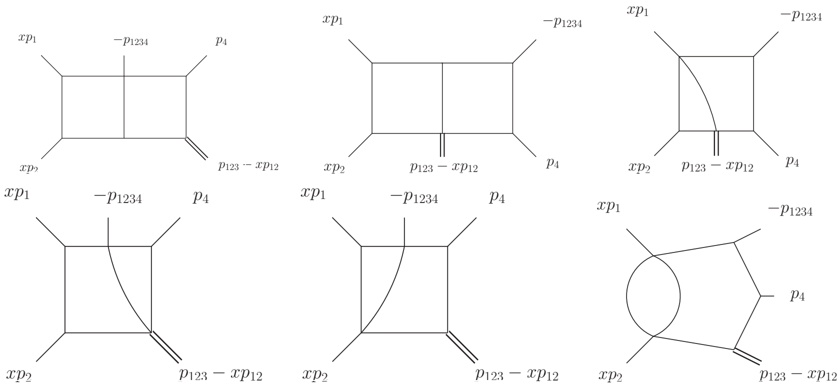

FIGURE : The five-point Feynman diagrams, besides the pentabox itself in Figure 1, that are contained in the family P_1. All external momenta are incoming.

5BOX - ONE LEG OFF-SHELL: P1

$$G^{P_1}_{a_1\cdots a_{11}}(x,s,\epsilon) := e^{2\gamma_E\epsilon}\int \frac{d^dk_1}{i\pi^{d/2}}\frac{d^dk_2}{i\pi^{d/2}}\frac{1}{k_1^{2a_1}(k_1+xp_1)^{2a_2}(k_1+xp_{12})^{2a_3}(k_1+p_{123})^{2a_4}}$$
$$\times\frac{1}{(k_1+p_{1234})^{2a_5}k_2^{2a_6}(k_2-xp_1)^{2a_7}(k_2-xp_{12})^{2a_8}(k_2-p_{123})^{2a_9}(k_2-p_{1234})^{2a_{10}}(k_1+k_2)^{2a_{11}}},$$

$P_1(74)$: {10000000101, 01000000101, 00100000101, 10000001001, 01000000011, 00100000011, 10100001100,
10100001010, 10100101000, 01000101001, 10100100100, 10100000102, 10100000101, 10100000011,
10000001102, 10000001101, 10000001011, 01000100101, 01000001101, 01000001011, 00100100102,
00100100101, 11100000101, 11100000011, 11000001102, 11000001101, 11000001012, 11000001011,
11000000111, 10100000112, 10000001111, 01100100102, 01100100101, 01100100011, 01100000111,
01000101102, 01000101101, 01000101011, 01000100111, 01000001111, 00100100111, 10100101100,
10100100101, 10100001101, 10100001011, 10100000111, 111m0000111, 110000m1111, 11000001111,
10100101110, 10100100111, 10100001111, 011001m0111, 01100100111, 010m0101111, 01000101111,
11100100101, 11100001101, 11100001011, 11100000111, 111m0101101, 111001m1101, 11100101101,
1110m101011, 11100101011, 111m0100111, 11100100111, 111000m1111, 111m0001111, 11100001111,
111001m0111, 11100101111, 111001m1111, 111m0101111},

m= −1

5box P1 - DE

$$\partial_x \mathbf{G} = \mathbf{M}\left(\{s_{ij}\}, \varepsilon, x\right) \mathbf{G}$$

$$\partial_x \mathbf{G}' = \mathbf{M}'\mathbf{G}' \qquad \mathbf{M}' = \mathbf{T}\mathbf{M}\mathbf{T}^{-1} + \left(\partial_x \mathbf{T}\right)\mathbf{T}^{-1} \qquad \mathbf{G}' = \mathbf{T}\mathbf{G}$$

$$\left(M_D\right)_{IJ} = \delta_{IJ} M_{II}\left(\varepsilon = 0\right), I, J = 1 \ldots 74$$

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$$1 - \frac{s_{34} - s_{51}}{s_{12}}, \quad \frac{s_{45} - s_{23}}{s_{12}}, \quad -\frac{s_{51}}{s_{12}}, \quad \frac{s_{45}}{-s_{23} + s_{45} + s_{51}}, \quad \frac{s_{45}}{s_{34} + s_{45}},$$

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$$\frac{s_{12}s_{23}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_1}}{2s_{12}\left(s_{23}+s_{34}-s_{51}\right)}, \quad \frac{s_{12}s_{45}\pm\sqrt{\Delta_3}}{s_{12}s_{34}+s_{12}s_{45}}, \quad \frac{s_{45}}{s_{12}+s_{23}},$$

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Letters (20):

$$0, \quad 1, \quad \frac{s_{45}}{s_{45}-s_{23}}, \quad \frac{s_{45}}{s_{12}}, \quad 1-\frac{s_{34}}{s_{12}}, \quad 1+\frac{s_{23}}{s_{12}},$$

$$1-\frac{s_{34}-s_{51}}{s_{12}}, \quad \frac{s_{45}-s_{23}}{s_{12}}, \quad -\frac{s_{51}}{s_{12}}, \quad \frac{s_{45}}{-s_{23}+s_{45}+s_{51}}, \quad \frac{s_{45}}{s_{34}+s_{45}},$$

$$\frac{s_{12}s_{23}-2s_{12}s_{45}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_1}}{2s_{12}\left(s_{23}-s_{45}-s_{51}\right)}, \quad \frac{s_{12}s_{23}-s_{12}s_{45}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_2}}{2s_{12}\left(s_{23}-s_{45}-s_{51}\right)},$$

$$\frac{s_{12}s_{23}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_1}}{2s_{12}\left(s_{23}+s_{34}-s_{51}\right)}, \quad \frac{s_{12}s_{45}\pm\sqrt{\Delta_3}}{s_{12}s_{34}+s_{12}s_{45}}, \quad \frac{s_{45}}{s_{12}+s_{23}},$$

5BOX P1 - DE

$$M_{IJ} = N_{IJ}(\varepsilon)\left(\sum_{i=1}^{20}\sum_{j=1}^{2}\sum_{k=0}^{1}\frac{C_{IJ;ijk}\varepsilon^k}{(x-l_i)^j} + \sum_{j=0}^{1}\sum_{k=0}^{1}\tilde{C}_{IJ;jk}\varepsilon^k x^j\right).$$

$$\int_0^x dt \frac{1}{(t-a_n)^2}\mathcal{G}(a_{n-1},\ldots,a_1,t) \qquad \int_0^x dt\; t^m\; \mathcal{G}(a_{n-1},\ldots,a_1,t)$$

Fuchsian

$N_{IJ}(\varepsilon) = n_J(\varepsilon)/n_I(\varepsilon)$, $G_I \to n_I(\varepsilon)\, G_I$

$$M_{IJ} = \left(\sum_{i=1}^{20}\sum_{j=1}^{2}\sum_{k=0}^{1}\frac{C_{IJ;ijk}\varepsilon^k}{(x-l_i)^j} + \sum_{j=0}^{1}\sum_{k=0}^{1}\tilde{C}_{IJ;jk}\varepsilon^k x^j\right).$$

$$\mathbf{G} \to (\mathbf{I}-\mathbf{K}_i)\,\mathbf{G}, \quad \mathbf{M} \to (\mathbf{M}-\partial_x\mathbf{K}_i-\mathbf{K}_i\mathbf{M})(\mathbf{I}-\mathbf{K}_i)^{-1} \quad i=1,2,3$$

$$\partial_x\mathbf{G} = \left(\varepsilon\sum_{a=1}^{19}\frac{\mathbf{M}_a}{(x-l_a)}\right)\mathbf{G}$$

Fuchsian
$N_{IJ}(\varepsilon) = n_J(\varepsilon)/n_I(\varepsilon), \; G_I \to n_I(\varepsilon)\, G_I$

$$M_{IJ} = \left(\sum_{i=1}^{20} \sum_{j=1}^{2} \sum_{k=0}^{1} \frac{C_{IJ;ijk}\varepsilon^k}{(x-l_i)^j} + \sum_{j=0}^{1} \sum_{k=0}^{1} \tilde{C}_{IJ;jk}\varepsilon^k x^j \right).$$

$$\mathbf{G} \to (\mathbf{I} - \mathbf{K}_i)\,\mathbf{G}, \quad \mathbf{M} \to (\mathbf{M} - \partial_x \mathbf{K}_i - \mathbf{K}_i \mathbf{M})(\mathbf{I} - \mathbf{K}_i)^{-1} \quad i = 1, 2, 3$$

$$\partial_x \mathbf{G} = \left(\varepsilon \sum_{a=1}^{19} \frac{\mathbf{M}_a}{(x - l_a)} \right) \mathbf{G}$$

Fuchsian

$N_{IJ}(\varepsilon) = n_J(\varepsilon)/n_I(\varepsilon)$, $G_I \to n_I(\varepsilon)\, G_I$

$$M_{IJ} = \left(\sum_{i=1}^{20} \sum_{j=1}^{2} \sum_{k=0}^{1} \frac{C_{IJ;ijk}\varepsilon^k}{(x-l_i)^j} + \sum_{j=0}^{1} \sum_{k=0}^{1} \tilde{C}_{IJ;jk}\varepsilon^k x^j \right).$$

$$\mathbf{G} \to (\mathbf{I} - \mathbf{K}_i)\,\mathbf{G}, \quad \mathbf{M} \to (\mathbf{M} - \partial_x \mathbf{K}_i - \mathbf{K}_i \mathbf{M})(\mathbf{I} - \mathbf{K}_i)^{-1} \quad i = 1,2,3$$

$\mathbf{M}(\varepsilon = 0)$ contains $(x - l_i)^{-2}$ and x^0

$$(\mathbf{K}_1)_{IJ} = \begin{cases} \int dx (\mathbf{M}(\varepsilon = 0))_{IJ} & I, J \neq 69, 74 \\ 0 & I, J = 69, 74 \end{cases}$$

$$(\mathbf{K}_2)_{IJ} = \begin{cases} \int dx (\mathbf{M}(\varepsilon = 0))_{IJ} & I, J \neq 74 \\ 0 & I, J = 74 \end{cases}$$

$$(\mathbf{K}_3)_{IJ} = \int dx (\mathbf{M}(\varepsilon = 0))_{IJ}$$

M.A. Barkatou and E.Pflügel, Journal of Symbolic Computation, **44** (2009),1017

$$\partial_x \mathbf{G} = \left(\varepsilon \sum_{a=1}^{19} \frac{\mathbf{M}_a}{(x - l_a)} \right) \mathbf{G}$$

5BOX P1 - DE

Fuchsian

$N_{IJ}(\varepsilon) = n_J(\varepsilon)/n_I(\varepsilon),\ G_I \to n_I(\varepsilon)\, G_I$

$$M_{IJ} = \left(\sum_{i=1}^{20} \sum_{j=1}^{2} \sum_{k=0}^{1} \frac{C_{IJ;ijk}\varepsilon^k}{(x-l_i)^j} + \sum_{j=0}^{1} \sum_{k=0}^{1} \tilde{C}_{IJ;jk}\varepsilon^k x^j \right).$$

$$\mathbf{G} \to (\mathbf{I} - \mathbf{K}_i)\,\mathbf{G}, \quad \mathbf{M} \to (\mathbf{M} - \partial_x \mathbf{K}_i - \mathbf{K}_i \mathbf{M})\,(\mathbf{I} - \mathbf{K}_i)^{-1} \quad i = 1,2,3$$

$$\partial_x \mathbf{G} = \left(\varepsilon \sum_{a=1}^{19} \frac{\mathbf{M}_a}{(x - l_a)} \right) \mathbf{G}$$

- Solution:

$$
\begin{aligned}
\mathbf{G} \quad = \quad & \varepsilon^{-2}\mathbf{b}_0^{(-2)} + \varepsilon^{-1}\left(\sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(-2)} + \mathbf{b}_0^{(-1)}\right) \\
+ \quad & \varepsilon^{0}\left(\sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(-2)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(-1)} + \mathbf{b}_0^{(0)}\right) \\
+ \quad & \varepsilon\left(\sum \mathcal{G}_{abc} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{b}_0^{(-2)} + \sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(-1)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(0)} + \mathbf{b}_0^{(1)}\right) \\
+ \quad & \varepsilon^{2}\left(\sum \mathcal{G}_{abcd} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{M}_d \mathbf{b}_0^{(-2)} + \sum \mathcal{G}_{abc} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{b}_0^{(-1)}\right. \\
+ \quad & \left.\sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(0)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(1)} + \mathbf{b}_0^{(2)}\right)
\end{aligned}
$$

$\mathbf{b}_0^{(k)}$, $k = -2, \ldots, 2$ representing the x-independent boundary terms in the limit $x = 0$ at order ε^k

$$
\mathbf{G} \underset{x \to 0}{\sim} \sum_{k=-2}^{2} \varepsilon^k \sum_{n=0}^{k+2} \mathbf{b}_n^{(k)} \log^n(x) + \text{subleading terms.}
$$

$\mathcal{G}_{a,b,\ldots} = \mathcal{G}(l_a, l_b, \ldots; x)$ with $a, b, c, d = 1, \ldots, 19$.

- Uniform transcendental: UT multi- vs one-parameter DE

$\mathbf{M}_a$ depend on kinematics, but eigenvalues not: $(x - l_a)^{-n_a\varepsilon}$, n_a positive integers, $x \to l_a$.

- Solution:

$$
\begin{aligned}
\mathbf{G} &= \varepsilon^{-2}\mathbf{b}_0^{(-2)} + \varepsilon^{-1}\left(\sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(-2)} + \mathbf{b}_0^{(-1)}\right) \\
&+ \varepsilon^{0}\left(\sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(-2)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(-1)} + \mathbf{b}_0^{(0)}\right) \\
&+ \varepsilon\left(\sum \mathcal{G}_{abc} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{b}_0^{(-2)} + \sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(-1)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(0)} + \mathbf{b}_0^{(1)}\right) \\
&+ \varepsilon^{2}\left(\sum \mathcal{G}_{abcd} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{M}_d \mathbf{b}_0^{(-2)} + \sum \mathcal{G}_{abc} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{b}_0^{(-1)}\right. \\
&+ \left.\sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(0)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(1)} + \mathbf{b}_0^{(2)}\right)
\end{aligned}
$$

$\mathbf{b}_0^{(k)}$, $k = -2, \ldots, 2$ representing the x-independent boundary terms in the limit $x = 0$ at order ε^k

$$
\mathbf{G} \underset{x \to 0}{\sim} \sum_{k=-2}^{2} \varepsilon^k \sum_{n=0}^{k+2} \mathbf{b}_n^{(k)} \log^n(x) + \text{subleading terms.}
$$

$\mathcal{G}_{a,b,\ldots} = \mathcal{G}(l_a, l_b, \ldots; x)$ with $a, b, c, d = 1, \ldots, 19$.

- Uniform transcendental: UT multi- vs one-parameter DE

$\mathbf{M}_a$ depend on kinematics, but eigenvalues not: $(x - l_a)^{-n_a\varepsilon}$, n_a positive integers, $x \to l_a$.

Baikov representation

Baikov, P. A., Phys. Lett. B385 (1996)

$$F_{\alpha_1 \ldots \alpha_N} = \int \left(\prod_{i=1}^{L} \frac{d^d k_i}{i\pi^{d/2}} \right) \frac{1}{D_1^{\alpha_1} \ldots D_N^{\alpha_N}}$$

$$D_a = \sum_{i=1}^{L} \sum_{j=i}^{M} A_a^{ij} s_{ij} + f_a = \sum_{i=1}^{L} \sum_{j=i}^{L} A_a^{ij} k_i \cdot k_j + \sum_{i=1}^{L} \sum_{j=L+1}^{M} A_a^{ij} k_i \cdot p_{j-L} + f_a, \quad a = 1, \ldots, N$$

$$F_{\alpha_1 \ldots \alpha_N} = C_N^L (G(p_1, \ldots, p_E))^{(-d+E+1)/2} \int \frac{dx_1 \ldots dx_N}{x_1^{\alpha_1} \ldots x_N^{\alpha_N}} P_N^L (x_1 - f_1, \ldots, x_N - f_N)^{(d-M-1)/2}$$

$$C_N^L = \frac{\pi^{-L(L-1)/4 - LE/2}}{\prod_{i=1}^{L} \Gamma\left(\frac{d-M+i}{2}\right)} \det\left(A_{ij}^a\right)$$

$$P_N^L(x_1, x_2, \ldots, x_N) = G(k_1, \ldots, k_L, p_1, \ldots, p_E) \Big|_{s_{ij} = \sum_{a=1}^{N} A_{ij}^a x_a \;\&\; s_{ji} = s_{ij}}$$

Baikov representation – IBP

P. A. Baikov, Nucl. Instrum. Meth. A **389**, 347 (1997) [hep-ph/9611449].

$$O_{ij} P_N^L = 0 \tag{2.5}$$

with the operators O_{ij} given by $(i = 1, \ldots, L)$

$$j \leq L \, (q_j = k_j) \qquad O_{ij} = d\delta_{ij} + \sum_{a=1}^{N} \sum_{b=1}^{N} \sum_{m=1}^{M} A_a^{mi} A_{mj}^{b} \left(1 + \delta_{mi}\right) \left(x_b - f_b\right) \frac{\partial}{\partial x_a} \tag{2.6}$$

and

$$j > L \, (q_j = p_{j-L}) \quad O_{ij} = \sum_{a=1}^{N} \left(\sum_{m=1}^{L} \sum_{b=1}^{N} A_a^{mi} A_{mj}^{b} \left(1 + \delta_{mi}\right) \left(x_b - f_b\right) + \sum_{m=L+1}^{M} A_a^{mi} s_{mj} \right) \frac{\partial}{\partial x_a} \tag{2.7}$$

Baikov representation – Integration Limits

H. Frellesvig and C. G. Papadopoulos, arXiv:1701.07356 [hep-ph].

$$F_{\alpha_1 \ldots \alpha_{N-1} 0} = C_N^1 G(p_1, \ldots, p_{N-1})^{(N-d)/2} \int \frac{dx_1 \ldots dx_{N-1}}{x_1^{\alpha_1} \ldots x_{N-1}^{\alpha_{N-1}}} \int_{x_N^-}^{x_N^+} dx_N {P_N^1}^{(d-N-1)/2}$$

$$= C_{N-1}^1 G(p_1, \ldots, p_{N-2})^{(N-1-d)/2} \int \frac{dx_1 \ldots dx_{N-1}}{x_1^{\alpha_1} \ldots x_{N-1}^{\alpha_{N-1}}} {P_{N-1}^1}^{(d-(N-1)-1)/2} \qquad (2.10)$$

where $P_N^1\left(x_N^+\right) = P_N^1\left(x_N^-\right) = 0$ and

$$\int_{x_N^-}^{x_N^+} dx_N {P_N^1}^{(d-N-1)/2} = \frac{2\pi^{1/2}\Gamma\left(\frac{d-N+1}{2}\right)}{\Gamma\left(\frac{d-N+2}{2}\right)} G\left(p_1, \ldots, p_{N-1}\right)^{(d-N)/2} G\left(p_1, \ldots, p_{N-2}\right)^{(N-1-d)/2}$$

$P_N^1 = \frac{1}{4} G\left(p_1, \ldots, p_{N-2}\right)\left(x_N^+ - x_N\right)\left(x_N - x_N^-\right)$ and $\left(x_N^+ - x_N^-\right)^2 = 16 \frac{G(p_1, \ldots, p_{N-1})}{G(p_1, \ldots, p_{N-2})^2} P_{N-1}^1$

J. Bosma, M. Sogaard and Y. Zhang, arXiv:1704.04255 [hep-th].

M. Harley, F. Moriello and R. M. Schabinger, arXiv:1705.03478 [hep-ph].

S. Abreu, R. Britto, C. Duhr and E. Gardi, arXiv:1702.03163 [hep-th].

Feynman-parameter representation

T. Binoth and G. Heinrich, Nucl. Phys. B **585** (2000) 741 [hep-ph/0004013].

$$\frac{1}{\prod A_l^{\lambda_l}} = \frac{\Gamma(\sum \lambda_l)}{\prod \Gamma(\lambda_l)} \int_0^1 \mathrm{d}\xi_1 \ldots \int_0^1 \mathrm{d}\xi_L \prod_l \xi_l^{\lambda_l - 1} \frac{\delta(\sum \xi_l - 1)}{(\sum A_l \xi_l)^{\sum \lambda_l}},$$

where $A_l = m_l^2 - p_l^2$.

$$F_\Gamma(q_1, \ldots, q_n; d) = (-1)^a \frac{\left(\mathrm{i}\pi^{d/2}\right)^h \Gamma(a - hd/2)}{\prod_l \Gamma(a_l)} \times \int_0^\infty \mathrm{d}\alpha_1 \ldots \int_0^\infty \mathrm{d}\alpha_L \, \delta\left(\sum \alpha_l - 1\right) \frac{\mathcal{U}^{a-(h+1)d/2} \prod_l \alpha_l^{a_l - 1}}{\mathcal{W}^{a - hd/2}}.$$

$$F(s, t, p_1^2, p_2^2; a_1, \ldots, a_4; d) = \mathrm{i}\pi^{d/2}(-1)^a \frac{\Gamma(a + \varepsilon - 2)}{\prod \Gamma(a_l)} \times \int_0^\infty \cdots \int_0^\infty \left(\prod_{l=1}^4 \alpha_l^{a_l - 1} \mathrm{d}\alpha_l\right) \delta\left(\sum_{l=1}^4 \alpha_l - 1\right) \times (-s\alpha_1\alpha_3 - t\alpha_2\alpha_4 - p_1^2\alpha_1\alpha_2 - p_2^2\alpha_2\alpha_3 - \mathrm{i}0)^{2-a-\varepsilon}.$$

Mellin-Barnes representation

V. A. Smirnov, Phys. Lett. B **460** (1999) 397 [hep-ph/9905323].

$$\frac{1}{(X_1+\cdots+X_n)^\lambda} = \frac{1}{\Gamma(\lambda)}\frac{1}{(2\pi i)^{n-1}}\int_{-i\infty}^{+i\infty}\cdots\int_{-i\infty}^{+i\infty} dz_2\ldots dz_n \prod_{i=2}^{n} X_i^{z_i} \times X_1^{-\lambda-z_2-\cdots-z_n}\Gamma(\lambda+z_2+\cdots+z_n)\prod_{i=2}^{n}\Gamma(-z_i). \quad (5.11)$$

$$F(s,t,p_1^2,p_2^2;a_1,\ldots,a_4;d) = \frac{i\pi^{d/2}(-1)^a}{\Gamma(4-2\varepsilon-a)\prod\Gamma(a_l)(-s)^{a+\varepsilon-2}} \times \frac{1}{(2\pi i)^3}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty} dz_2 dz_3 dz_4 \frac{(-p_1^2)^{z_2}(-p_2^2)^{z_3}(-t)^{z_4}}{(-s)^{z_2+z_3+z_4}} \times \Gamma(a+\varepsilon-2+z_2+z_3+z_4)\Gamma(a_2+z_2+z_3+z_4)\Gamma(a_4+z_4) \times \Gamma(2-\varepsilon-a_{234}-z_3-z_4)\Gamma(2-\varepsilon-a_{124}-z_2-z_4) \times \Gamma(-z_2)\Gamma(-z_3)\Gamma(-z_4). \quad (5.12)$$

$$\frac{\partial}{\partial X}F_{\alpha_1\ldots\alpha_N} = \left(\frac{-d+E+1}{2}\right)\left(\frac{1}{G}\frac{\partial G}{\partial X}\right)F_{\alpha_1\ldots\alpha_N} \qquad (4.1)$$
$$+ \; C_N^L G^{(-d+E+1)/2}\int \frac{dx_1\ldots dx_N}{x_1^{\alpha_1}\ldots\, x_N^{\alpha_N}} P_N^{L\,(d-M-1)/2}\left[\left(\frac{d-M-1}{2}\right)\frac{1}{P_N^L}\frac{\partial P_N^L}{\partial X}\right]$$

$$b\frac{\partial P_N^L}{\partial X} + \sum_a c_a \frac{\partial P_N^L}{\partial x_a} = 0$$

$$\frac{\partial}{\partial X}F_{\alpha_1\ldots\alpha_N} = \left(\frac{-d+E+1}{2}\right)\frac{1}{G}\frac{\partial G}{\partial X}F_{\alpha_1\ldots\alpha_N}$$
$$+ \; C_N^L G^{(-d+E+1)/2}\int \frac{dx_1\ldots dx_N}{x_1^{\alpha_1}\ldots\, x_N^{\alpha_N}}\left(-\sum_a \frac{c_a}{b}\frac{\partial}{\partial x_a}P_N^{L\,(d-M-1)/2}\right)$$

$$\frac{\partial}{\partial X}F_{\alpha_1\ldots\alpha_N} = \left(\frac{-d+E+1}{2}\right)\frac{1}{G}\frac{\partial G}{\partial X}F_{\alpha_1\ldots\alpha_N} \qquad (4.4)$$
$$+ \; C_N^L G^{(-d+E+1)/2}\int dx_1\ldots dx_N P_N^{L\,(d-M-1)/2}\left\{\sum_a \frac{\partial}{\partial x_a}\left(\frac{c_a}{b}\frac{1}{x_1^{\alpha_1}\ldots\, x_N^{\alpha_N}}\right)\right\}$$

sygyzy equation

Definition:

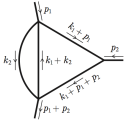
$$F_{\alpha_1\ldots\alpha_N}|_{n\times\text{cut}} \equiv C_N^L (G)^{(-d+E+1)/2} \left(\prod_{a=n+1}^{N} \int dx_a\right) \left(\prod_{c=1}^{n} \oint_{x_c=0} dx_c\right) \frac{1}{x_1^{\alpha_1}\ldots x_N^{\alpha_N}} P_N^{L\,(d-M-1)/2}$$

$$\frac{\partial}{\partial X_j} F_i = \sum_{l=1}^{I} M_{il}^{(j)} F_l \qquad \frac{\partial}{\partial X_j} F_i|_{n\times\text{cut}} = \sum_{l=1}^{I} M_{il}^{(j)} F_l|_{n\times\text{cut}}$$

cut-integrals satisfy the same DE.

A. Primo and L. Tancredi, Nucl. Phys. B **916** (2017) 94 [arXiv:1610.08397 [hep-ph]].

Canonical Basis



$$I_1 = \epsilon R_{12} F_{11210} \qquad I_2 = \left(s F_{1221-1} - \tfrac{1}{2}\epsilon(p_1^2 - p_2^2 - s) F_{11210} \right)$$

$$\begin{aligned} I_{1\,|4\times\mathrm{cut}} &= \frac{2^{4\epsilon-3}\epsilon\cos(\pi\epsilon)\Gamma\left(\epsilon+\frac{1}{2}\right)}{\pi^2\Gamma\left(\frac{3}{2}-\epsilon\right)} \left(p_1^2\right)^{-2\epsilon} x^{-\epsilon}(x+1)^{-\epsilon}(y-1)(xy+1)^{-\epsilon} \\ &\times\ {}_2F_1(1-\epsilon,\epsilon+1;2-2\epsilon;1-y) \end{aligned}$$

$$I_{2\,|4\times\mathrm{cut}} = \frac{4^{2\epsilon-1}}{\pi\Gamma\left(\frac{1}{2}-\epsilon\right)^2}\left(p_1^2\right)^{-2\epsilon} x^{-\epsilon}(x+1)^{-\epsilon}(xy+1)^{-\epsilon}\,{}_2F_1(-\epsilon,\epsilon;-2\epsilon;1-y)$$

$$\begin{aligned} N_\epsilon I_{1\,|4\times\mathrm{cut}} &= \epsilon\log(y) + \epsilon^2\left(-2\,\mathrm{Li}_2(1-y) - \log^2(y)\right) + \epsilon^3\Big(-4\,\mathrm{Li}_3(1-y) - 2\,\mathrm{Li}_3(y) \\ &- \mathrm{Li}_2(y)\log(y) + \frac{2}{3}(\log(y) - 3\log(1-y))\log^2(y) + 2\,\zeta(3)\Big) + \mathcal{O}(\epsilon^4) \quad \text{(B.29)} \end{aligned}$$

$$\begin{aligned} N_\epsilon I_{2\,|4\times\mathrm{cut}} &= 1 - \frac{1}{2}\epsilon\log(y) + \frac{1}{2}\epsilon^2\left(\log^2(y) - \pi^2\right) + \frac{1}{12}\epsilon^3\Big(36\,\mathrm{Li}_3(y) + 18\,\mathrm{Li}_2(1-y)\log(y) \\ &- 4\log^3(y) + 18\log(1-y)\log^2(y) - 3\pi^2\log(y) - 92\,\zeta(3)\Big) + \mathcal{O}(\epsilon^4) \qquad \text{(B.30)} \end{aligned}$$

with $N_\epsilon = e^{2\gamma_E\epsilon}(p_1^2)^\epsilon x^\epsilon (x+1)^\epsilon (xy+1)^\epsilon$.

Canonical Basis

R. Bonciani, V. Del Duca, H. Frellesvig, J. M. Henn, F. Moriello and V. A. Smirnov, JHEP **1612** (2016) 096 [arXiv:1609.06685 [hep-ph]].

$$F_{\text{box-triangle}} = \int \frac{d^d k_1\, d^d k_2}{(i\pi^{d/2})^2} \frac{1}{x_1 x_2 x_3 x_4 x_5 x_6} \qquad \text{(B.31)}$$

with

$$\begin{aligned} &x_1 = k_1^2 - m^2 && x_2 = (k_1+p_1)^2 - m^2 && x_3 = (k_1+p_1+p_2)^2 - m^2 \\ &x_4 = (k_2-p_4)^2 - m^2 && x_5 = k_2^2 - m^2 && x_6 = (k_1-k_2)^2 \\ &x_7 = (k_1-p_4)^2 \end{aligned} \qquad \text{(B.32)}$$

$$F_1(z) = m^4 - 2m^2 p_4^2 + p_4^4 - 2m^2 z - 2p_4^2 z + z^2$$
$$F_2(z) = s(m^4 s + 2m^2(2tu + s(t-z)) + s(t-z)^2)$$

$$F_{\text{box-triangle}|6\times\text{cut}} = C \int_{r_-}^{r_+} \frac{\mathrm{d}z}{\sqrt{F_1(z)F_2(z)}} + \mathcal{O}(\epsilon)$$

$$F_{\text{box-triangle}|6\times\text{cut}} = \frac{2iC}{\sqrt{X}} K\left(\frac{-16m^2\sqrt{-p_4^2 stu}}{X}\right) + \mathcal{O}(\epsilon)$$

$$X = s(p_4^2 - t)^2 - 4m^2\left(p_4^2 s - tu + 2\sqrt{-p_4^2 stu}\right)$$

Canonical Basis

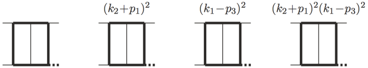


Figure 5. The four master integrals of the elliptic sector $I^{A}_{1,1,1,1,1,1,1,0,0}$.

$$F_{\text{ell. double-box}} = \int \frac{d^d k_1\, d^d k_2}{(i\pi^{d/2})^2} \frac{1}{x_1 x_2 x_3 x_4 x_5 x_6 x_7} \qquad \text{(B.40)}$$

$$\begin{aligned} &x_1 = k_1^2 - m^2 && x_2 = (k_1+p_1)^2 - m^2 && x_3 = (k_1+p_1+p_2)^2 - m^2 \\ &x_4 = (k_2+p_1+p_2)^2 - m^2 && x_5 = (k_2-p_4)^2 - m^2 && x_6 = k_2^2 - m^2 \\ &x_7 = (k_1-k_2)^2 && x_8 = (k_1-p_4)^2 && \end{aligned} \qquad \text{(B.41)}$$

$$F_{\text{ell. double-box}} = \frac{-\pi^{-3}}{\Gamma^2\left(\frac{d-3}{2}\right)} \frac{\det(A^{-1})}{\sqrt{-G_1}} \int \frac{1}{x_1 \cdots x_7} \frac{\lambda_{22}^{(d-5)/2}\, \lambda_{11}^{(d-5)/2}}{\sqrt{-G_2}} \mathrm{d}^8 x \qquad \text{(B.42)}$$

$$F_{\text{ell. double-box}|7\times\text{cut}} = \frac{C}{\sqrt{s(s-4m^2)}} \int_{r_-}^{r_+} \frac{\mathrm{d}z}{z\sqrt{f(z)}} + \mathcal{O}(\epsilon) \qquad \text{(B.43)}$$

$$f(z) = s(4m^2 tu + s(t-z)^2) \qquad r_{\mp} = t \mp 2\sqrt{-m^2 stu}/s$$

$$F_{\text{ell. double-box}|7\times\text{cut}} = \frac{-i}{4\pi^3} \frac{1}{s\sqrt{(4m^2-s)t(st+4m^2u)}} + \mathcal{O}(\epsilon)$$

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