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Integrability conditions of some cosmological models in f(R) gravity

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$f(R)$ gravitation

$f(R)$ models are a sub-class of 4^{th}-order theories of gravitation, with an action of the form

$$\mathcal{A} = \frac{1}{2}\int d^4x\sqrt{-g}\,[f(R) + 2\mathcal{L}_m]$$

- Simplest generalizations to GR
- An extra degree of freedom
- Cosmological viability
  - observational constraints
  - theoretical constraints: analysis of the integrability conditions on the field equations

Field equations

The $f(R)$-generalized Einstein field equations can be given by

$$f' G_{ab} = T^m_{ab} + \frac{1}{2}(f - Rf')g_{ab} + \nabla_b \nabla_a f' - g_{ab} \nabla_c \nabla^c f'$$

- Generic viability conditions on f:
  - To ensure gravity remains attractive
    $$f' > 0 \;\; \forall R$$
  - For stable matter-dominated and high-curvature cosmological regimes (nontachyonic scalaron)
    $$f'' > 0 \;\; \forall R \gg f''$$
  - GR-like law of gravitation in the early universe (BBN, CMB constraints)
    $$\lim_{R \to \infty} \frac{f(R)}{R} = 1 \Rightarrow f' < 1$$
  - At recent epochs
    $$|f' - 1| \ll 1$$

Covariant thermodynamics

The matter-energy content of the Universe is specified by

$$T_{ab} = (\mu + p)u_a u_b + p g_{ab} + q_{(a}u_{b)} + \pi_{ab}$$

- Curvature and total fluid thermodynamics

$$\mu_R = \frac{1}{f'}\left[\frac{1}{2}(Rf' - f) - \Theta f'' \dot{R} + f'' \tilde{\nabla}^2 R\right]$$

$$p_R = \frac{1}{f'}\left[\frac{1}{2}(f - Rf') + f'' \ddot{R} + f''' \dot{R}^2 \right.$$
$$\left. + \frac{2}{3}\left(\Theta f'' \dot{R} - f'' \tilde{\nabla}^2 R - f''' \tilde{\nabla}^a R \tilde{\nabla}_a R\right)\right]$$

$$q_a^R = -\frac{1}{f'}\left[f''' \dot{R} \tilde{\nabla}_a R + f'' \tilde{\nabla}_a \dot{R} - \frac{1}{3} f'' \Theta \tilde{\nabla}_a R\right]$$

$$\pi_{ab}^R = \frac{1}{f'}\left[f'' \tilde{\nabla}_{\langle a} \tilde{\nabla}_{b\rangle} R + f''' \tilde{\nabla}_{\langle a} R \tilde{\nabla}_{b\rangle} R - \sigma_{ab} \dot{R} f''\right]$$

$$\mu \equiv \frac{\mu_m}{f'} + \mu_R\ , \quad p \equiv \frac{p_m}{f'} + p_R\ , \quad q_a \equiv \frac{q_a^m}{f'} + q_a^R\ , \quad \pi_{ab} \equiv \frac{\pi_{ab}^m}{f'} + \pi_{ab}^R$$

The covariant derivative of the timelike vector u^a is decomposed into its irreducible parts as

Acceleration, Shear, Rotation, Expansion

$$\nabla_a u_b = -A_a u_b + \tfrac{1}{3} h_{ab}\Theta + \sigma_{ab} + \epsilon_{abc}\omega^c$$

$$A_a \equiv \dot{u}_a\ , \quad \Theta \equiv \tilde{\nabla}_a u^a\ , \quad \sigma_{ab} \equiv \tilde{\nabla}_{\langle a} u_{b\rangle}\ , \quad \omega^a \equiv \epsilon^{abc}\tilde{\nabla}_b u_c$$

The trace-free part of the Riemann tensor defines the *Weyl conformal curvature tensor*

$$C^{ab}{}_{cd} = R^{ab}{}_{cd} - 2g^{[a}{}_{[c}R^{b]}{}_{d]} + \frac{R}{3} g^{[a}{}_{[c}g^{b]}{}_{d]}$$

- Split into its symmetric, trace-free "electric" and "magnetic" parts, E_{ab} and H_{ab} respectively given by

$$E_{ab} \equiv C_{agbh}u^g u^h, \qquad H_{ab} \equiv \tfrac{1}{2}\eta_{ae}{}^{gh}C_{ghbd}u^e u^d$$

E_{ab} represents the free gravitational field (tidal forces); H_{ab} is responsible for gravitational waves, no Newtonian analogue

Evolution equations

- $1+3$ covariant splitting of the Bianchi and Ricci identities

$$\nabla_{[a}R_{bc]d}{}^{e}=0\,,\quad (\nabla_a\nabla_b-\nabla_b\nabla_a)u_c=R_{abc}{}^{d}u_d$$

  result in propagation and constraint equations

- The evolution equations uniquely determine the covariant variables on some initial hypersurface S_0 at t_0:

$$\dot{\mu}_m=-(\mu_m+p_m)\Theta-\tilde{\nabla}^a q^m_a-2A_a q^a_m-\sigma^a_b\pi^b_{a(m)}$$

$$\dot{\mu}_R=-(\mu_R+p_R)\Theta+\frac{\mu_m f''}{f'^2}\dot{R}-\tilde{\nabla}^a q^R_a-2A_a q^a_R-\sigma^a_b\pi^b_{a(R)}$$

$$\dot{\Theta}=-\tfrac{1}{3}\Theta^2-\tfrac{1}{2}(\mu+3p)+\tilde{\nabla}_a A^a-A_aA^a-\sigma_{ab}\sigma^{ab}+2\omega_a\omega^a$$

$$\dot{q}^m_a=-\tfrac{4}{3}\Theta q^m_a-(\mu_m+p_m)A_a-\tilde{\nabla}_a p_m-\tilde{\nabla}^b\pi^m_{ab}$$

$$-\sigma^b_a q^m_b-A^b\pi^m_{ab}-\epsilon_{abc}\omega^b q^c_m$$

Evolution equations...

$$\dot{q}^R_a = -\tfrac{4}{3}\Theta q^R_a + \frac{\mu_m f''}{f'^2}\tilde{\nabla}_a R - \tilde{\nabla}_a p_R - \tilde{\nabla}^b \pi^R_{ab} - \sigma_a^{\ b} q^R_b$$
$$- (\mu_R + p_R) A_a - A^b \pi^R_{ab} - \epsilon_{abc}\omega^b q^c_R$$
$$\dot{\omega}_a = -\tfrac{2}{3}\Theta\omega_a - \tfrac{1}{2}\epsilon_{abc}\tilde{\nabla}^b A^c + \sigma_a^{\ b}\omega_b \tag{1.1}$$
$$\dot{\sigma}_{ab} = -\tfrac{2}{3}\Theta\sigma_{ab} - E_{ab} + \tfrac{1}{2}\pi_{ab} + \tilde{\nabla}_{\langle a}A_{b\rangle} + A_{\langle a}A_{b\rangle} - \sigma^c_{\ \langle a}\sigma_{b\rangle c}$$
$$- \omega_{\langle a}\omega_{b\rangle} \tag{1.2}$$
$$\dot{E}_{ab} + \tfrac{1}{2}\dot{\pi}_{ab} = \epsilon_{cd\langle a}\tilde{\nabla}^c H^d_{b\rangle} - \Theta\left(E_{ab} + \tfrac{1}{6}\pi_{ab}\right) - \tfrac{1}{2}\left(\mu + p\right)\sigma_{ab} - \tfrac{1}{2}\tilde{\nabla}_{\langle a}q_{b\rangle}$$
$$+ 3\sigma^{\langle c}_{a}\left(E_{b\rangle c} - \tfrac{1}{6}\pi_{b\rangle c}\right) - A_{\langle a}q_{b\rangle} + \epsilon_{cd\langle a}\left[2A^c H^d_{b\rangle} + \omega^c(E^d_{b\rangle} + \tfrac{1}{2}\pi^d_{b\rangle})\right] \tag{1.3}$$
$$\dot{H}_{ab} = -\Theta H_{ab} - \epsilon_{cd\langle a}\tilde{\nabla}^c E^d_{b\rangle} + \tfrac{1}{2}\epsilon_{cd\langle a}\tilde{\nabla}^c\pi^d_{b\rangle}$$
$$+ 3\sigma^{\langle c}_{a}H_{b\rangle c} + \tfrac{3}{2}\omega_{\langle a}q_{b\rangle} - \epsilon_{cd\langle a}\left[2A^c E^d_{b\rangle} - \tfrac{1}{2}\sigma^c_{b\rangle}q^d - \omega^c H^d_{b\rangle}\right] \tag{1.4}$$

Constraints

- Restrict the initial data to be specified; must remain satisfied on any hypersurface S_t for all t

$$
\begin{aligned}
(C^1)_a &:= \tilde{\nabla}^b\sigma_{ab} - \tfrac{2}{3}\tilde{\nabla}_a\Theta + \epsilon_{abc}\left(\tilde{\nabla}^b\omega^c + 2A^b\omega^c\right) + q_a = 0 \\
(C^2)_{ab} &:= \epsilon_{cd(a}\tilde{\nabla}^c\sigma_{b)}{}^d + \tilde{\nabla}_{\langle a}\omega_{b\rangle} - H_{ab} - 2A_{\langle a}\omega_{b\rangle} = 0 \qquad (1.5) \\
(C^3)_a &:= \tilde{\nabla}^b H_{ab} + (\mu + p)\omega_a + \epsilon_{abc}\left[\tfrac{1}{2}\tilde{\nabla}^b q^c + \sigma_{bd}\left(E^d{}_c + \tfrac{1}{2}\pi^d{}_c\right)\right] \\
&\qquad + 3\omega_b\left(E^{ab} - \tfrac{1}{6}\pi^{ab}\right) = 0 \\
(C^4)_a &:= \tilde{\nabla}^b E_{ab} + \tfrac{1}{2}\tilde{\nabla}^b\pi_{ab} - \tfrac{1}{3}\tilde{\nabla}_a\mu + \tfrac{1}{3}\Theta q_a \\
&\qquad - \tfrac{1}{2}\sigma_a^b q_b - 3\omega^b H_{ab} - \epsilon_{abc}[\sigma^{bd}H_d^c - \tfrac{3}{2}\omega^b q^c] = 0 \\
(C^5) &:= \tilde{\nabla}^a\omega_a - A_a\omega^a = 0 \qquad (1.6)
\end{aligned}
$$

Rotating and expanding universes

AA, Goswami, Dunsby. Phys. Rev. D 84 124027

Classic GR result (Gödel, Ellis): shear-free perfect-fluid cosmological models (homogeneous, inhomogeneous) cannot rotate and expand simultaneously, *i.e.,*

$$\Theta\omega^a = 0$$

- Turning off the shear from the propagation equations results in a new constraint equation

$$(C^6)_{ab} := E_{ab} - \tfrac{1}{2}\pi_{ab} - \tilde{\nabla}_{\langle a}A_{b\rangle} = 0$$

- Demanding consistent spatial (curl) and temporal (time derivative) propagations results in

$$\Theta\omega^a\Big\{\Big[\frac{(1-w)P}{3}\tilde{R} + \frac{(1+w)}{f'}\frac{(3w+5)f' + 4f''Q}{6f'}\mu_m\Big] + \frac{Z}{P}\Big[\Big(\frac{1+w}{f'}\Big)\mu_m\Big]\Big\} = 0 \qquad (2.1)$$

In the above result, we have defined

$$\Theta \equiv 3\frac{\dot{a}}{a}, \quad q \equiv -\frac{\ddot{a}a}{\dot{a}^2}, \quad j \equiv \frac{\dddot{a}\,a^2}{\dot{a}^3}, \quad s \equiv \frac{a^3}{\dot{a}^4}\frac{d^4 a}{dt^4}$$

$$Q \equiv \frac{1}{3}\Theta^2(j-q-2)+\tilde{R}$$

$$P \equiv \frac{f''}{f'}Q+\frac{3w}{2}$$

$$Z \equiv \frac{2}{3}\left(\frac{f'''}{f'}-(\frac{f''}{f'})^2\right)Q^2+\frac{f''}{9f'}\left((4+5q+j+jq+s)\Theta^2+6\tilde{R}\right)$$

- It follows that we must have either $\omega^a\Theta = 0$ or the expression in the curly brackets of Eq. (2.1) must vanish
- Notice that if the 3-curvature vanishes $\tilde{R}$, then the GR result can always be avoided for vacuum universes ($\mu_m = 0$), *i.e.*, a shear-free, spatially flat vacuum universe in any $f(R)$ theory can rotate and expand simultaneously in the linearized regime

- For the non-vacuum case, it can be shown that using flat Milne universe solutions

$$\mu_m = \frac{\mu_0}{a^{3(1+w)}} \,, \quad \dot{\Theta} = -\tfrac{1}{3}\Theta^2, \quad R = \tfrac{2}{3}\Theta^2 \,, \quad a(R) = \frac{1}{\sqrt{R}}$$

into the Friedmann equation

$$\frac{1}{3}\Theta^2 = \frac{1}{f'}\left[\mu_m + \frac{Rf' - f}{2} - \Theta\dot{R}f''\right] \,,$$

one gets

$$-R^2\frac{d^2 f(R)}{dR^2} + \frac{f(R)}{2} - \frac{\mu_0}{a(R)^{3(1+w)}} = 0 \,,$$

which has the following general solution:

$$\boxed{f(R) = C_1 R^{\frac{1+\sqrt{3}}{2}} + C_2 R^{\frac{1-\sqrt{3}}{2}} - \frac{4\mu_0}{1+12w+9w^2} R^{\frac{3(1+w)}{2}}} \qquad (2.2)$$

If we consider the R^n toy model, the term in the curly brackets of Eq. (2.1) reduces to

$$\frac{(1+w)\mu_m}{6f'}\left[3w+9-4n\right]=0 \tag{2.3}$$

- Comparing solutions (2.3) and the particular solution of Eq. (2.2), we get $w=1$ if $\mu_m \neq 0$, *i.e.,* for a stiff fluid in R^3 gravity, there exists a flat Milne-universe solution which can rotate and expand simultaneously at the level of linearised perturbation theory
- This suggests that there are situations where linearized fourth-order gravity shares properties with Newtonian theory not valid in GR

Classes of non-rotating fluid models

AA, Elmardi. Int. J. Geom. Methods Mod. Phys. 12 1550118

Fluid flows with vanishing vorticity $\omega_a = 0$ will have the evolution equation (1.1) turned into a new constraint

$$(C^{6*})_a := \epsilon_{abc}\tilde{\nabla}^b A^c = 0 \implies A_a = \tilde{\nabla}_a \psi$$

for some scalar ψ. Taking the curl and temporal derivative of this constraint results in the mathematical identities

$$\left(\epsilon_{abc}\tilde{\nabla}^b A^c\right)^{\cdot} = 0$$
$$curl(curl(A_a)) = \tilde{\nabla}_a\left(\tilde{\nabla}^2\psi\right) - \tilde{\nabla}^2\left(\tilde{\nabla}_a\psi\right) + \frac{2}{3}\left(\mu - \frac{1}{3}\Theta^2\right)\tilde{\nabla}_a\psi = 0$$

- Generic irrotational fluid models in $f(R)$ gravity are self-consistent!

On the other hand, if we specialize to dust models

$$w = 0 = p_m \,, \qquad q^m_a = 0 = A_a \,, \qquad \pi^m_{ab} = 0 \,,$$

then some interesting integrability conditions arise. For example, in shear-free dust models, *i.e.,* $\sigma_{ab} = 0$, Eq. (1.2) turns into a new constraint

$$(C^{6d})_{ab} := E_{ab} - \tfrac{1}{2}\pi^R_{ab} = 0 \tag{3.1}$$

- Unlike in GR, E_{ab} does not vanish because π^R_{ab} is nonzero, but H_{ab} does vanish, leading to a modified constraint from Eq. (1.4), which obviously is an identity:

$$(C^{7d})_{ab} := \epsilon_{cd\langle a}\tilde{\nabla}^c E^d_{b\rangle} - \tfrac{1}{2}\epsilon_{cd\langle a}\tilde{\nabla}^c \pi^d_{b\rangle}{}^d = 0$$

- Since q^R_a becomes irrotational, it can be shown that for some scalar field ϕ and some spatially constant scalar C:

$$q^R_a = \tilde{\nabla}_a \phi \,, \quad \phi = \tfrac{2}{3}\Theta + C \tag{3.2}$$

An interesting consequence of the above result is the integrability condition

$$\tfrac{2}{3}f'\tilde{\nabla}_a\Theta + \left(f'''\dot{R} - \tfrac{1}{3}\Theta f''\right)\tilde{\nabla}_a R + f''\tilde{\nabla}_a\dot{R} = 0$$

- In the GR limit, we get a spatially homogeneous expansion

$$\tilde{\nabla}_a\Theta = 0$$

- Propagating the new constraint above results in the new equation

$$\dot{\pi}^R_{ab} + \tfrac{2}{3}\Theta\pi^R_{ab} - \tfrac{1}{2}\tilde{\nabla}_{\langle a}q^R_{b\rangle} = 0\,,$$

  implying that irrotational shear-free dust spacetimes governed by $f(R)$ gravitational physics evolve consistently if

$$\left[\frac{3}{2}\left(\frac{f'''}{f'} - \frac{f''^2}{f'^2}\right)\dot{R} - \frac{\Theta f''}{6f'}\right]\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}R + \tfrac{3f''}{2f'}\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}\dot{R} = 0\,. \quad (3.3)$$

- The GR limit of the above equation is an identity since the left-hand side vanishes identically

Now since for any scalar field ψ,

$$\epsilon_{cda}\tilde{\nabla}^{c}\tilde{\nabla}_{\langle b}\tilde{\nabla}^{d\rangle}\psi=\epsilon_{cda}\tilde{\nabla}^{c}\tilde{\nabla}_{(b}\tilde{\nabla}^{d)}\psi=\epsilon_{cda}\tilde{\nabla}^{c}\tilde{\nabla}_{b}\tilde{\nabla}^{d}\psi=0$$

taking the curl of Eq. (3.3) results in another identity:

$$\left[\frac{3}{2}\left(\frac{f'''}{f'}-\frac{f''^{2}}{f'^{2}}\right)\dot{R}-\frac{\Theta f''}{6f'}\right]\epsilon_{cda}\tilde{\nabla}^{c}\tilde{\nabla}_{\langle b}\tilde{\nabla}^{d\rangle}R+\tfrac{3f''}{2f'}\epsilon_{cda}\tilde{\nabla}^{c}\tilde{\nabla}_{\langle b}\tilde{\nabla}^{d\rangle}\dot{R}=0 \tag{3.4}$$

- This suggests that all irrotational shear-free dust spacetimes in $f(R)$-gravity are self-consistent
- For the conformally flat metric, *i.e.*, if $E_{ab}=0$ as well, the following new linearized constraints emerge:

$$\tilde{\nabla}_{\langle a}q^{R}_{b\rangle}=0=\left(\dot{R}f'''-\tfrac{1}{3}\Theta f''\right)\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}R+f''\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}\dot{R}$$
$$\pi^{R}_{ab}=0=f''\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}R$$

Irrotational dust spacetimes with div $H_{ab} = 0$

A necessary condition for the propagation of gravitational waves is the vanishing of the divergence of a non-zero H_{ab}.

- Prescribing this condition on the field equations results in a generalized constraint of the irrotational q_a^R term we saw a few slides back, Eq. (3.2):

$$\tilde{\nabla}_a \phi = \tfrac{2}{3} \tilde{\nabla}_a \Theta - \tilde{\nabla}^b \sigma_{ab} \qquad (3.5)$$

- A subclass of such models, called “purely radiative” dust spacetimes, is a divergence-free E_{ab}. Such models in $f(R)$ gravity are constrained further as

$$\boxed{\tilde{\nabla}_a \mu_m + f' \tilde{\nabla}_a \mu_R + f' \Theta q_a^R - \tfrac{3f'}{2} \tilde{\nabla}^b \pi_{ab}^R = 0} \qquad (3.6)$$

  - In GR purely radiative irrotational dust spacetimes are spatially homogeneous:

$$\tilde{\nabla}_a \mu_m = 0 \qquad (3.7)$$

Non-expanding spacetimes

Here we want to explore the (in)consistencies that emerge assuming theoretical cases of a non-expanding spacetime, i.e., $\Theta = 0$.

- One can immediately conclude, for example, that a new constraint arises from the Raychaudhuri equation:

$$(C^{6s}) := \tilde{\nabla}_a A^a - \frac{1}{2f'}(1+3w)\mu_m - \frac{1}{2}(\mu_R + 3p_R) = 0 \qquad (3.8)$$

For dust models ($A_a = 0 = q^m_a$), this would mean a vanishing active gravitational mass: $\mu + 3p = 0$. Furthermore, the conservation equation would guarantee that $\mu_d(t) = \text{const}$, and hence that $\mu_R + 3p_R = \text{const}$, as well. Combining this with the *trace equation*

$$3f''\ddot{R} + 3\dot{R}^2 f''' + 3\Theta\dot{R}f'' - 3f''\tilde{\nabla}^2 R - Rf' + 2f - \mu_m + 3p_m = 0$$

we conclude that

$$\boxed{f - 2f''\tilde{\nabla}^2 R = \text{const}} \qquad (3.9)$$

- Any non-rotating, non-expanding dust spacetime in $f(R)$ cosmology should have a gravitational Lagrangian that satisfies Eq. (3.9)

Summary

- In this presentation, we have
  - looked at the consistency relations of linearized perturbations of FLRW universes arising as a result of imposing special restrictions to the field equations in $f(R)$ gravity
  - shown that, contrary to the results of GR, simultaneously rotating and expanding spacetimes exist in modified gravity
  - explored different classes of non-rotating fluid models in $f(R)$ gravity, and their corresponding GR implications
- The take-home message here is that, even at the theoretical level, if the exact evolutionary history of the Universe is known, one can, *in principle*, constrain the gravitational action

Some Differential Identities

- Standard $1+3$-*covariant approach*, dynamical and kinematical quantities decomposed into temporal and spatial projections
- Two derivatives defined
  - the *4-velocity vector* u^a is used to define the *covariant time derivative* for any tensor $S^{a..b}_{c..d}$ along an observer's worldlines:

$$\dot{S}^{a..b}_{c..d} = u^e \nabla_e S^{a..b}_{c..d} \tag{4.1}$$

  - the *projection tensor* h_{ab} is used to define the fully orthogonally *projected covariant derivative* $\tilde{\nabla}$ for any tensor $S^{a..b}_{c..d}$:

$$\tilde{\nabla}_e S^{a..b}_{c..d} = h^a_f h^p_c ... h^b_g h^q_d h^r_e \nabla_r S^{f..g}_{p..q} \tag{4.2}$$

  $\longrightarrow$ total projection on all the free indices
- Round brackets (ab) indicate symmetrization over the indices a and b whereas square brackets $[ab]$ denote anti-symmetrization over these indices

- Angled brackets denote orthogonal projections of vectors and the orthogonally *projected symmetric trace-free* part of tensors is defined as

$$V^{\langle a \rangle} = h^a_b V^b \qquad S^{\langle ab \rangle} = \left[h^{(a}_c h^{b)}_d - \tfrac{1}{3} h^{ab} h_{cd} \right] S^{cd} \quad (4.3)$$

- The volume element for the 3-restspaces orthogonal to u^a is defined by :

$$\epsilon_{abc} = u^d \eta_{dabc} = -\sqrt{|g|}\delta^0_{[a}\delta^1_b\delta^2_c\delta^3_{d]}u^d \Rightarrow \epsilon_{abc} = \epsilon_{[abc]},\ \epsilon_{abc}u^c = 0, \quad (4.4)$$

where η_{abcd} is the 4-dimensional volume element such that

$$\eta_{abcd} = \eta_{[abcd]} = 2\epsilon_{ab[c}u_{d]} - 2u_{[a}\epsilon_{b]cd} \quad (4.5)$$

- The covariant spatial divergence and curl of vectors and tensors are defined as

$$\mathrm{div}\, V = \tilde{\nabla}^a V_a \qquad (\mathrm{div}\, S)_a = \tilde{\nabla}^b S_{ab} \quad (4.6)$$

$$\mathrm{curl}\, V_a = \epsilon_{abc}\tilde{\nabla}^b V^c \qquad \mathrm{curl}\, S_{ab} = \epsilon_{cd(a}\tilde{\nabla}^c S_{b)}{}^d \quad (4.7)$$

- We define the Riemann tensor as

$$R^a_{bcd} = \Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^e_{bd}\Gamma^a_{ce} - \Gamma^f_{bc}\Gamma^a_{df} \tag{4.8}$$

  - Christoffel symbols:

$$\Gamma^a_{bd} = \frac{1}{2}g^{ae}\left(g_{be,d} + g_{ed,b} - g_{bd,e}\right) \tag{4.9}$$

  - The Ricci tensor:

$$R_{ab} = g^{cd}R_{cadb} \tag{4.10}$$

  - the Ricci scalar

$$R = R^a{}_a \tag{4.11}$$

- For any scalar ϕ

$$[\tilde{\nabla}_a\tilde{\nabla}_b - \tilde{\nabla}_b\tilde{\nabla}_a]\phi = 2\epsilon_{abc}\omega^c\dot{\phi}$$
$$\epsilon^{abc}\tilde{\nabla}_b\tilde{\nabla}_c\phi = 2\omega^a\dot{\phi} \qquad (4.12)$$

- To linear order

$$[\tilde{\nabla}^a\tilde{\nabla}_b\tilde{\nabla}_a - \tilde{\nabla}_b\tilde{\nabla}^2]\phi = \frac{1}{3}\tilde{R}\tilde{\nabla}_b\phi \qquad (4.13)$$
$$[\tilde{\nabla}^2\tilde{\nabla}_b - \tilde{\nabla}_b\tilde{\nabla}^2]\phi = \frac{1}{3}\tilde{R}\tilde{\nabla}_b\phi + 2\epsilon_{dbc}\tilde{\nabla}^d(\omega^c\dot{\phi}) \qquad (4.14)$$

  - $\tilde{R} = \frac{6K}{a^2} = 2\left(\mu - \frac{1}{3}\Theta^2\right)$ is the 3-curvature scalar
  - $K = -1, 0$ or 1 depending on whether the Universe is open, flat or closed
  - $a = a(t)$ is the cosmological scale factor

- For any first order 3-vector $V^a = V^{\langle a\rangle}$

$$[\tilde{\nabla}^a\tilde{\nabla}_b - \tilde{\nabla}_b\tilde{\nabla}^a]V_a = \frac{1}{3}\tilde{R}h^a_{[a}V_{b]} \qquad (4.15)$$
$$h^a_c h^d_b(\tilde{\nabla}_d V^c)^{\cdot} = \tilde{\nabla}_b \dot{V}^{\langle a\rangle} - \frac{1}{3}\Theta\tilde{\nabla}_b V^a \qquad (4.16)$$
$$h^a_c(\tilde{\nabla}^2 V^c)^{\cdot} = \tilde{\nabla}_b(\tilde{\nabla}^{\langle b}V^{a\rangle})^{\cdot} - \frac{1}{3}\Theta\tilde{\nabla}^2 V^a \qquad (4.17)$$