Quantum Corrections
and
Cosmological Singularities

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Outline

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0. Introduction:

- Penrose-Hawking's singularity theorems assured the existence of singularities for any gravitational system starting from generic initial distribution of energy/matter.

- In the absence of a concrete framework for dealing with singularities one might smooth out these singularities through modifying either the gravitational theory or the EoS near the singularity!

- For both cases we have several choices and there is no clear guiding principle on which EoS or a gravity theory one should choose.

- *On the other hand, QFT on curved spaces was successful in inflation and in producing important phenomena such as Hawking radiation.*

- *What is the effect of quantum corrections (in the matter field) on the initial singularity in cosmology* (i.e., big bang singularity)

- *Here we consider a specific class of QFT on curved spaces, which produces a specific higher-curvature terms due to quantum corrections. This theory works as a modified theory of gravity with no arbitrary parameters apart from the number of scalars, spinors and vectors fields.*

I. What is Weyl anomaly?

- Consider a theory with massless fields (ϕ,ψ, A_μ) coupled to a background metric $g_{\mu\nu}$.
- A classical theory is called Weyl/Conformally invariant if its action is invariant under Weyl transformation

$$g'_{\mu\nu} = \Omega^2 g_{\mu\nu}\,, \qquad \psi' = \Omega^d \psi, \quad \text{d is the conformal weight}$$

- As a consequence we have $T_\mu{}^\mu = 0$.
- Quantum one-loop effects breaks this symmetry. The trace takes the form

$$\langle T^\mu_\mu \rangle = c_1 E_4 + c_2 I_4 + c_3 \Box R. \quad *$$

- The symmetry is anomalous and c's are spin-dependent coefficients, R is the Ricci scalar, E_4 *is* the Euler density and I_4 is the Weyl scalar.

$$E_4 = \frac{1}{64}\left(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right)$$

$$I_4 = -\frac{1}{64}\left(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2\right).$$

* D. Capper and Duff, Nouvo Cimento Soc. Ital. Fis. 23A,173 (1974); M. Duff, Nucl. Phys. B125, 334 (1977)

I. What is Weyl anomaly?

- In the special case of conformally flat background, $g_{\mu\nu} = \chi\, \eta_{\mu\nu}$, stress tensor is

$$\langle T(g)^{(ren)}{}_{\mu\nu}\rangle = T^{(m)}_{\mu\nu} + \alpha H_{\mu\nu}{}^{(1)} + \beta H_{\mu\nu}{}^{(3)} \qquad *$$

- We are interested in this tensor since it modifies Einstein FE;

$$\frac{2}{\sqrt{-g}}\frac{\delta S_{eff}}{\delta g_{\mu\nu}} = G^{\mu\nu} - \langle T^{\mu\nu}\rangle = 0$$

- where $T^{(m)}{}_{\mu\nu}$ is a local (not geometric) conserved traceless tensor and $H^{(1)}$ and $H^{(3)}$ are given by

$$H_{\mu\nu}{}^{(1)} = 2R_{;\mu\nu} - 2g_{\mu\nu}\Box R - \frac{1}{2}g_{\mu\nu}R^2 + 2RR_{\mu\nu}$$
$$H_{\mu\nu}{}^{(3)} = \frac{1}{12}R^2 g_{\mu\nu} - R^{\rho\sigma}R_{\rho\mu\sigma\nu}$$

- Then the trace is $$\langle T^{(ren)\mu}{}_{\mu}\rangle = -6\alpha\,\Box R - \beta\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right),$$

- α is a regularization-scheme dependent (and gauge dependent too). Later we will set $\alpha = 0$.

* J. Brown and N. Cassidy, Phys. Rev. D 15, 2810 (1977)

II. Cosmology with Weyl anomaly

FLRW Cosmology:

- Consider a generic conformal field theory of scalars, spinors and gauge fields coupled to a flat FLRW metric

$$ds^2 = -dt^2 + a(t)^2[dr^2 + r^2(d\theta^2 + \sin\theta^2 d\phi^2)]$$

- Here we are interested in regularization-independent back-reaction (α=0) of these fields on the geometry at early times. Einstein field equations is given by

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T^{(m)}_{\mu\nu} + \kappa\,\beta\left[\frac{R^2}{12}g_{\mu\nu} - R^{\rho\sigma}R_{\rho\mu\sigma\nu}\right]$$

- where $T_{\mu\nu}^{(m)}$=*diag*(ρ, p, p, p), *EoS consistent with conformal symmetry* $P=1/3\ \rho$,
- $\beta = \frac{-1}{2880\pi^2}(n_s + 11\, n_f + 62\, n_v) < 0$, for the above field theory.

II. Cosmology with Weyl anomaly: FLRW cosmology

- Two independent field eqn.'s are

$$\kappa\rho - 3[1 + \kappa\beta H^2]\, H^2 = 0$$

$$2\ddot{a}a(1 + 2\kappa\beta H^2) + a^2 H^2 (1 - \kappa\beta H^2) + \kappa\, a^2 P = 0,$$

- the last eqn. can also take the form

$$\dot{H}(1 + 2\kappa\,\beta H^2) = -\frac{\kappa}{2}(\rho + P),$$

- Solving the first eqn. for *H* we get

$$H = \pm\sqrt{\frac{-1 \pm \sqrt{1 + \frac{4}{3}\beta\kappa^2\rho}}{2\beta\kappa}}.$$

- $\beta < 0$ implies that density and Hubble rate have max. values and scale factor has a min.

II. Cosmology with Weyl anomaly

A mechanical analogue:

- Solving the continuity equation $\rho(a) = c\,a^{-4}$
- The model is analogues to a simple mechanical model that can be defines through the equations;

$$KE + PE = \frac{1}{2}\eta'^2 - \frac{1}{2}\eta^2(1 \mp \sqrt{1-\eta^{-4}}) = 0$$

$$\eta'' = \eta\frac{(\sqrt{1-\eta^{-4}} \mp 1)}{\sqrt{1-\eta^{-4}}}.$$

- Soln. is $$\tau + c_1 = \pm\left[\frac{\sqrt{2}}{4}tanh^{-1}(2^{-1/2}\sqrt{1 \mp \sqrt{1-\eta^{-4}}}) + \frac{1}{2\sqrt{1 \mp \sqrt{1-\eta^{-4}}}}\right]$$

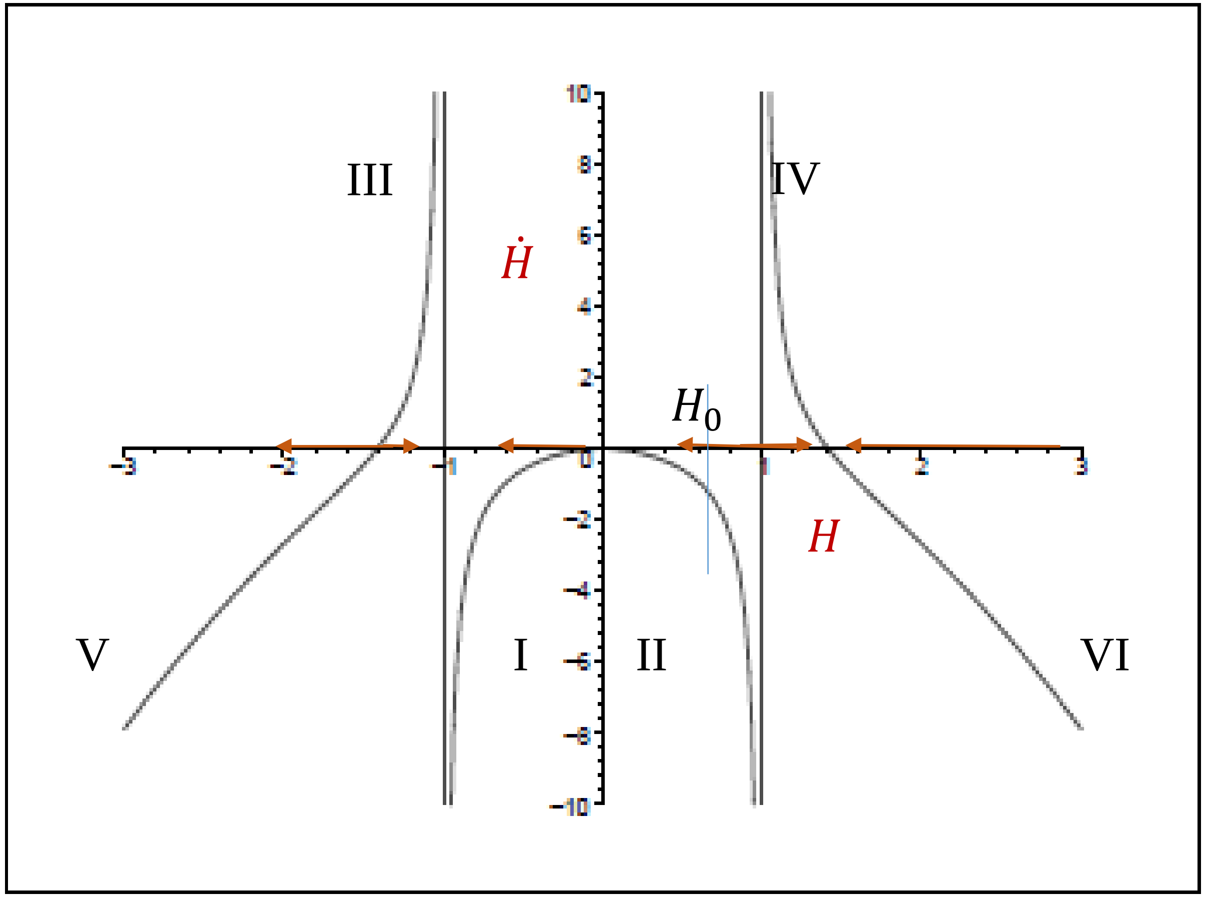
where $$t = \tau\sqrt{2\beta\kappa},\ a/a_c = \eta, \text{ and } a_c = 4\beta\kappa^2 c/3.$$

II. Cosmology with Weyl anomaly: a mechanical analogue

- Cosmological eqn. takes the form

$$\dot{H} = -2\left(\frac{1+\kappa\,\beta H^2}{1+2\kappa\,\beta H^2}\right)H^2$$

$$H = \pm\sqrt{\frac{-1 \pm \sqrt{1+\frac{4}{3}\beta\kappa^2\rho}}{2\beta\kappa}}.$$



Branches of solutions

II. Cosmology with Weyl anomaly: a mechanical analogue

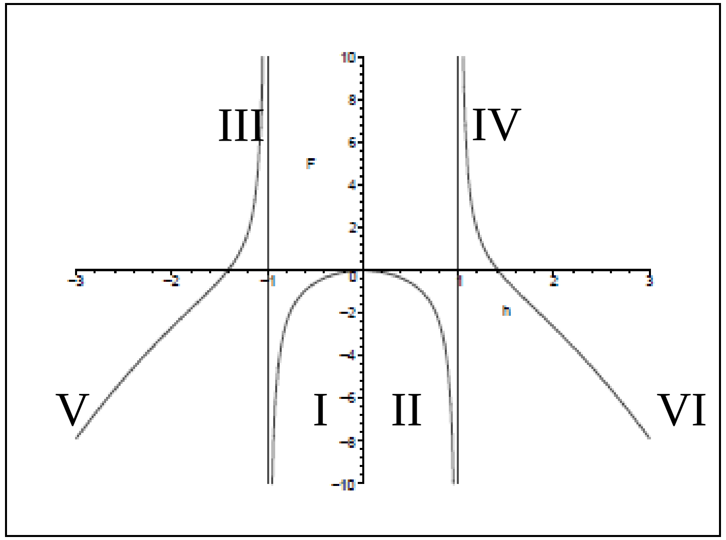
- Notice that the validity of the semi-classical approximation requires that

$$R \ll R_p = l_P^{-2}$$

- But

$$H = \pm\sqrt{\frac{-1 \pm \sqrt{1 + \frac{4}{3}\beta\kappa^2\rho}}{2\beta\kappa}}.$$

- Therefore, we are considering branches with $H \leq (2\kappa|\beta|)^{-1/2}$. *
- In this model H *and* ρ are bounded, but $R \sim \frac{1}{\sqrt{t}}$ near t=0.
- This singularity is milder than big bang singularity ($R \sim \frac{1}{t^2}$).
- Although the force/acceleration is divergent at t=0, the mechanical system needs only a finite amount of work to go from the singularity to any closeby point since pot. energy at t=0 is finite.



* This shows that $H_{max} \sim \frac{M_P}{\sqrt{\beta}}$, is the effective cutoff scale of gravity (also predicted by Antoniadis in arXiv:1410.8845v2)

II. Cosmology with Weyl anomaly

Types of singularity:

- FLRW Singularities have been classified *

I- Big rip: $a \to \infty$, $\dot{a} \to \infty$, $\ddot{a} \to \infty$.

II- Sudden: $a \to a_s$, $\dot{a} \to \dot{a}_s$, $\ddot{a} \to \infty$.

III- Big freeze: $a \to a_s$, $\dot{a} \to \infty$, $\ddot{a} \to \infty$.

IV- Big bang: $a \to 0$, $\dot{a} \to \infty$, $\ddot{a} \to \infty$.

- Our singularity is a sudden singularity since $\ddot{a} \to \infty$;

$$\eta'' = \eta \frac{(\sqrt{1-\eta^{-4}} \mp 1)}{\sqrt{1-\eta^{-4}}}.$$

* S. Nojiri and Odintsov, Phys. Rev. D71, 063004, (2005)

III. Strength of a singularity

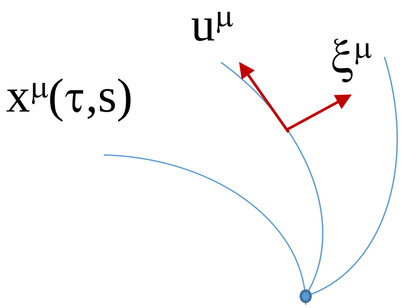
Penrose-Hawking Theorems:

- Geodesic incompleteness of nonspacelike geodesics is the criterion for a singular spacetime.
- Penrose-Hawking theorems show geodesics inextendibility of spacetimes with certain energy conditions and global properties.
- Energy conditions are needed to show gravitational focusing through Raychaudhuri's eqn.

$$\frac{d\theta}{d\tau} = -R_{lk}v^l v^k - \frac{\theta^2}{3}, \qquad \text{(expansion parameter } \theta = -3\,H, \text{ for } FLRW)$$

- Formation of acoustic (or conjugate points) is an essential ingredient in singularity theorems, where $\theta \rightarrow -\infty$.
- In our model $H \rightarrow (2\beta\kappa)^{-1/2}$ at the singularity, therefore, this is not a singularity a la Penrose-Hawking.

III. Strength of a singularity



$$u^{\mu} = \frac{\partial x^{\mu}}{\partial \tau}\ ,\ \xi^{\mu} = \frac{\partial x^{\mu}}{\partial s}$$

Tipler's and Krolak's criteria for a strong singularity:

- Soln. of geodesic deviation eqn. are called Jacobi fields

$$\frac{D^2\xi^{\mu}}{d\tau^2} = R^{\mu}_{\alpha\beta\gamma}\, u^{\alpha} u^{\beta} \xi^{\gamma}$$

- *Tipler's criterion for a strong singularity*; a singularity is called strong if the volume spanned by three orthonormal Jacobi fields shrunk to zero size at the singularity.

- *Krolak's criterion for a strong singularity* is similar but based on the rate of change of the volume w.r.t. affine parameter instead of the volume at the singularity.

III. Strength of a singularity

- The singularity is Tipler strong for a null geodesic iff the following integral diverges

$$\lim_{\lambda\to\lambda_0}\int_0^{\lambda} d\lambda' \int_0^{\lambda'} d\lambda'' R_{ab}u^a u^b,$$

- But the singularity is Krolark strong iff the following integral diverges

$$\lim_{\lambda\to\lambda_0}\int_0^{\lambda} d\lambda' R_{ab}u^a u^b,$$

- For our spacetime close to t=0, (int. cond. $\lambda_0 = 0$ at t=0) solving geodesic eqn.'s we get

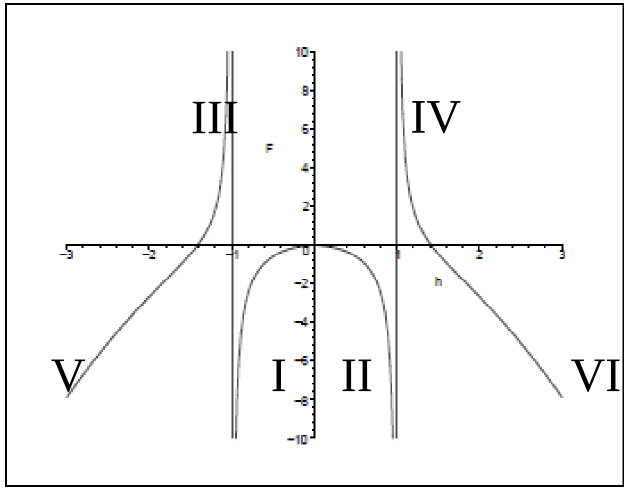
$$\frac{dt}{d\lambda} = \pm\sqrt{s + \frac{v^2}{a^2}} = g(t),$$

$$\frac{dx^i}{d\lambda} = \frac{v^i}{a^2} = f^i(\lambda),$$

$$\longrightarrow \quad a(\lambda) = a_0\left[1 + \chi H_0 \lambda\right] + O(\lambda^2), \quad t(\lambda) = \chi\lambda + O(\lambda^2).$$

- Now evaluating both integrals in the limit $\lambda \to 0$, we find that both integrals vanishes.

IV. Extending spacetime and singularity crossing



Geodesic extension:

- Geodesic equations for FLRW metric are

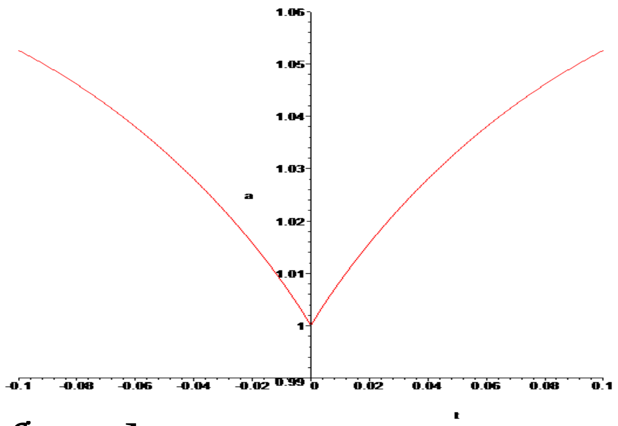
$$\frac{d^2x^i}{d\lambda^2} = 2H\,\frac{dt}{d\lambda}\frac{dx^i}{d\lambda}, \qquad \frac{dt^2}{d\lambda^2} = a^2\,H\left(\frac{dx^i}{d\lambda}\right)^2.$$

- These eqn.'s can be solved to get

$$\frac{dt}{d\lambda} = \pm\sqrt{s+\frac{v^2}{a^2}} = g(t), \qquad \frac{dx^i}{d\lambda} = \frac{v^i}{a^2} = f^i(\lambda),$$

- Where s and v^i are integration constants and λ is a nonspacelike affine parameter. s =1,0 for timelike and null affine parameter.

IV. Extending spacetime and singularity crossing: geodesic extension



- For these first-order eqn.'s Picard-Lindelof theorem states that if f^i and g are continues in λ and Lipshitz continues in t, there exist a unique soln. for the first-order eqn.'s
- Now joining the two branches (soln's for t ≥ 0 and t < 0) together leads to following scale factor

$$a(t) = a_0\left[1 + |H_0 t| - \frac{2}{3}|H_0 t|^{3/2}\right] + O(t^2).$$

- Notice: First, the above theorem shows the possibility of geodesic extensions for sudden singularities in general.
- Furthermore, the invariance of Raychaudhuri eqn. under t → -t, and H → -H leads to existence of time-revered soln. which works as a natural extension for FLRW for t<0.

IV. Extending spacetime and singularity crossing: geodesic extension

- Now integrating geodesic eqn.'s such that t=0 at λ=0, one obtains

$$t(\lambda) = \chi\,\lambda - sign(\lambda)\frac{H_0 v^2}{2a_0^2}\lambda^2 + O(\lambda^3). \qquad x^i(\lambda) = x_0{}^i + \frac{v^i}{a_0^2}\,\lambda + sign(\lambda)\,\frac{H_0 v^2}{a_0^2}\,\chi\lambda^2 + O(\lambda^3).$$

Notice:

- These geodesics are C^1 and defined for all values of λ. Therefore, geodesics are complete and nonspacelike test objects does not get destroyed crossing the singularity.

- To have a consistent gravitational description it is not enough to have a C^1 geodesic extension, we have to check the consistency of this extension with the field equations.

IV. Extending spacetime and singularity crossing

Junction conditions for effective higher-curvature gravity:

- Here we use Gauss-Codazzi equations to derive junction condition for this higher-curvature gravity.
- Let us start with Gauss's normal coordinates near a hypersurface $\sum$ with metric $\tilde{g}$;

$$ds^2 = \epsilon dw^2 + \tilde{g}_{ij} dx^i dx^j ,$$

where n^μ is a normal vector to $\sum$ with n.n = ε = -1, 1 (for spacelike or timelike surface)

- Extrinsic curvature is defined as $K_{ij} = -\frac{1}{2}\ \tilde{g}_{ij,w}$
- Gauss-Codazzi equations reads

$$\begin{aligned} R^l{}_{ijk} &= \tilde{R}^l{}_{ijk} + \epsilon\,(K_{ij}K_k{}^l - K_{ik}K^l{}_j) \\ R^w{}_{ijk} &= -\epsilon\,(K_{ij|k} - K_{ik|j}) \\ R^w{}_{iwj} &= \epsilon\,(K_{ij,w} + K_{il}K^l{}_k). \end{aligned}$$

$\Longrightarrow$

$$\begin{aligned} R^i{}_j &= \tilde{R}^i{}_j + \epsilon\,(K^i{}_{j,w} - K\,K^i{}_j) \\ R^w{}_j &= -\epsilon\,(K^i{}_{j|i} - K_{|j}) \\ R^w{}_w &= \epsilon\,(K_{,w} - trK^2), \end{aligned}$$

where $K = K^i{}_i$ and $trK^2 = K_{ij}K^{ij}$.

IV. Extending spacetime and singularity crossing:

Junction conditions for effective higher-curvature gravity:

- Now we want to express everything in terms of K_{ij} and its derivatives. Field eqn.'s

$$G_{\mu\nu} + H_{\mu\nu} = \kappa T^{(m)}_{\mu\nu}, \qquad H_{\mu\nu} = -\kappa\,\beta\left[\frac{R^2}{12} g_{\mu\nu} - R^{\rho\sigma} R_{\rho\mu\sigma\nu}\right].$$

- Let us start with Einstein tensor;

$$\begin{aligned} G^w{}_w &= -\frac{1}{2}\tilde{R} + \frac{1}{2}\epsilon\,[K^2 - trK^2] \\ G^w{}_i &= -\epsilon\,[K_i{}^m{}_{|m} - K_{|i}] \\ G^i{}_j &= \tilde{G}^i{}_j + \epsilon\left[(K^i{}_j - \delta^i{}_j K)_{,w} - KK^i{}_j + \frac{1}{2}\delta^i{}_j K^2 + \frac{1}{2}\delta^i{}_j\, trK^2\right] \end{aligned}$$

- Now for joining two spacetimes at the hypersurface w=0, we must have a continues $\tilde{g}_{ij}$.
- Integrating the above eqn.'s one obtains

$$\lim_{\sigma\to 0}\int_{-\sigma}^{\sigma} G^w{}_w\, dw = [\mathbf{G}^w{}_w] = 0 \qquad \lim_{\sigma\to 0}\int_{-\sigma}^{\sigma} G^i{}_j\, dw = [\mathbf{G}^i{}_j] = \epsilon\,([K^i{}_j] - \delta^i{}_j\,[K])$$

$$\lim_{\sigma\to 0}\int_{-\sigma}^{\sigma} G^w{}_i\, dw = [\mathbf{G}^w{}_i] = 0 \qquad \text{or } [K_{ij}] \neq 0$$

IV. Extending spacetime and singularity crossing:

Junction conditions for effective higher-curvature gravity:

- Now let us express $H_{\mu\nu}$ in terms of K_{ij};

$$
\begin{aligned}
H^{w}{}_{w} = \quad & \epsilon\,\beta\,\kappa \hat{K}_{ij,w}\,\tilde{R}^{ij} + \epsilon^2\,\beta\,\kappa\left[\hat{K}_{ij,w}\,\hat{K}^{ij}_{,w} + \hat{K}^{ij}_{,w}\left(\frac{4}{3}K\,\hat{K}_{ij} + \hat{K}_i^m\,\hat{K}_{jm}\right)\right. \\
& \left. -\hat{K}_{ij,w}\left(3\,K\,\hat{K}^{ij} + 2\,\hat{K}^{jr}\,\hat{K}^i_r\right)\right], \\
H^{w}{}_{i} = \quad & \epsilon^2\,\beta\,\kappa\left[\hat{K}_{ij,w}\left(\hat{K^{mj}}_{|m} - \frac{2}{3}\,\hat{K}^j_{|}\right) - \hat{K}^{rs}_{,w}\,\hat{K}_{ri|s} - \hat{K}^{rs}_{,w}\,\hat{K}_{rs|i}\right], \\
H^{i}{}_{j} = \quad & -\frac{1}{3}\epsilon\,\beta\,\kappa\left[K_{,w}\,\tilde{R}\,\delta^i{}_j - K_{,w}\,\tilde{R}^i_j - 3\,\hat{K^{rs}}_{,w}\,\tilde{R}^i_{rjs}\right] + \epsilon^2\,\beta\,\kappa\left[\frac{K_{,w}}{3}\left(3\,\hat{K}^i{}_{j,w}\right.\right. \\
& \left. -2\,\hat{K}^{ir}\,\hat{K}_{rj} - \frac{5}{3}K\,\hat{K}^i{}_j - \frac{4}{9}K^2\,\delta^i{}_j\right) - \hat{K}^i{}_{j,w}\left(\frac{2}{9}K^2 + tr\hat{K}^2\right) \\
& \left. +\hat{K^{rs}}_{,w}\left(\hat{K_{rj}}\,\hat{K^i}_s - \hat{K_{rs}}\,\hat{K^i}_j - \frac{\delta^i{}_j}{3}K\,\hat{K_{rs}}\right) + \hat{K}^s{}_{j,w}\,\hat{K}^i{}_s\,K\right].
\end{aligned}
$$

IV. Extending spacetime and singularity crossing:

Junction conditions for effective higher-curvature gravity:

- Since $H_w{}^w$ and $H_i{}^w$ depends quadratically on $K_{ij,w}$ one might choose $[K_{ij}]=0$, but this doesn't allow for surface layer to form.
- It is more convenient to split K_{ij} into a trace and traceless part (this splitting is adopted in F(R) theories for junction condition too);

$$\hat{K}_{ij} = K_{ij} - \frac{\tilde{g}_{ij}}{3} K.$$

- Now the conditions reads

$$[\hat{K}_{ij}] = 0, \qquad [K] \neq 0.$$

- These conditions leads to

$$\kappa\,[\mathbf{T}^w{}_w] = [\mathbf{G}^w{}_w] + [\mathbf{H}^w_w] = 0,$$
$$\kappa\,[\mathbf{T}^w{}_i] = [\mathbf{G}^w{}_i] + [\mathbf{H}^w_i] = 0,$$
$$\kappa\,[\mathbf{T}^i{}_j] = [\mathbf{G}^i{}_j] + [\mathbf{H}^i_j] = \mathbf{S}^i_j$$

$$\mathbf{S}^i_j = \lim_{\sigma\to 0}\int_{-\sigma}^{\sigma} dw K_{,w}\left[\frac{\epsilon}{3}\left(\beta\kappa[\tilde{R}^i_j - \tilde{R}\delta^i{}_j] - 2\delta^i{}_j\right) + \epsilon^2\beta\kappa\left(\hat{K}^i_{j,w} - \frac{4}{27}K^2\delta^i{}_j\right)\right]$$

IV. Extending spacetime and singularity crossing:
Junction conditions for effective higher-derivative gravity:

- The extended spacetime scale factor is (for all times)

$$a(t) = a_0 \left[1 + |H_0 t| - \frac{2}{3}|H_0 t|^{3/2}\right] + O(t^2),$$

where, $w = t$, $\varepsilon = -1$ and $\tilde{g}_{ij} = a(t)^2\, \delta_{ij}$ and

$$\hat{K}_{ij} = 0, \qquad K = -3\,H(t), \qquad K_{,t} = -3\,\dot{H}(t).$$

- Calculating S_{ij}

$$\begin{aligned} \mathbf{S}^i_j &= \lim_{\sigma\to 0}\int_{-\sigma}^{\sigma} dt\, K_{,t}\left[\frac{2}{3} - \frac{4\beta\kappa}{27}K^2\right]\delta^i{}_j \\ &= -4\,H_0\left[1 - \frac{H(0)^2}{H_0{}^2}\right]\delta^i_j, \end{aligned}$$

V. Application to Cosmology

- *Clearly, this results suggest that FLRW has a contracting phase prior to its expanding phase.*

- *This Construction is similar to big bounce cosmology in which there is a sudden singularity at the bounce but* $\ddot{a} > 0$. *Here we have* $\ddot{a} < 0$.

- *An important features of bounces, including ours, is that they evade the trans-Planckian problem, and they can produce a scale invariant power spectrum.*

- *There might be a need for some cutoff scale to deform slightly the scale invariant power spectrum to account for the smaller number of satellite galaxies observed, compared to the number predicted by the LambdaCDM.*

VI. Conclusion

- *We have considered the role of Weyl anomaly in modifying FLRW cosmology at early times and showed the following;*
- *The singularity of the model is weak, therefore, might admits geodesic extension.*
- *Quantum corrections changed the nature of the singularity from big bang to sudden singularity.*
- *Joining the two disjoint branches of soln.'s provides us with a C^1 extension to geodesics that leave the spacetime geodesically complete.*
- *Using Gauss-Codazzi eqn.'s one can derive junction conditions for this higher-curvature gravity which are consistent with the geodesic extension.*
- *These results suggest that FLRW has a contracting phase prior to its expanding phase.*