Maplesoft
Mathematics • Modeling • Simulation

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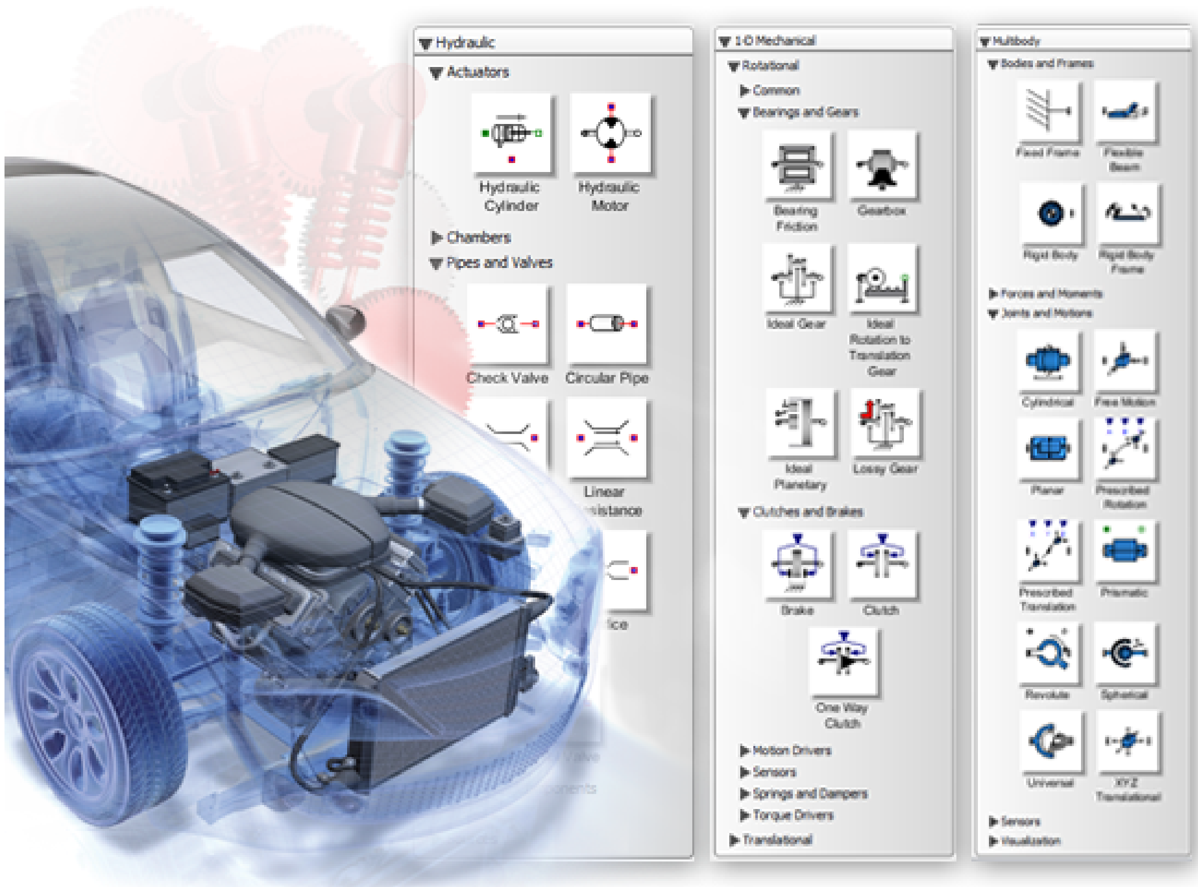
HiggsTools & Maplesoft

- Giulia Gonella - interactive equation solving
- Agnieszka Ilnicka - generalizing bivariate limit code & extending Feynman diagram code
- Raquel Gomez - evaluating *Magma* library for matrix algebra on GPUs (like LAPACK)
- Hjalte Frellesvig - implement the generalized polylogarithm in Maple

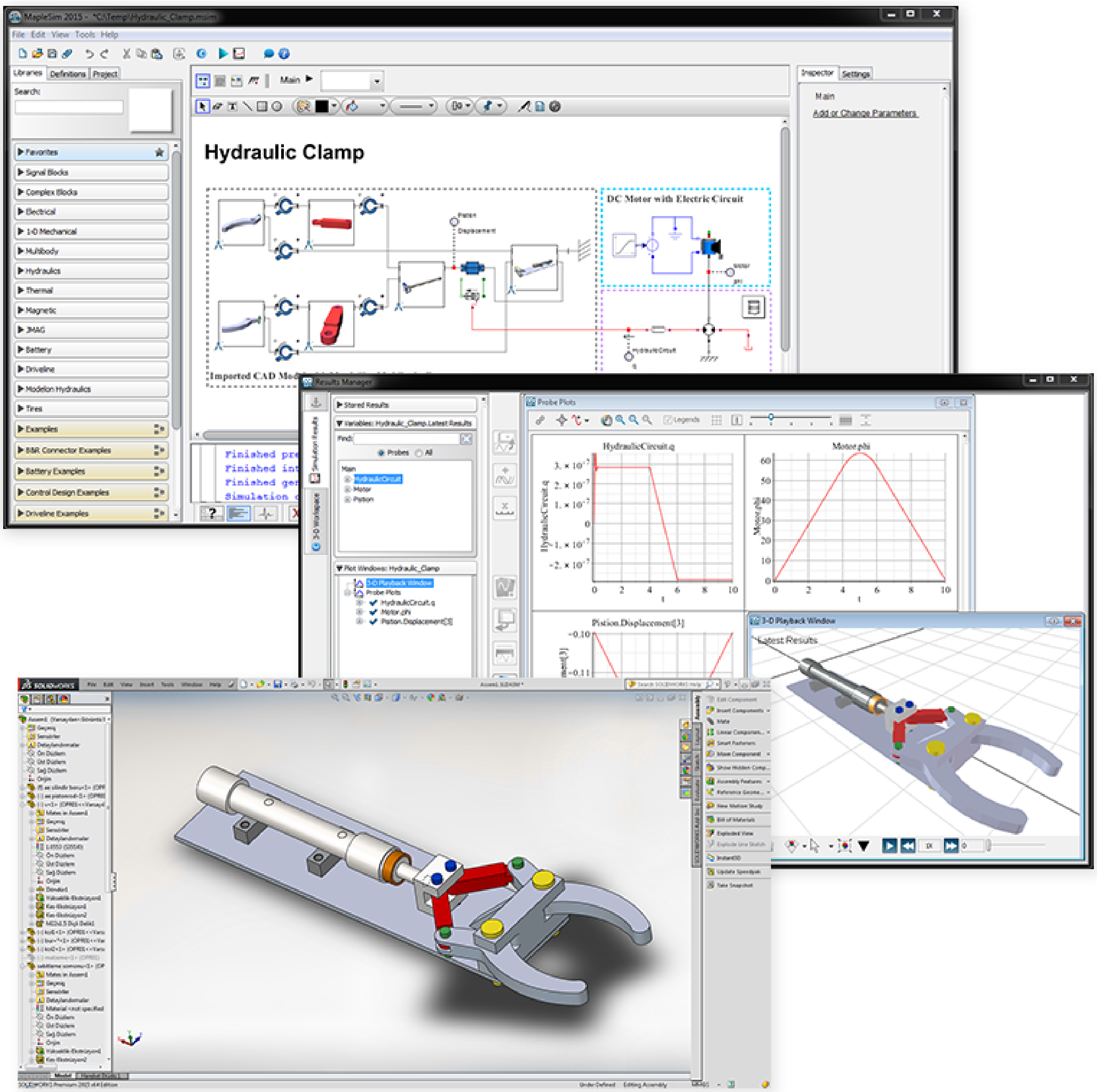
What's new at Maplesoft

MapleSim

- Multi-Domain Physical Modeling and Simulation



- Fast simulation and visualization



Möbius

- Online courseware environment that focuses on science, technology, engineering, and mathematics
- Automated assessment
- Gradebook and analytics

Maple

- At the heart of it all

What's new with Maple

Kernel

- Multithreaded garbage collection, mostly in a thread separate from the "main" thread
- Updated builtin libraries for polynomial arithmetic

GUI

• Construct collections of user interface elements programmatically - see this example
• Workbook: collection of worksheets and data files

Import("this://IRIS.csv");

	1	2	3	4	5	6	7	8	...
1	5.1	0.222222222	3.5	0.625	1.4	0.06779661	0.2	0.041666667	...
2	4.9	0.166666667	3	0.416666667	1.4	0.06779661	0.2	0.041666667	...
3	4.7	0.111111111	3.2	0.5	1.3	0.050847458	0.2	0.041666667	...
4	4.6	0.083333333	3.1	0.458333333	1.5	0.084745763	0.2	0.041666667	...
5	5	0.194444444	3.6	0.666666667	1.4	0.06779661	0.2	0.041666667	...
6	5.4	0.305555556	3.9	0.791666667	1.7	0.118644068	0.4	0.125	...
7	4.6	0.083333333	3.4	0.583333333	1.4	0.06779661	0.3	0.083333333	...
8	5	0.194444444	3.4	0.583333333	1.5	0.084745763	0.2	0.041666667	...
...	...	...	...	...	...	...	...	...	...

(3.2.1)

Math library

• Most of the stuff that will interest you most

What's new with the math library

ThermophysicalData

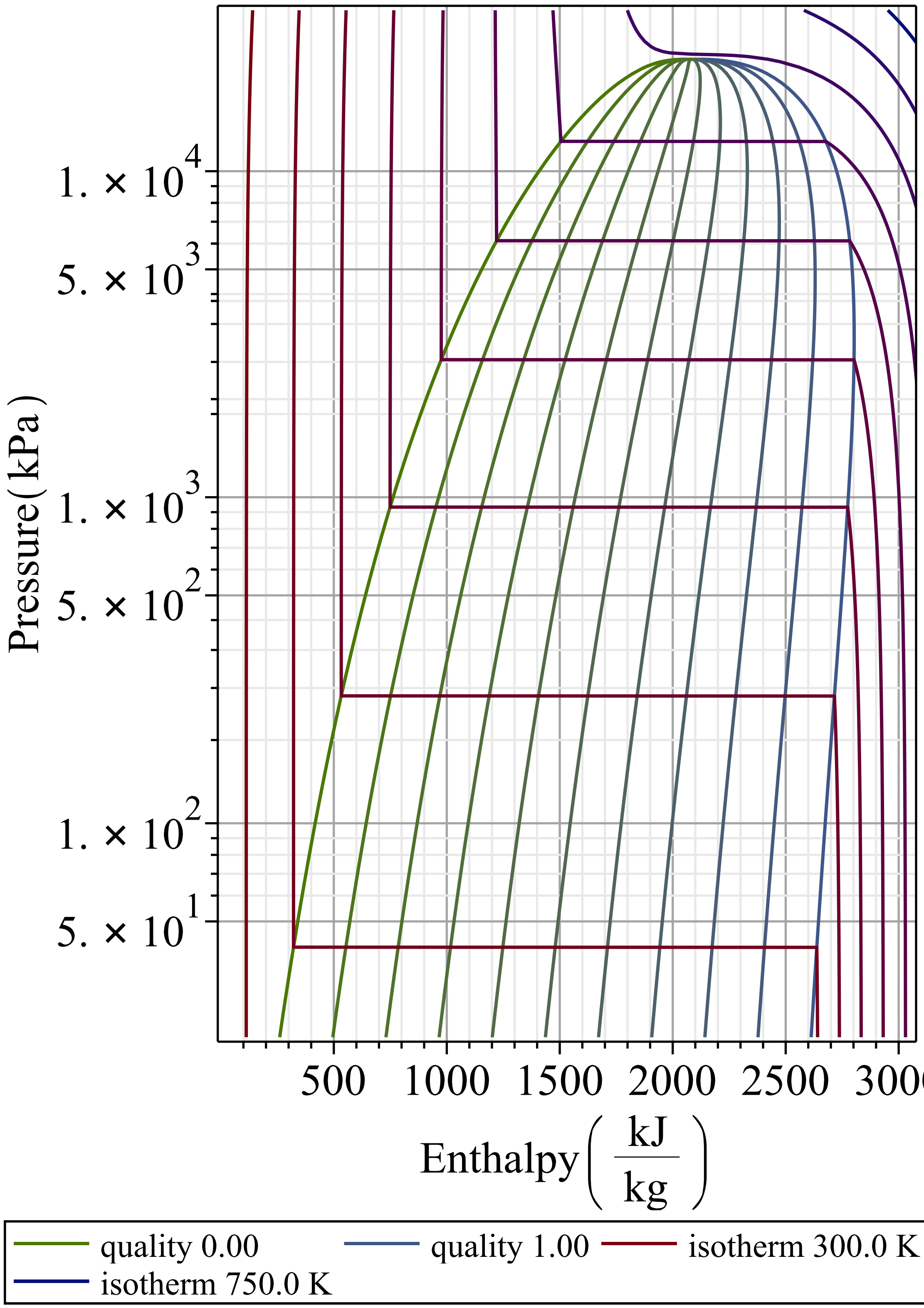
with(*ThermophysicalData*) :
Property(*temperature_triple*, *Water*, *useunits*);

$$273.1600000 \text{ K} \qquad \textbf{(4.1.1)}$$

$$Property\left(temperature\_dew\_point, HumidAir, temperature\_dry\_bulb = 300, pressure = 1 \text{ atm}, humidity = \frac{1}{2}\right)$$

$$288.7139414 \text{ K} \qquad \textbf{(4.1.2)}$$

PHTChart(*Water*)



See also this demo.

Bivariate limits

$$expr := \frac{x^3 \cdot y}{x^6 + y^2}$$

$$expr := \frac{x^3\,y}{x^6+y^2} \tag{4.2.1}$$

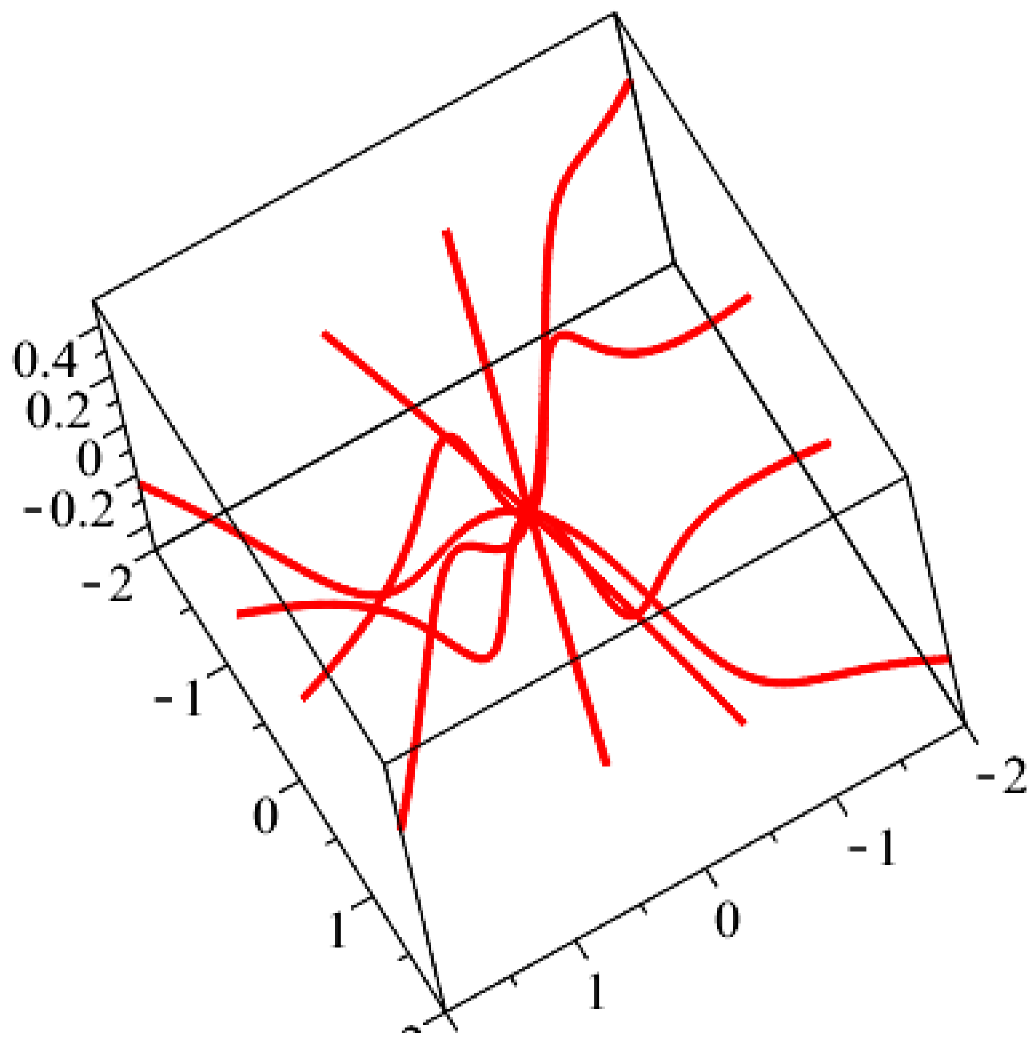
$expr\_line := expr\Big|_{y=a\,x}$

$$expr\_line := \frac{x^4\,a}{x^6+a^2\,x^2} \tag{4.2.2}$$

$\lim_{x \to 0} expr\_line$

$$0 \tag{4.2.3}$$

$curves := plots[display]\left(seq\left(plots[spacecurve]\left([x, a\,x, expr\_line], x = \max\left(-2, -\frac{2}{|a|}\right) .. \min\left(2, \frac{2}{|a|}\right), color = red, thickness = 3\right), a = \left[-4, -1, -\frac{1}{4}, \frac{1}{4}, 1, 4\right]\right)\right)$

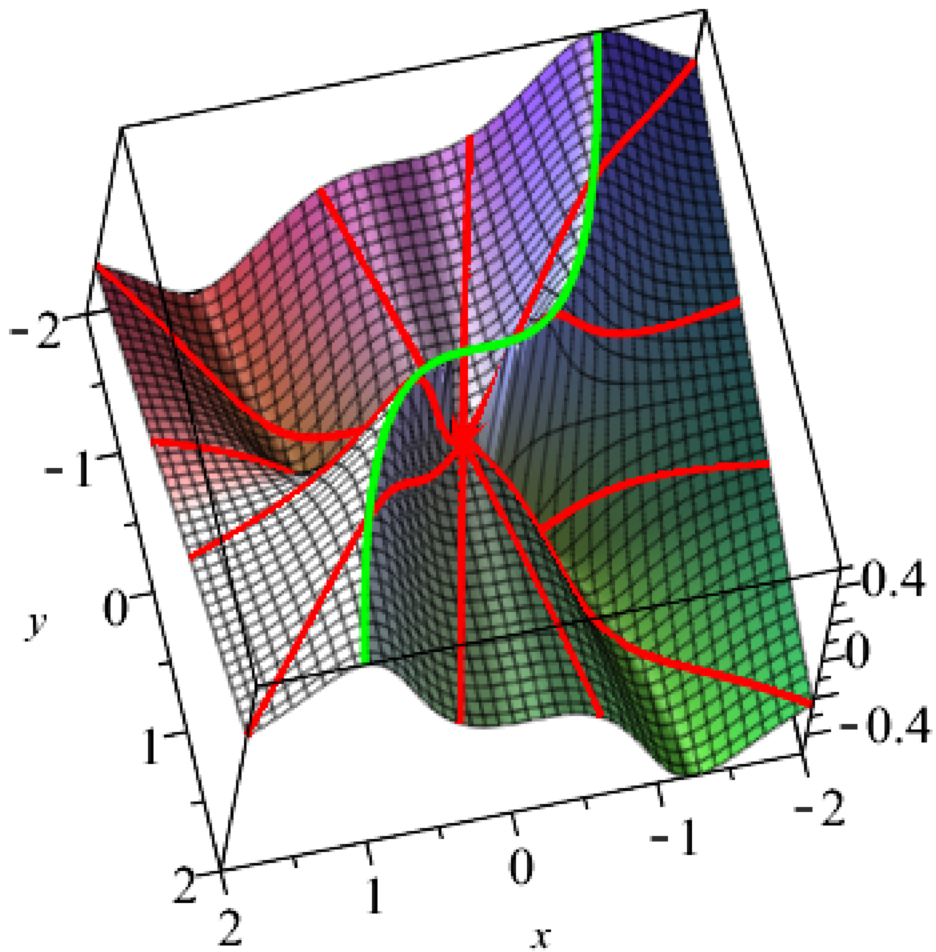


$limit(expr, \{x = 0, y = 0\})$

$$undefined \tag{4.2.4}$$

$plots[display]\Big(plot3d(expr, x = -2 ..2, y = -2 ..2, grid = [200, 200]), curves, plots\text{:-}$

$spacecurve\left([x, x^3, eval(expr, y = x^3)], x = -2^{\frac{1}{3}}..2^{\frac{1}{3}}, color = green, thickness = 3\right)$



$expr := \frac{x^3 \cdot \sin(y)}{x^6 + y^2}$

$$expr := \frac{x^3 \sin(y)}{x^6 + y^2} \quad \textbf{(4.2.5)}$$

$limit(expr, \{x = 0, y = 0\})$

$$undefined \quad \textbf{(4.2.6)}$$

This will be computed correctly in Maple 2017!

Partial differential equations

$pde := \frac{\partial}{\partial x} f(x, y, z) + \frac{\partial}{\partial y} f(x, y, z) + \frac{\partial}{\partial z} f(x, y, z) = f(x, y, z)$

$$pde := \frac{\partial}{\partial x} f(x, y, z) + \frac{\partial}{\partial y} f(x, y, z) + \frac{\partial}{\partial z} f(x, y, z) = f(x, y, z) \quad \textbf{(4.3.1)}$$

$pdsolve(pde)$

$$f(x, y, z) = \_F1(-x + y, -x + z)\, e^x \quad \textbf{(4.3.2)}$$

$bc := f(\alpha+\beta, \alpha-\beta, 1) = \alpha\cdot\beta$

$$bc := f(\alpha+\beta, \alpha-\beta, 1) = \alpha\,\beta \tag{4.3.3}$$

$pdsolve([bc, pde])$

$$f(x, y, z) = \frac{(x-2z+2+y)\,(x-y)\,e^{z-1}}{4} \tag{4.3.4}$$

How does this work?

$eval(\textbf{(4.3.2)}, [x = \mathrm{alpha}+\mathrm{beta}, y = \mathrm{alpha}-\mathrm{beta}, z = 1])$

$$f(\alpha+\beta, \alpha-\beta, 1) = \_F1(-2\,\beta, -\alpha-\beta+1)\,e^{\alpha+\beta} \tag{4.3.1.1}$$

$eval(\%, bc)$

$$\alpha\,\beta = \_F1(-2\,\beta, -\alpha-\beta+1)\,e^{\alpha+\beta} \tag{4.3.1.2}$$

$isolate(\%, op(indets(\%, specfunc(\_F1))))$

$$\_F1(-2\,\beta, -\alpha-\beta+1) = \frac{\alpha\,\beta}{e^{\alpha+\beta}} \tag{4.3.1.3}$$

$solve([op(lhs(\textbf{(4.3.1.3)}))] =\sim [x0, y0], [\mathrm{alpha}, \mathrm{beta}]);$

$$\left[\left[\alpha = 1 - y0 + \frac{x0}{2}, \beta = -\frac{x0}{2}\right]\right] \tag{4.3.1.4}$$

$eval(\textbf{(4.3.1.3)}, \%[1])$

$$\_F1(x0, y0) = -\frac{\left(1 - y0 + \frac{x0}{2}\right) x0}{2\,e^{1-y0}} \tag{4.3.1.5}$$

$eval(\textbf{(4.3.2)}, \_F1 = unapply(rhs(\%), [x0, y0]))$

$$f(x, y, z) = -\frac{\left(\frac{x}{2} - z + 1 + \frac{y}{2}\right)(-x+y)\,e^{x}}{2\,e^{x-z+1}} \tag{4.3.1.6}$$

$simplify(\%)$

$$f(x, y, z) = \frac{(x-2z+2+y)\,(x-y)\,e^{z-1}}{4} \tag{4.3.1.7}$$

$pde := 3\,(u(x,y)-y)^2 \left(\frac{\partial}{\partial x}\,u(x,y)\right) - \left(\frac{\partial}{\partial y}\,u(x,y)\right) = 0$

$$pde := 3\,(u(x,y)-y)^2 \left(\frac{\partial}{\partial x}\,u(x,y)\right) - \left(\frac{\partial}{\partial y}\,u(x,y)\right) = 0 \tag{4.3.5}$$

$pdsolve(pde)$

$$u(x, y) = RootOf(-y^3 + 3\,y^2\,\_Z - 3\,y\,\_Z^2 + \_Z^3 - \_F1(\_Z) - x) \tag{4.3.6}$$

$DETools[remove\_RootOf](\%)$

$$-y^3 + 3\,y^2\,u(x,y) - 3\,y\,u(x,y)^2 + u(x,y)^3 - \_F1(u(x,y)) - x = 0 \tag{4.3.7}$$

$eval(\%, \_F1 = \sin)$

$$-y^3 + 3\,y^2\,u(x,y) - 3\,y\,u(x,y)^2 + u(x,y)^3 - \sin(u(x,y)) - x = 0 \tag{4.3.8}$$

$pdetest(\%, pde)$

$$0 \tag{4.3.9}$$

$bc := u(0, \alpha) = \alpha$

$$bc := u(0, \alpha) = \alpha \qquad \textbf{(4.3.10)}$$

$pdsolve([pde, bc])$

$$u(x, y) = x^{1/3} + y,\ u(x, y) = -\frac{x^{1/3}}{2} - \frac{I\sqrt{3}\,x^{1/3}}{2} + y,\ u(x, y) = -\frac{x^{1/3}}{2} + \frac{I\sqrt{3}\,x^{1/3}}{2} + y \qquad \textbf{(4.3.11)}$$

Units

Consider:

$d := 5\ \mathrm{m}$

$$d := 5\ \mathrm{m} \qquad \textbf{(4.4.1)}$$

$t := 5\ \mathrm{s}$

$$t := 5\ \mathrm{s} \qquad \textbf{(4.4.2)}$$

$(x + y \cdot d) \cdot (y + t \cdot x)$

This is a violation of unit consistency: the first factor means that $\frac{x}{y}$ has dimension length, whereas the second factor implies that $\frac{y}{x}$ has dimension time. That cannot be.

$Units\text{:-}TestDimensions((x + y \cdot d) \cdot (y + t \cdot x));$

$$false \qquad \textbf{(4.4.3)}$$

How does this work? Every expression is taken apart and its dimension expressed in terms of the dimensions of its subexpressions; concrete units are expanded in terms of independent base dimensions. Subexpressions that we don't know anything about (such as x, y, $f(\ldots)$) remain; conceptually, $y \cdot d$ gets turned into $dimension(y) \cdot length$. Inequalities, sums, and equations are recorded: their operands all have the same dimension; we turn that into expressions that must be dimensionless. In the example above:

$exprs := \left[\frac{dimension(x)}{dimension(y) \cdot length}, \frac{dimension(y)}{dimension(x) \cdot time}\right]$: might get represented as

$A := \langle 1, -1; -1, 1 \rangle$

$$A := \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \textbf{(4.4.4)}$$

$C := \langle -1, 0; 0, -1 \rangle$

$$C := \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \textbf{(4.4.5)}$$

If we knew the dimensions of x and y, then we could express them in the same way; as an Ansatz, suppose x is a velocity and y is an area:

$B := \langle 1, -1; 2, 0 \rangle$

$$B := \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} \qquad \textbf{(4.4.6)}$$

Now $A . B + C$ give us the dimension of the expressions in *exprs*:

$A . B + C$

$$\begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix} \quad \textbf{(4.4.7)}$$

Clearly we have failed to make these expressions dimensionless. And we have just shown that $A\,.\,B = -C$ has a solution if and only if there is a consistent assignment of dimensions to our atomic expressions. In this case, there is no solution:

$LinearAlgebra{:}\text{-}LinearSolve(A, -C);$

Error, (in LinearAlgebra:-BackwardSubstitute) inconsistent system

For another example, replace one y with z: $(x + y^2 \cdot d) \cdot (z + t \cdot x)$. Now

$exprs = \left[\dfrac{dimension(x)}{dimension(y)^2 \cdot length}, \dfrac{dimension(z)}{dimension(x) \cdot time} \right]$,

$A := \langle 1, -2, 0; -1, 0, 1 \rangle$

$$A := \begin{bmatrix} 1 & -2 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad \textbf{(4.4.8)}$$

$C := \langle -1, 0; 0, -1 \rangle$

$$C := \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \textbf{(4.4.9)}$$

$B := LinearAlgebra{:}\text{-}LinearSolve(A, -C);$

$$B := \begin{bmatrix} \_t_{1,1} & -1 + \_t_{1,2} \\ -\frac{1}{2} + \frac{\_t_{1,1}}{2} & -\frac{1}{2} + \frac{\_t_{1,2}}{2} \\ \_t_{1,1} & \_t_{1,2} \end{bmatrix} \quad \textbf{(4.4.10)}$$

$B := eval(B, indets(B) =\sim 1)$

$$B := \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \quad \textbf{(4.4.11)}$$

$A\,.\,B + C$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \textbf{(4.4.12)}$$

$Units{:}\text{-}TestDimensions\left((x + y^2 \cdot d) \cdot (z + t\,x) \right);$

$$true \quad \textbf{(4.4.13)}$$

Physics

See the Physics examples help page:

```
> restart;
  with(Physics);
  Setup(mathematicalnotation = true);
```

$[`*`, `.`, Annihilation, AntiCommutator, Antisymmetrize, Assume, Bra, Bracket, Check, Christoffel, Coefficients, Commutator, CompactDisplay, Coordinates, Creation, D\_, Dagger, Decompose, Define, D\gamma, Einstein, EnergyMomentum, Expand, ExteriorDerivative, Factor, FeynmanDiagrams, Fundiff, Geodesics, GrassmannParity, Gtaylor, Intc, Inverse, Ket, KillingVectors, KroneckerDelta, LeviCivita, Library, LieBracket, LieDerivative, Normal, Parameters, PerformOnAnticommutativeSystem, Projector, Psigma, Redefine, Ricci, Riemann, Setup, Simplify, SpaceTimeVector, StandardModel, SubstituteTensor, SubstituteTensorIndices, SumOverRepeatedIndices, Symmetrize, TensorArray, Tetrads, ThreePlusOne, ToFieldComponents, ToSuperfields, Trace, TransformCoordinates, Vectors, Weyl, `\wedge`, dAlembertian, d\_, diff, g\_, gamma\_]$

$$[mathematicalnotation = true] \tag{4.5.1}$$

Consider two conjugate observables Q, P, and the corresponding Hermitian operators satisfying $[Q, P]_- = I\hbar$.

```
> macro(h = `&hbar;`):
> Setup(hermitianoperators = {Q,P}, %Commutator(Q,P) = I*h);
```

$$[algebrarules = \{[Q, P]_- = I\hbar\}, hermitianoperators = \{P, Q\}] \tag{4.5.2}$$

Suppose now that the system where Q and P act is in some state $|\psi\rangle$ normalized to 1, and set $|\psi\rangle$ as the default state for computing Brackets.

```
> Ket(psi);
```

$$|\psi\rangle \tag{4.5.3}$$

```
> Dagger(%) = Bra(psi);
```

$$\langle\psi| = \langle\psi| \tag{4.5.4}$$

```
> Bra(psi) . Ket(psi);
```

$$\langle\psi|\psi\rangle \tag{4.5.5}$$

```
> Bracket(psi, psi);
```

$$\langle\psi|\psi\rangle \tag{4.5.6}$$

```
> Setup(Bracket(psi, psi) = 1, bracketbasis = psi);
```

$$[bracketbasis = \psi, bracketrules = \{\langle\psi|\psi\rangle = 1\}] \tag{4.5.7}$$

We now have:

```
> Ket(psi);
```

$$|\psi\rangle \tag{4.5.8}$$

```
> Bra(psi) . %;
```

$$1 \tag{4.5.9}$$

The mean values of the operators Q and P in the state $|\psi\rangle$ are then given by:

```
> Qm := Bracket(Q);  #Shortcut for Bracket(psi, Q, psi) after
  having set the bracketbasis to psi
```

$$Qm := \langle Q\rangle \tag{4.5.10}$$

```
> Pm := Bracket(P);
```

$$Pm := \langle P\rangle \tag{4.5.11}$$

Let's introduce another Hermitian operator, Δ, and denote $\Delta(Q)$ and $\Delta(P)$ the operators representing the observable deviations from these mean values by $\langle Q\rangle$ and $\langle P\rangle$.

```
> Setup(hermitianoperators = Delta);
```
$$[hermitianoperators = \{\Delta, P, Q\}] \quad (4.5.12)$$
```
> DefDQ := Delta(Q) = Q - Bracket(Q);
```
$$DefDQ := \Delta(Q) = Q - \langle Q \rangle \quad (4.5.13)$$
```
> DefDP := Delta(P) = P - Bracket(P);
```
$$DefDP := \Delta(P) = P - \langle P \rangle \quad (4.5.14)$$

The value of the Commutator between $\Delta(Q)$ and $\Delta(P)$ is a consequence of the value of the Commutator between Q and P, and so it can be computed by rewriting the deviations in terms of Q and P.
```
> %Commutator(Delta(Q), Delta(P));
```
$$[\Delta(Q), \Delta(P)]_- \quad (4.5.15)$$
```
> eval(%, {DefDQ, DefDP});
```
$$[Q - \langle Q \rangle, P - \langle P \rangle]_- \quad (4.5.16)$$
```
> expand(%);
```
$$Q\,P - P\,Q \quad (4.5.17)$$
```
> Simplify(%);
```
$$\mathrm{I}\,\hbar \quad (4.5.18)$$
```
> eval(Commutator(Delta(Q), Delta(P)), {DefDQ, DefDP});
```
$$\mathrm{I}\,\hbar \quad (4.5.19)$$

Track this result as an algebra rule, so that in what follows we compute directly with $\Delta(Q)$ and $\Delta(P)$.
```
> Setup((4.5.15) = (4.5.19));
```
$$[algebrarules = \{[Q, P]_- = \mathrm{I}\,\hbar, [\Delta(Q), \Delta(P)]_- = \mathrm{I}\,\hbar\}] \quad (4.5.20)$$

To show now that $[Q, P]_- = \mathrm{I}\,\hbar$ implies $\frac{\hbar^2}{4} \leq \langle \Delta(P)^2 \rangle \langle \Delta(Q)^2 \rangle$, consider the action of these deviation operators $\Delta(Q)$ and $\Delta(P)$ on the state of the system $|\psi\rangle$, and construct with them a new Ket involving a real parameter λ.
```
> Ket(Psi, lambda) := (Delta(Q) + I*lambda*Delta(P)) . Ket(psi)
  ;
```
$$|\Psi_\lambda\rangle := \Delta(Q) \cdot |\psi\rangle + \mathrm{I}\,\lambda\,(\Delta(P) \cdot |\psi\rangle) \quad (4.5.21)$$

The square of the norm of $|\Psi_\lambda\rangle$, for λ real, is
```
> Dagger(%) . % assuming lambda::real;
```
$$\langle \Delta(P)^2 \rangle \lambda^2 - \mathrm{I}\,\langle \Delta(P)\,\Delta(Q) \rangle \lambda + \mathrm{I}\,\lambda\,\langle \Delta(Q)\,\Delta(P) \rangle + \langle \Delta(Q)^2 \rangle \quad (4.5.22)$$

Simplify this norm, taking into account the commutator $[\Delta(Q), \Delta(P)]_- = \mathrm{I}\,\hbar$, set in **(4.5.20)**
```
> Simplify(%);
```
$$-\hbar\,\lambda + \langle \Delta(P)^2 \rangle \lambda^2 + \langle \Delta(Q)^2 \rangle \quad (4.5.23)$$

This is a polynomial in λ of second degree; its discriminant is negative or zero.
```
> discrim(%, lambda) <= 0;
```
$$\hbar^2 - 4\,\langle \Delta(P)^2 \rangle \langle \Delta(Q)^2 \rangle \leq 0 \quad (4.5.24)$$

isolating $\frac{\hbar^2}{4}$, we obtain the lower bound for $\langle \Delta(P)^2 \rangle \langle \Delta(Q)^2 \rangle$.

```
> isolate(%, h^2)/4;
```

$$\frac{\hbar^2}{4} \leq \langle \Delta(P)^2 \rangle \langle \Delta(Q)^2 \rangle \quad \textbf{(4.5.25)}$$

Note that this result is a consequence of $[\Delta(Q), \Delta(P)]_- = I\,\hbar$, which in turn is a consequence of $[Q, P]_- = I\,\hbar$, so that Q and P too satisfy $\frac{\hbar^2}{4} \leq \langle P^2 \rangle \langle Q^2 \rangle$, and in fact the product of *any* two conjugate Hermitian operators, as well as of the root-mean square deviations of them, satisfy this inequality.