Constant-roll Inflation in $f(T)$ Teleparallel Gravity

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Introduction

Unquestionably, the inflationary era is of profound importance for the description of the primordial cosmological evolution of our Universe [1–3], and many theoretical frameworks can successfully incorporate various version of this early-time acceleration [4–9]. Most common is the scalar-tensor description, in which a scalar field slowly rolls a nearly plateau like potential, however modified gravity in various forms can describe an inflationary era. With regards to the modified gravity description of inflation, it is also possible to describe early and late-time acceleration within the same theoretical framework [10].
One questionable feature of the scalar field description of inflation, is that it is not possible to describe non-Gaussianities. The existence of non-Gaussianities may be verified by future observations of the primordial density perturbations. In some sense, there are debatable arguments against the existence of non-Gaussianities (see Ref. [11] for a review on non-Gaussianities), that to our opinion are philosophically aligned with the Occam's razor way of thinking, that is, the non-correlation of the primordial modes is the simplest answer, and therefore non-Gaussianities should be absent in the power spectrum. However, this is a unilateral approach in the scientific problem at hand, and a theory that aims to describe successfully the Universe should be robust against any opposing future observation. In this context, it was shown in Refs. [12–28] that if the slow-roll condition is modified, non-Gaussianities can be predicted even in the context of scalar-tensor theories of inflation. Also in Refs. [23, 24, 28] several transition between constant and slow-roll eras

were successfully described. The implications of the constant-roll condition in $F(R)$ gravity were firstly studied in Ref. [25], and also in a later publication [29], an alternative approach was considered.

In this paper we shall investigate the implications of a constant-roll inflationary era in the context of $f(T)$ teleparallel gravity. The theoretical framework of $f(T)$ teleparallel gravity has proved to be quite useful in cosmological and also astrophysical applications, and for recent reviews we refer to [9, 30]. Particularly, late-time acceleration in $f(T)$ gravity was studied in Refs. [31–44], and also inflationary and bouncing cosmology scenarios were studied in [45–54]. Also various astrophysical aspects of $f(T)$ gravity were addressed in Refs. [55–60] and in addition, the thermodynamics of $f(T)$ and other modified gravities were studied in Ref. [61]. In view of the various successful description of $f(T)$ gravity in both at a local and global scales in the Universe, with this work we aim to investigate thoroughly the implications of a constant-roll condition in $f(T)$ gravity. We shall assume a scalar constant-roll condition holds true, and we shall perform an in depth analysis of the various implications on the $f(T)$ inflationary era. Particularly, we shall demonstrate how the cosmological equations are altered in view of the constant-roll condition, and we shall show that the resulting formalism is actually a reconstruction mechanism that enables us to either, fix the cosmological evolution and find the corresponding $f(T)$ gravity which realizes the given cosmological evolution, or to fix the $f(T)$ gravity and seek for the Hubble rate

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solution that corresponds to this $f(T)$ gravity. In both cases, we shall assume the existence of a scalar field which acts as the inflaton, and in both cases we shall calculate the scalar potential that corresponds to the constant-roll scenario under study. In the case that the $f(T)$ gravity is fixed, we shall also calculate the cosmological indexes corresponding to the power spectrum of the primordial curvature perturbations, and particularly, the spectral index and the scalar-to-tensor ratio, and accordingly we confront the results with the latest Planck [62] and BICEP2/Keck Array data [63]. As we will show, the parameter that quantifies the constant-roll era, enters to the final expressions of the observational indices.

This paper is organized as follows: In section 2 we briefly review the essential features of $f(T)$ teleparallel gravity and in section 3 we present some characteristic results from the minimally coupled scalar-$f(T)$ theory, that will be needed in the sections to follow. In section 4 we present the reconstruction mechanism of constant-roll $f(T)$ gravity, and we discuss how the constant-roll condition alters the formalism of teleparallel gravity. In section 5, we fix the Hubble evolution and we investigate which teleparallel gravity and which potential can generate such an evolution. Also we perform a thorough phase space analysis of the cosmological dynamical system, discussing the physical meaning of the resulting fixed points. In section 6 we fix the functional form of the $f(T)$ gravity and we investigate which Hubble evolution this generates, and also we calculate in detail the observational indices of the corresponding cosmological theory. Finally the results follow in the end of this paper.

Essential Features of Teleparallel Geometry

Before we get into the core of this paper, let us briefly present some fundamental features of teleparallel geometry and gravity, for details we refer the reader to the reviews [9, 30]. We consider a 4-dimensional smooth manifold (M, h_a), with h_a $(a = 1, \cdots, 4)$ being four independent vector (tetrad) fields defined globally on M, with the last condition actually being the realization of absolute parallelism. The tetrad vector fields satisfy the tensor relation $h_a{}^\mu h^a{}_\nu = \delta^\mu_\nu$ and also $h_a{}^\mu h^b{}_\mu = \delta^b_a$, where $(\mu = 1, \cdots, a)$ are the coordinate components of the a-th vector field h_a. By using the tetrad field, we can construct a curvature-less (Weitzenböck) linear connection of the following form $\Gamma^\alpha{}_{\mu\nu} \equiv h_a{}^\alpha \partial_\nu h^a{}_\mu = -h^a{}_\mu \partial_\nu h_a{}^\alpha$. Notably, the tetrad fields fulfill the teleparallel condition $\nabla_\nu h_a{}^\mu \equiv 0$, where the operator ∇_ν is the covariant derivative with respect to the Weitzenböck connection we defined above.

Also, the tetrad field can be used to construct the metric tensor on the manifold M by using $g_{\mu\nu} \equiv \eta_{ab} h^a{}_\mu h^b{}_\nu$ with η_{ab} being an induced Minkowski metric on the tangent space of M. The inverse metric is equal to $g^{\mu\nu} = \eta^{ab} h_a{}^\mu h_b{}^\nu$ and subsequently the Levi-Civita symmetric connection is $\overset{\circ}{\Gamma}{}^\alpha{}_{\mu\nu} = \frac{1}{2} g^{\alpha\sigma} \left(\partial_\nu g_{\mu\sigma} + \partial_\mu g_{\nu\sigma} - \partial_\sigma g_{\mu\nu}\right)$ can be defined, and in effect, a Riemannian geometry can be defined. The torsion and the contorsion tensors of the Weitzenböck connection are defined as follows, $T^\alpha{}_{\mu\nu} \equiv \Gamma^\alpha{}_{\nu\mu} - \Gamma^\alpha{}_{\mu\nu} = h_a{}^\alpha \left(\partial_\mu h^a{}_\nu - \partial_\nu h^a{}_\mu\right)$ and $K^\alpha{}_{\mu\nu} \equiv \Gamma^\alpha{}_{\mu\nu} - \overset{\circ}{\Gamma}{}^\alpha{}_{\mu\nu} = h_a{}^\alpha \overset{\circ}{\nabla}_\nu h^a{}_\mu$., where the covariant derivative $\overset{\circ}{\nabla}_\nu$ is defined with respect to the Levi-Civita connection. The torsion tensor can be written in terms of the

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contorsion tensor as $T_{\alpha\mu\nu} = K_{\alpha\mu\nu} - K_{\alpha\nu\mu}$, while the inverse is equal to $K_{\alpha\mu\nu} = \frac{1}{2}\left(T_{\nu\alpha\mu} + T_{\alpha\mu\nu} - T_{\mu\alpha\nu}\right)$, where $T_{\mu\nu\sigma} = g_{\epsilon\mu}\, T^{\epsilon}{}_{\nu\sigma}$ and $K_{\mu\nu\sigma} = g_{\epsilon\mu}\, K^{\epsilon}{}_{\nu\sigma}$.
In teleparallel geometry, the torsion scalar is defined as follows,
$T = \frac{1}{4} T^{\alpha\mu\nu} T_{\alpha\mu\nu} + \frac{1}{2} T^{\alpha\mu\nu} T_{\mu\alpha\nu} - T^{\alpha} T_{\alpha}$, where $T^{\alpha} = T_{\rho}{}^{\alpha\rho}$. The torsion scalar can be written in a compact form in the following way,

$$T \equiv T^{\alpha}{}_{\mu\nu} S_{\alpha}{}^{\mu\nu}, \tag{2.1}$$

where the superpotential tensor $S_{\alpha}{}^{\mu\nu}$ is defined as follows,

$$S_{\alpha}{}^{\mu\nu} = \frac{1}{2}\left(K^{\mu\nu}{}_{\alpha} + \delta^{\mu}_{\alpha} T^{\beta\nu}{}_{\beta} - \delta^{\nu}_{\alpha} T^{\beta\mu}{}_{\beta}\right), \tag{2.2}$$

which is skew symmetric in the last pair of indices. The teleparallel torsion scalar is equivalent to the Riemannian curvature scalar R, up to a total derivative term. Consequently, when T is used in a Lagrangian instead of R in Einstein-Hilbert action, the resulting field equations are equivalent, and this is actually the Teleparallel Equivalent of General Relativity (TEGR) theory of gravity.

Inflaton Minimally Coupled to $f(T)$ Teleparallel Gravity

In the context of $f(T)$ teleparallel gravity, the most successful inflationary theories are those for which the inflaton ϕ is minimally coupled to gravity, with the action being,

$$\mathcal{S} = \int d^4x|h| \left(\mathcal{L}_g + \mathcal{L}_\phi\right), \tag{3.1}$$

where $|h| = \sqrt{-g} = \det\left(h_\mu{}^a\right)$. The scalar field part of the Lagrangian in Eq. (3.1), namely $\mathcal{L}_\phi$, is defined as follows,

$$\mathcal{L}_\phi = \frac{1}{2}\partial_\mu\phi\, \partial^\mu\phi - V(\phi), \tag{3.2}$$

where $\partial^\mu = g^{\mu\nu}\partial_\nu$. By varying $\mathcal{L}_\phi$ with respect to the metric, or equivalently with respect to the tetrad fields [64], enables us to define the stress-energy tensor as follows,

$$\overset{\phi}{\mathfrak{T}}{}_\mu{}^\nu = h^a{}_\mu\left(-\frac{1}{h}\frac{\delta\mathcal{L}_\phi}{\delta h^a{}_\nu}\right) = \partial_\mu\phi\, \partial^\nu\phi - \delta^\nu_\mu\mathcal{L}_\phi\,, \tag{3.3}$$

which describes the matter content of the theory. We assume the stress-energy tensor to have a perfect fluid form, so it can be expressed as follows,

$$\overset{\phi}{\mathfrak{T}}{}_{\mu\nu} = \rho_\phi u_\mu u_\nu + p_\phi(u_\mu u_\nu + g_{\mu\nu}), \tag{3.4}$$

where u^μ is the 4-velocity unit vector of the fluid. In most cosmological applications, where a massless scalar field ϕ with potential $V(\phi)$ is used, the unit vector is chosen to be normal to spacelike hypersurfaces defined by $\phi =$ constant. In effect, the stress-energy tensor (3.4) defines the scalar field density ρ_ϕ and the corresponding pressure p_ϕ in its rest frame, as follows,

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \quad p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{3.5}$$

In the above, we ignored an extra term which is generated by existing anisotropies.
In the spirit of $f(R)$-gravity, in the context of which, one replaces R by an arbitrary function $f(R)$ in the Einstein-Hilbert action, the TEGR has been generalized by replacing T by an arbitrary function $f(T)$ [32, 33, 39, 65]. In the natural units ($c = \hbar = k_B = 1$), the $f(T)$ Lagrangian is equal to,

$$\mathcal{L}_g = \frac{M_p^2}{2} f(T), \tag{3.6}$$

where $M_p = 2.4 \times 10^{18}$ GeV is the reduced Planck mass, which can be related to the gravitational constant G via $M_p = 1/\sqrt{8\pi G} \equiv 1/\kappa$. Upon varying the action containing the Lagrangian $\mathcal{L}_g$, with respect to the tetrad fields, we obtain the tensor,

$$\tilde{H}_\mu{}^\nu = h^a{}_\mu \left(\frac{1}{h} \frac{\delta \mathcal{L}_g}{\delta h^a{}_\nu} \right) = \frac{M_p^2}{2} h^a{}_\mu \left(\frac{1}{h} \frac{\delta f(T)}{\delta h^a{}_\nu} \right) \tag{3.7}$$

Upon rescaling, the tensor above takes the form ${H_\mu}^\nu = \frac{1}{2} M_p^{-2} {\tilde{H}_\mu}{}^\nu$, and in effect we have,

$$ {H_\mu}^\nu = {S_\mu}^{\rho\nu} \partial_\rho T f_{TT} + \left[\frac{{h_\mu}^a}{h} \partial_\rho \left(h {S_a}^{\rho\nu} \right) - T_\rho {S_\mu}^{\nu\rho} \right] f_T + \frac{1}{4} \delta^\nu_\mu f(T), \tag{3.8} $$

where f_T and f_{TT}, stand for $f_T = \frac{df(T)}{dT}$ and $f_{TT} = \frac{d^2 f(T)}{dT^2}$ respectively. By varying the action (3.1) with respect to the tetrad fields, using Eqs. (3.7) and (3.3), gives the following field equations of $f(T)$ teleparallel gravity,

$$ {H_\mu}^\nu = \frac{1}{2} M_p^{-2} \overset{\phi}{\mathfrak{T}}{}_\mu{}^\nu, \tag{3.9} $$

or equivalently, by substituting from Eq. (3.8), we obtain,

$$ {S_a}^{\rho\nu} \partial_\rho T f_{TT} + \left[\frac{1}{h} \partial_\rho \left(h {S_a}^{\rho\nu} \right) - h^\lambda_a {T^\mu}_{\rho\lambda} {S_\mu}^{\nu\rho} \right] f_T + \frac{1}{4} h^\nu_a f(T) = \frac{M_p^{-2}}{2} h^\mu_a \overset{\phi}{\mathfrak{T}}{}_\mu{}^\nu, \tag{3.10} $$

It is clear that the general relativistic limit is recovered by setting $f(T) = T$. We will assume that the background metric is a flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric, with line element,

$$ ds^2 = dt^2 - a(t)^2 \delta_{ij} dx^i dx^j, \tag{3.11} $$

where $a(t)$ is the scale factor of the Universe. Thus, the vierbein may take the following diagonal form,

$$h_\mu{}^a = \text{diag}\left(1, a(t), a(t), a(t)\right). \tag{3.12}$$

This directly relates the teleparallel torsion scalar (2.1) to Hubble rate as follows,

$$T = -6H^2, \tag{3.13}$$

where $H \equiv \dot{a}/a$ is Hubble rate, and the "dot" denotes differentiation with respect to the cosmic time t. Inserting the vierbein (3.12) into the field equations (3.10) for the scalar field matter fluid (3.4), the modified Friedmann equations of the $f(T)$-gravity are,

$$\rho_\phi = \frac{M_p^2}{2}\left[f(T) + 12H^2 f_T\right], \tag{3.14}$$

$$p_\phi = -\frac{M_p^2}{2}\left[f(T) + 4(3H^2 + \dot{H})f_T - 48\dot{H}H^2 f_{TT}\right]. \tag{3.15}$$

Independently from the above equations, one could choose an equation of state to relate ρ_ϕ and p_ϕ. Here, we choose the simple barotropic case $p_\phi \equiv p_\phi(\rho_\phi)$. Generally, any modified theory of gravity should be recognized as a correction of the standard general relativistic gravity, so it is convenient to transform from the matter frame, we have been using, to the effective frame, which yields Einstein's gravity, in addition to the higher order $f(T)$ teleparallel gravity. So we rewrite the modified Friedmann equations in the case of $f(T)$-gravity, as follows,

$$\begin{aligned} H^2 &= \frac{M_p^{-2}}{3}\left(\rho_\phi + \rho_T\right) \equiv \frac{M_p^{-2}}{3}\rho_{eff}, && (3.16)\\ 2\dot{H} + 3H^2 &= -M_p^{-2}\left(p_\phi + p_T\right) \equiv -M_p^{-2} p_{eff}. && (3.17)\end{aligned}$$

In this case, the density and pressure of the torsional counterpart of $f(T)$ are defined by,

$$\begin{aligned} \rho_T &= \frac{M_p^2}{2}\left(2Tf_T - T - f(T)\right), && (3.18)\\ p_T &= \frac{M_p^2}{2}\frac{f(T) - Tf_T + 2T^2 f_{TT}}{f_T + 2Tf_{TT}}. && (3.19)\end{aligned}$$

At the GR limit ($f(T) = T$), we have $\rho_T = 0$ and $p_T = 0$. In the barotropic case, the torsion will have an equation of state,

$$w_T = \frac{p_T}{\rho_T} = \frac{(f(T) - 2Tf_T)(f_T + 2Tf_{TT} - 1)}{(f(T) + T - 2Tf_T)(f_T + 2Tf_{TT})}. \quad (3.20)$$

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To fulfill the conservation law, when the scalar field and the torsion are minimally coupled, we have the following continuity equations,

$$\dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) = 0, \tag{3.21}$$

$$\dot{\rho}_T + 3H(\rho_T + p_T) = 0. \tag{3.22}$$

Also the effective equation of state (EoS) parameter is defined as follows,

$$w_{eff} \equiv \frac{p_{eff}}{\rho_{eff}} = -1 - \frac{2}{3}\frac{\dot{H}}{H^2}. \tag{3.23}$$

Slow-roll and Constant-roll Inflation in $f(T)$ Teleparallel Gravity

In this section we shall investigate the qualitative and quantitative consequences of a constant-roll inflationary era in $f(T)$ teleparallel gravity.
It is a known fact that the modification of the Friedmann equations due to $f(T)$ gravity, can be written as a one-dimensional autonomous system of the form $\dot{H} \equiv \dot{H}(H) = \mathcal{F}(H)$, for a general barotropic equation of state [66]. In this case, it is more convenient to use the Hubble rate H as an independent variable instead of the torsion scalar T. Using Eq. (3.13), the modified Friedmann equations (3.14) and (3.15) can be written as follows,

$$\rho_\phi = \frac{M_p^2}{2}\left(f - Hf'\right), \tag{4.1}$$

$$p_\phi = -\frac{M_p^2}{2}\left(f - Hf' - \frac{1}{3}\dot{H}f''\right) = \frac{M_p^2}{6}\dot{H}f'' - \rho_\phi, \tag{4.2}$$

where $f \equiv f(H)$, $f' \equiv \frac{df}{dH}$ and $f'' \equiv \frac{d^2 f}{dH^2}$. Interestingly, Eq. (4.2) shows that the Hubble parameter does not only decrease as in the GR limit, but it can also increase without violating the weak energy condition $\rho_\phi + p_\phi > 0$. In particular, we have $\dot{H} > 0$ with $f'' > 0$, while $\dot{H} < 0$ with $f'' < 0$. The last case includes the particular value $f'' = -12$ which

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produces the GR limit. By using Eq. (3.5), we obtain the inflaton's kinetic term and the scalar potential,

$$\frac{M_p^2}{2}(f - Hf') = \frac{\dot{\phi}^2}{2} + V(\phi), \tag{4.3}$$

$$\frac{M_p^2}{6}\dot{H}f'' = \dot{\phi}^2. \tag{4.4}$$

Also, the continuity equation (3.21) is nothing but the Klein-Gordon equation of motion for the inflaton in the FLRW background,

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0, \tag{4.5}$$

which also can be obtained directly by varying the inflaton Lagrangian $\mathcal{L}_\phi$, appearing in Eq. (3.2), with respect to the scalar field ϕ. In the context of the slow-roll inflation approximation, two conditions have been imposed to the dynamical variables, firstly the condition which guarantees an accelerated expansion phase,

$$\frac{1}{2}\dot{\phi}^2 \ll V(\phi), \tag{4.6}$$

and secondly, the condition which makes the duration of the inflationary era prolonged, which is,

$$|\ddot{\phi}| \ll \left|\frac{\partial V}{\partial \phi}\right|. \tag{4.7}$$

Substituting Eqs. (4.4), (4.6) and (4.7) into Eq. (4.5), it is easy to show that the slow-roll potential in terms of the $f(T)$ gravity can be obtained as follows,

$$V' \simeq -\frac{M_p^2}{2} H f'' \,, \tag{4.8}$$

so by integrating we get,

$$V(H) = V_0 + \frac{M_p^2}{2}\left(f - Hf'\right). \tag{4.9}$$

In the GR limit, that is when $f = -6H^2$, the above equation reproduces the well-known relation

$$V(\phi) \simeq V_0 + 3M_p^2 H^2. \tag{4.10}$$

It is useful to parameterize the inflationary Universe by defining the first Hubble slow-roll index [9] and its running

$$\epsilon_1 = -\frac{d \ln H/dt}{H}, \quad \epsilon_{n+1} = \frac{d \ln \epsilon_n/dt}{H}. \tag{4.11}$$

Since H is almost constant during slow-roll inflationary era, the first slow-roll index is $\epsilon_1 < 1$. However, the slow-roll inflation particularly requires also ϵ_n to be small, that is, $|\epsilon_n| \ll 1$. In some other inflationary models, the slow-roll conditions have been modified, and replaced by a condition in which the term $|\ddot{\phi}|$ is no longer negligible. This is known as the so-called ultra slow-roll condition [13],

$$\ddot{\phi} = -3H\dot{\phi}. \tag{4.12}$$

This condition has been introduced in order to produce a potential with an exact flat plateau (4.5). It has been shown that the condition (4.12) violates the slow-roll

approximation, where the running of the first Hubble slow-roll index $|\epsilon_2|$ is no longer small [13]. This makes the ultra slow-roll models in fact to be some sort of a fast-roll inflation model, like the ones of Ref. [17]. Interestingly enough, the ultra slow-roll inflation can produce a scale invariant power spectrum [14]. However, the scalar (curvature) perturbations grow on the super-horizon energy scale unlike the slow-roll inflation [14, 15] which disfavors the ultra slow-roll condition [16].
Along the research line of the ultra slow-roll inflation, the constant-roll inflation scenario [12–28] is a modification of the slow-roll inflation scenario, in which case the following condition holds true,

$$\ddot{\phi} = \beta H \dot{\phi}, \tag{4.13}$$

where the slow-roll condition can be recovered if $\beta \ll 1$, while the ultra slow-roll is recovered by setting $\beta = -3$. Remarkably, it has been shown that $\beta = -3/2$ is a critical value, where the scalar perturbations grow for $\beta < -\frac{3}{2}$ and decay for $\beta > -\frac{3}{2}$ at the super-horizon scale. Moreover, the first Hubble slow-roll parameter satisfies $\epsilon \ll 1$ during the constant-roll inflation era, but its running satisfies $|\epsilon_n| > 1$. Furthermore, in spite of using a single inflaton in the constant-roll inflationary scenario, the local non-Gaussianity consistency relation can be violated which makes the constant-roll inflation phenomenologically distinguishable from the slow-roll scenario.

The constant roll inflation has been studied in the $f(R)$ gravity context in two different ways. In the first approach, the constant-roll condition (4.13) is applied to the $f(R)$ modified Friedmann equations [25], and in this way one can obtain the $f(R)$ gravity which generates a constantly rolling scalar field, or the construction a constant-roll potential for a given $f(R)$ gravity. Also, in extended studies, the possible transition between slow-roll and constant-roll and also between constant-roll eras, has been investigated in Refs. [23, 24, 28]. In the second approach, the condition $\ddot{F} = \beta H_J \dot{F}$ is considered as some sort of generalization of the constant-roll condition, where $F = df/dR_J$ and H_J and R_J are Hubble and Ricci scalar in Jordan frame [29]. In this paper, we investigate the condition (4.13) within the framework of the $f(T)$ teleparallel gravity.

In this section we shall present a fundamental technique that will enable us to derive the function Hubble rate function $H(\phi)$ in the context of constant-roll $f(T)$ teleparallel gravity. This will also enable us to obtain the scalar potential $V(\phi)$.
Plugging $\dot{H} = \dot{\phi}\frac{dH}{d\phi}$ in Eq. (4.4), we obtain,

$$\dot{\phi} = \frac{M_p^2}{6} f'' H_\phi, \tag{4.14}$$

where $H_\phi = \frac{dH}{d\phi}$, $H_{\phi\phi} = \frac{d^2H}{d\phi^2}$ and also we used the fact that the second derivative of the inflaton field with respect to the cosmic time is,

$$\ddot{\phi} = \frac{M_p^2}{6}\left[f'' H_{\phi\phi} + f''' H_\phi^2\right]\dot{\phi}. \tag{4.15}$$

Then by applying the constant-roll condition (4.13), we obtain the following differential equation,

$$f'' H_{\phi\phi} + f''' H_\phi^2 - \frac{6\beta}{M_p^2} H = 0. \tag{4.16}$$

The above equation represents a modified version of the original work of the constant-roll inflation [16] due to the contribution of the torsional counterpart of $f(T)$ gravity. For a given

$f(T)$ theory, we can solve Eq. (4.16) to obtain the generating function $H(\phi)$ analytically. For example, at the GR limit, that is when, $f(H) = -6H^2$, the above differential equation reduces to the simple harmonic oscillator differential equation, where $H(\phi)$ can be obtained analytically, as a linear combination of exponentials of the form $e^{\pm\sqrt{-\beta/2}\phi}$. Since the teleparallel torsion (Hubble) can be related directly to the inflaton field, then we have $f' = f_\phi/H_\phi$, $f'' = \left(f_{\phi\phi}H_\phi - f_\phi H_{\phi\phi}\right)/H_\phi^3$, and so on. In effect, the differential equation (4.16) can be rewritten as follows,

$$H_\phi^2\, f_{\phi\phi\phi} - 2H_\phi\, H_{\phi\phi}\, f_{\phi\phi} + \left(2H_{\phi\phi}^2 - H_\phi\, H_{\phi\phi\phi}\right) f_\phi - \frac{6}{M_p^2} H_\phi^3 H = 0, \tag{4.17}$$

which has a general solution,

$$f(\phi) = c_3 + \int \left(c_2 + c_1\phi + \frac{6\beta}{M_p^2} \int\int H d\phi d\phi\right) H_\phi\, d\phi, \tag{4.18}$$

where c_1, c_2 and c_3 are integration constants. It is clear that the c_2 term $\propto H$, acts as a divergence term in the action (3.1). Hereafter we omit this term, and also for simplicity we take $c_3 = 0$. For a given cosmic evolution $H(\phi)$, the above differential equation determines the corresponding constant-roll $f(T)$ gravity.

Turning our focus to the scalar potential, by inserting (4.14) into (4.3), the constant-roll potential can be written as follows,

$$V(\phi) = \frac{M_p^2}{2}\left[f - Hf' - \frac{M_p^2}{36} f''^2 H_\phi^2\right]. \tag{4.19}$$

The above equation represents a modified version of the constant-roll potential which has been previously obtained in for example, in the context of a canonical scalar field gravity. This can be shown clearly by taking $f(H) = -6H^2 + F(H)$, in effect Eq. (4.19) becomes,

$$V(\phi) = M_p^2\left[3H^2 - 2M_p^2 H_\phi^2\right] + \underbrace{\frac{M_p^2}{2}\left[F - HF' - \frac{M_p^2}{36}(F'' - 24)F'' H_\phi^2\right]}_{f(T)\text{ modification}}. \tag{4.20}$$

In the above expression for the scalar potential, the second term on the right hand side, is essentially the contribution of $f(T)$ gravity in the constant-roll potential. For $F(H) = 0$, the constant-roll potential takes the usual scalar tensor form appearing in the related literature. We note that for any $f(T)$ gravity, the constant-roll inflationary era can be quantified by making use of Eqs. (4.16) and (4.19).

Alternatively, applying the constant-roll condition (4.13) to the Klein-Gordon equation (4.5), we obtain,

$$(3+\beta)H\dot{H}\phi'^2 + V' = 0, \tag{4.21}$$

where H here is an independent variable. Since $\dot{\phi} = \dot{H}\phi'$, Eq. (4.4) becomes $\dot{H}\phi'^2 = \frac{M_p^2}{6} f''$. Thus, the above differential equation takes the form,

$$V' = -\frac{M_p^2}{6}(3+\beta)Hf''. \tag{4.22}$$

$$V(H) = V_0 + \frac{M_p^2}{6}(3+\beta)\left[f - Hf'\right]. \tag{4.23}$$

The equation (4.23) is an equivalent alternative to Eq. (4.19). However, it can be used to reconstruct constant-roll potentials directly for a given $f(T)$ gravity, without knowing the generating function $H(\phi)$. Conversely, it enables us to reconstruct the $f(T)$ gravity, which generates a given constant-roll potential $V(H)$. We also note that the constant-roll inflation can be fully determined by combining Eqs. (4.16) and (4.22).

In the sections to follow, we investigate some particular constant-roll inflationary models.

Our investigation is two-fold: first, we shall assume a particular constant-roll generating function $H(\phi)$ which has been obtained in Ref. [17]. According to the $f(T)$ contribution in

(4.16), a modified constant-roll potential will be obtained. This will eventually constrain the constant-roll parameter β when the observational indices are taken into account. Second, we shall assume a particular form of $f(T)$ teleparallel gravity, and we shall construct the corresponding constant-roll potential. Also, we shall examine the compatibility of the model obtained, with the current observational data.

Reconstruction of $f(T)$ Gravity for Constant-roll Inflation

In this section, we shall specify the Hubble rate as a function of the scalar field $H(\phi)$, and by using this and the reconstruction techniques we presented in the previous sections, we shall find the scalar potential and the $f(T)$ gravity that may generate $H(\phi)$. We assume that $H(\phi)$ has the following form [17],

$$H(\phi) = M \cos\left(\sqrt{\frac{\beta}{2}}\frac{\phi}{M_p}\right), \tag{5.1}$$

where M is a scale characteristic of $H(\phi)$. In order to assure the validity of the spacetime description of the model, we assume that $M \leq M_p$, otherwise, quantum gravity effects should be also taken into account. Substituting Eq. (5.1) in Eq. (4.17), we get,

$$f(\phi) = -3M^2\left[1 + \cos\left(\sqrt{2\beta}\frac{\phi}{M_p}\right)\right] + c_1\left[2M_p \sin\left(\sqrt{\frac{\beta}{2}}\frac{\phi}{M_p}\right) - \sqrt{2\beta}\phi\cos\left(\sqrt{\frac{\beta}{2}}\frac{\phi}{M_p}\right)\right]. \tag{5.2}$$

The above equation will enable us to reconstruct the corresponding constant-roll potential $V(\phi)$ and also, it can be used to reconstruct the corresponding $f(T)$ theory that generates $H(\phi)$.

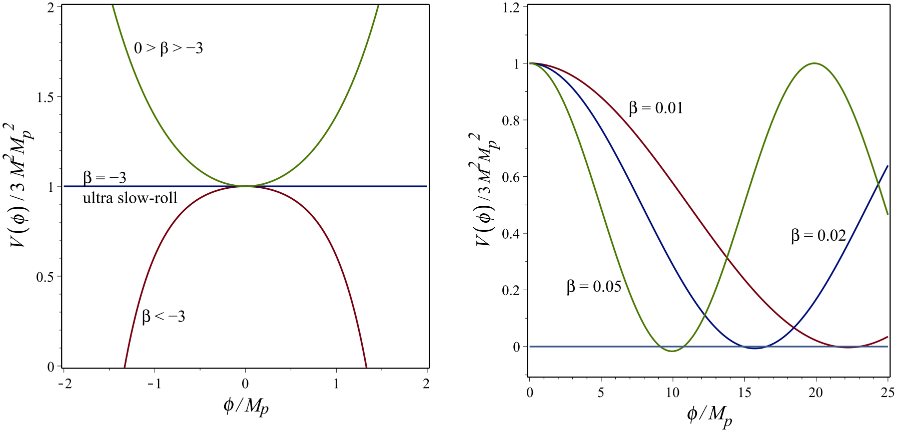
Reconstruction of $f(T)$ Gravity for Constant-roll Inflation

By substituting Eqs. (5.1) and (5.2) in Eq. (4.19), we obtain the scalar potential,

$$\begin{aligned} V(\phi) &= 3M^2 M_p^2 \left[1 - \frac{3+\beta}{6}\left\{1 - \cos\left(\sqrt{2\beta}\frac{\phi}{M_p}\right)\right\}\right] \\ &- \frac{c_1 M_p^2}{36}\left[c_1\beta\frac{M_p^2}{M^2} - 12M_p(3+\beta)\sin\left(\sqrt{\frac{\beta}{2}}\frac{\phi}{M_p}\right)\right]. \end{aligned} \tag{5.3}$$

The quantity in the second line of Eq. (5.3) corresponds to the $f(T)$ contribution to the cosmic evolution (5.1). For $c_1 = 0$, the potential reduces to the one which has been obtained in Ref. [17]. This case matches the cosine natural inflation model [67] with a negative cosmological constant. In Fig. 1, we show the role of the constant role parameter β in obtaining different patterns of the potential (5.3).

Reconstruction of $f(T)$ Gravity for Constant-roll Inflation



(a) $\beta < 0$ (b) $\beta > 0$

Figure 1: possible potential patterns of Eq. (5.3): (a) For $\beta < -3$ the potential has no attractor. For $\beta = -3$, the constant potential of the ultra slow roll model is

achieved. For $0 < \beta < -3$ the potential has an attractor; (b) For $\beta > 0$, the cosine potential pattern is achieved where β determines its frequency.

Using Eqs. (4.3) and (4.4), we obtain the following differential equation,

$$\dot{H} = 6\,\frac{f - Hf' - 2V(\phi)}{f''}. \tag{5.4}$$

Also, the constant-roll potential in the $f(T)$ gravity is given by Eq. (4.23). Then, Eq. (5.4) reads,

$$\dot{H} = -2\beta\,\frac{f - Hf'}{f''} = \mathcal{F}(H). \tag{5.5}$$

In fact, the above equation represents a one-dimensional autonomous system, since $\dot{H}$ can be written explicitly in terms of H. The above relation can clearly dictate the role of the constant-roll parameter β. As it can be easily shown, the parameter β is strongly related to the inflaton equation of state. By assuming a linear barotropic equation of state, that is, of the form $p_\phi = w_\phi \rho_\phi$, by combining Eqs. (4.3), (4.4) and (5.5), we obtain,

$$\beta = -\frac{3}{2}(1 + w_\phi). \tag{5.6}$$

Reconstruction of $f(T)$ Gravity for Constant-roll Inflation

By using Eq. (5.6), we classified all the different inflationary scenarios in Table 1. Interestingly enough, the constant-roll inflationary scenario becomes identical to the slow-roll, when $w_\phi = -1$, in which case $\beta = 0$. Also, when the constant-roll parameter β takes positive values, then the inflaton has a phantom EoS $w_\phi < -1$. It is worth mentioning that by choosing a dynamically varying EoS, the Universe can interpolate between different scenarios. This is an indirect approach to the transition problems studied in Refs. [23, 24, 28].

Table 1: Classifications of possible inflationary scenarios according to Eq. (5.6).

EoS	β	Model	Curvature perturbations
$w_\phi > 1$	$\beta < -3$	Non-attractor	growing
$w_\phi = 1$	$\beta = -3$	ultra slow-roll	growing
$0 < w_\phi < 1$	$-3 < \beta < -3/2$	attractor	growing
$w_\phi = 0$	$\beta = -3/2$	Cosh potential	growing
$-1 < w_\phi < 0$	$-3/2 < \beta < 0$	attractor	decaying
$w_\phi = -1$	$\beta = 0$	slow-roll	decaying
$w_\phi < -1$	$\beta > 0$	Cosine potential	decaying

For the model at hand, by using the inverse of Eq. (5.1), which yields $\phi \equiv \phi(H)$, we can rewrite Eq. (5.2) in terms of H as follows,

$$f(H) = -6H^2 + \frac{2c_1 M_p}{M}\left[\sqrt{M^2 - H^2} - 2H \arccos\left(\frac{H}{M}\right)\right]. \tag{5.7}$$

Thus, the constant-roll $f(T)$ gravity imposes the constraint $|H| \leq M \leq M_p$ on the Hubble parameter. This constraint ensures the physical consistency of the theory, since the Hubble rate cannot exceed the Planck mass M_p. The first term in Eq. (5.7) reproduces the GR limit, while the term proportional to c_1, corresponds to the $f(T)$ gravity modification. In effect, the differential equation of Eq. (5.5) reads,

$$\dot{H} = -2\beta \frac{\left(c_1 M_p \sqrt{M^2 - H^2} + 3MH^2\right)\sqrt{M^2 - H^2}}{c_1 M_p - 6M\sqrt{M^2 - H^2}}. \tag{5.8}$$

Obviously, the choice $\beta = 0$ implies that $\dot{H} = 0$, which matches exactly the de Sitter solution of slow-roll inflation. On the other hand, the choice $c_1 = 0$ implies that $\dot{H} = \beta H^2$, which matches exactly the standard cosmology. In this sense, the non-null values of c_1 and β could provide a cosmic evolution interpolating between these two cases. In Fig. 2, we plot different phase portraits corresponding to solutions of Eq. (5.8), for various choices of the parameters $c_1 \neq 0$ and $\beta \neq 0$.

Analysis of the Phase Space

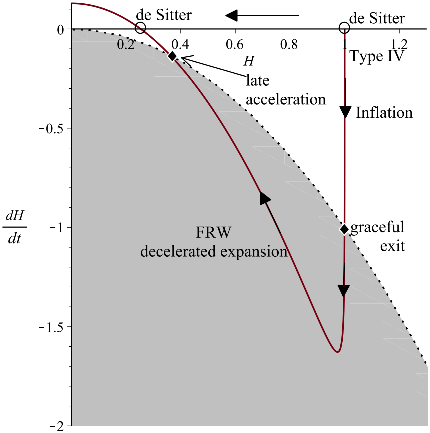
Let us now study in some detail the phase space of the cosmological dynamical system of Eq. (5.8). For $c_1 < 0$ and $\beta < 0$, and during the era for which $H > 0$, the dynamical system has two fixed points at,

$$H_{upper-fix} = M, \quad H_{lower-fix} = \frac{\sqrt{-2c_1 M_p}}{6M}\sqrt{c_1 M_p + \sqrt{c_1^2 M_p^2 + 36M^4}},$$

Obviously, the choice $c_1 = 0$ shifts the lower fixed point to be that of a Minkowski Universe $H = 0$, just as in the GR limit. However, the choice $c_1 < 0$, enforces the Universe to evolve towards the de Sitter solution $H > 0$ instead of the Minkowski. Actually, the modification that the $f(T)$ gravity introduces to the cosmological equations, namely (5.7), makes easy to interpret the de Sitter solution at the small H regime as late-time acceleration. In Fig. 2(a), the plot shows that the Universe interpolates between two de Sitter phases, as the Hubble rate decreases. In the standard model of cosmology, the Universe begins with an initial crushing-type singularity where the Hubble rate and its derivative blow-up, that is, $H \to \infty$ and $\dot{H} \to \infty$ at $t = 0$ finite time. Interestingly enough, our model imposes an upper bound for the Hubble parameter to be $H_{max} = M$, which can be chosen to be consistent with a maximum energy density at the Planck scale. Although, this maximum value of Hubble rate is at a fixed point, this can be reached at a finite time, which is,

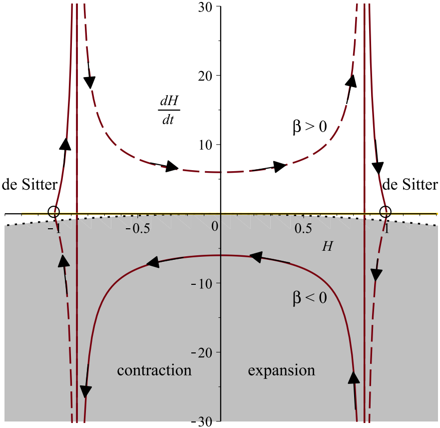
$$t_i = \int_{H_i}^{M} \frac{dH}{\dot{H}} = finite, \quad H_{lower-fix} < H_i < M. \tag{5.9}$$

Analysis of the Phase Space



(a) $c_1 < 0$

Analysis of the Phase Space



(b) $c_{1>0}$

Figure 2: Possible cosmic evolutions of the phase portrait (5.8); (a) For $c_1 < 0$, $H > 0$, and $\beta < 0$: The universe interpolates between two de Sitter (fixed points)

spaces, instead of big bang initial singularity there is type IV de Sitter state; (b) For $c_1 > 0$ and $\beta < 0$: The universe interpolates between two sudden (type II) singularities where junction conditions are applicable in this model, which leads to a cyclic universe.

In fact, at the upper fixed point the slope of the phase portrait $\frac{d\dot{H}}{dH}$ diverges which indicates the presence of a finite-time singularity of Type IV at that point [52, 68–70]. Alternatively, from Eq. (5.8), it can be shown that at the upper fixed point $H = M = finite$, $\dot{H} = 0 = finite$ but $\ddot{H}$ diverges. In this case, we call this fixed point a de Sitter of Type IV. In conclusion, the model replaces the initial big bang singularity with Type IV de Sitter singularity. As it is clear from Fig. 2(a), at the large Hubble rate regime, the Universe undergoes an accelerating expansion (unshaded region) at early-time, and also it exits into a FRW deceleration era (shaded region). The Hubble rate value at the graceful exit time instance, can be identified as the cutting point of the phase portrait with the zero acceleration curve, that is, when $\dot{H} = -H^2$. For $\beta = -2$, the phase portrait matches exactly the radiation phase portrait of standard Big Bang cosmology. This feature is important in order to ensure a successful thermal history. In addition, at the small Hubble rate regime, the phase portrait cuts the zero acceleration curve once more towards a future de Sitter fixed point H_{lo}, and

Analysis of the Phase Space

this fixed point is reached asymptotically. In effect the Universe is free from any future finite time singularities. At the second cutting point, the Hubble rate value can be chosen as $H_{tr} \sim 100 - 120$ km/s/Mpc, so at a redshift $z_{tr} \sim 0.6 - 0.8$, in order to be comparable with the ΛCDM model at late times.

In summary, our model can provide a unified cosmic history of the early and late-time acceleration eras. Also, the FRW decelerated phase is compatible with the standard cosmology. Moreover, the late-time acceleration, compatible with ΛCDM model, is realized without using a cosmological constant.

Analysis of the Phase Space

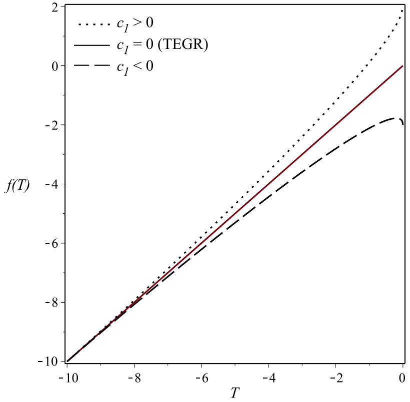


Figure 3: The evolution of the $f(T)$ gravity, Eq. (5.10). At large T, the theory reduces to the TEGR limit. However, at small T, the theory deviates from the TEGR limit. In this Fig. we take $M = 1$ and $M_p = 1$.

Analysis of the Phase Space

Now let us discuss another interesting solution of our dynamical system, and for $c_1 > 0$, the fixed points of (5.8) are $H_{fix} = \pm M$. The phase portraits of the $c_1 > 0$ case are given in Fig. 2(b). We consider the more physically appealing scenario $\beta < 0$, where the Universe is in the $\dot{H} < 0$ regime and trapped between two singularities at

$$H_{s\pm} = \pm \frac{\sqrt{36M^4 - c_1^2 M_p^2}}{6M}.$$

At these points, the Hubble parameter is finite but $\dot{H}$ diverges. Therefore, we conclude that the Universe has singularities of type II at these points. However, the geodesics are well behaved, whereas the first derivative of the scale factor is finite and subsequently the Christoffel symbols are regular. In this scenario, junction conditions in $f(T)$ gravity can be applied at $H_{s\pm}$ and a cyclic Universe can be obtained in principle [71, 72].
Now let us turn our focus on the $f(T)$ gravity that generates the cosmic evolution (5.1), so by substituting Eq. (3.13) in (5.7), we obtain,

$$f(T) = T + \sqrt{\frac{2}{3}}\frac{c_1 M_p}{M}\left[\sqrt{6M^2 + T} - \sqrt{-6T}\,\arccos\left(\frac{\sqrt{-6T}}{6M}\right)\right]. \tag{5.10}$$

Analysis of the Phase Space

It is clear that the GR limit is recovered by setting $c_1 = 0$. It is interesting to note that there is no β-dependence in the resulting $f(T)$ gravity. Also, as it can be seen from Fig. 3, the $f(T)$ theory reduces to the GR limit at large T and therefore, the deviation of the theory at hand from the Einstein-Hilbert case will actually occur at late times.

In this section, we shall fix the functional form of the $f(T)$ gravity and we shall investigate which constant-roll potentials does the $f(T)$ gravity generates. We consider the power-law $f(T)$ gravity of the form,

$$f(T) = T_0 \left(\frac{T}{T_0}\right)^n . \tag{6.1}$$

Substituting Eq. (6.1) in Eqs. (4.17) and (4.19), we obtain,

$$H(\phi) = H_0 \left[\frac{\beta(n-1)^2(\phi-\phi_0)^2}{2n^2 M_p^2(1-2n)}\right]^{\frac{1}{2(n-1)}} , \tag{6.2}$$

and also,

$$V(\phi) = V_0 + (2n-1)(3+\beta)H_0^2 M_p^2 \left[\frac{\beta(n-1)^2(\phi-\phi_0)^2}{2n^2 M_p^2(1-2n)}\right]^{\frac{n}{n-1}} . \tag{6.3}$$

It is clear that $V(\phi) = V_0 = constant$ and for simplicity, we take $V = 0$ at $\phi = \phi_0$, and hence $V_0 = 0$ and $\phi_0 = 0$. However, by substituting from (6.1) in the constant-roll differential equation (5.5), we get,

$$H(t) = -\frac{n}{\beta t - n t_i}, \quad a(t) = a_i(\beta t - n t_i)^{-\frac{n}{\beta}}, \tag{6.4}$$

where a_i and t_i are integration constants. Notably, from Eqs. (6.2) and (6.4) one can specify the explicit form of $\phi(t)$.
Substituting from Eq. (6.4) in Eq. (4.11), we obtain,

$$\epsilon_1 = -\frac{\beta}{n}, \quad \epsilon_{n>1} = 0. \tag{6.5}$$

Also, the speed of sound of the scalar perturbation in the context of $f(T)$ gravity is equal to[1],

$$c_s^2 = \frac{f_H}{H f_{HH}}\,, \tag{6.6}$$

and for the power-law theory of Eq. (6.1), it reads,

$$c_s = \frac{1}{\sqrt{2n-1}}. \tag{6.7}$$

Thus, the causality condition ($c_s \leq 1$) sets the constraint that $n < \frac{1}{2}$ or $n \geq 1$. However, if both the stability and the causality conditions are imposed ($0 \leq c_s \leq 1$), the parameter n is constrained as $n \geq 1$.

Similar to k-inflation models [75, 76], the running of the speed of sound should be introduced in the $f(T)$ perturbative analysis, as an additional slow-roll parameter [73, 74],

$$s_1 \equiv -\frac{d \ln c_s/dt}{H}, \quad s_{n+1} \equiv \frac{d \ln s_n/dt}{H}. \tag{6.8}$$

Using (6.7), it is easy to find that all the sound speed slow-roll parameters of the power law $f(T)$ gravity are null.

In the FRW cosmological background, small deviations from homogeneity $\delta\varphi(t, \vec{r})$ can be transformed to Fourier space, in which case each Fourier mode evolves in an independent way from the other modes, as it can be seen below,

$$\delta\varphi(t, \vec{k}) = \int d^3\vec{r} e^{-i\vec{k}.\vec{r}} \delta\varphi(t, \vec{r}),$$

where $\vec{r}$ and $k = |\vec{k}|$ are the comoving coordinates and the comoving wavenumber, respectively. Then, $1/k$ defines the comoving wavelength, and the physical mode wavelength is $\lambda(t) = a(t)/k$. At sub-horizon scale, the physical wavelength satisfies $\lambda \ll \lambda_H$, where $\lambda_H = H^{-1}$ is the Hubble radius, to which we refer to as “the horizon”. However, in $f(T)$ gravity, due to the contribution of the speed of sound of the scalar fluctuations, it is convenient to modify this condition to be $\lambda \ll \lambda_s$, where $\lambda_s = (c_s H)^{-1}$ is the sound horizon.

In observational cosmology, it is convenient to expand the power spectrum of scalar (tensor) perturbations as follows,

$$\mathcal{P}_s(k) = A_s \left(\frac{k}{k_*}\right)^{n_s - 1 + \frac{1}{2}\frac{dn_s}{d\ln k}\ln\left(\frac{k}{k_*}\right) + \frac{1}{6}\frac{d^2 n_s}{d\ln k^2}\left(\ln\left(\frac{k}{k_*}\right)\right)^2 + \cdots}, \tag{6.9}$$

$$\mathcal{P}_t(k) = A_t \left(\frac{k}{k_*}\right)^{n_t + \frac{1}{2}\frac{dn_t}{d\ln k}\ln\left(\frac{k}{k_*}\right) + \cdots}, \tag{6.10}$$

where A_s (A_t) are the scalar (tensor) amplitude and n_s (n_t), $\frac{dn_s}{d\ln k}$ $\left(\frac{dn_t}{d\ln k}\right)$ and $\frac{d^2 n_s}{d\ln k^2}$ are the scalar (tensor) spectral index of primordial curvature perturbations, the running of the scalar (tensor) spectral index, and the running of the running of the scalar spectral index, respectively.

We restrict ourselves to the lowest order in the slow-roll parameters, in which case the primordial power spectrum of the primordial curvature scalar perturbations is equal to, [45, 73, 74]

$$\mathcal{P}_s = \frac{1}{8\pi^2 M_p^2} \frac{H^2}{c_s^3 \epsilon_1}\bigg|_{\lambda=\lambda_s} = -\frac{1}{8\pi^2 M_p^2} \frac{(2n-1)^{3/2} n^3}{\beta(\beta t - n t_i)^2}\bigg|_{\lambda=\lambda_s}, \tag{6.11}$$

The sound horizon crossing can be expressed as $\lambda = \lambda_s$, which determines the time of the sound horizon exit,

$$t_s = \frac{n}{\beta k}\left(kt_i - \left[\frac{(2n-1)^2 a_i k}{n}\right]^{\frac{\beta}{n+\beta}}\right). \tag{6.12}$$

Inserting the above expression in Eq. (6.11), we can evaluate the scalar power spectrum in terms of the comoving wavenumber k, and the resulting expression is,

$$\mathcal{P}_s(k) = -\frac{1}{8\pi^2 M_p^2 \beta}\left((2n-1)^2 n\right)^{\frac{3n+\beta}{n+\beta}}\left(\frac{k}{a_i}\right)^{\frac{2\beta}{n+\beta}}. \tag{6.13}$$

By comparing with (6.9), we easily obtain the resulting expression for the spectral index of primordial curvature perturbations, which is,

$$n_s - 1 = \frac{2\beta}{n+\beta} \tag{6.14}$$

Remarkably, the above relation gives a modified version of the general relativistic power spectrum. We restrict ourselves by choosing $n_s = 0.96$ to fulfill the observational constraints from the joint analysis of Planck and BICEP2/Keck Array collaborations. Consequently, we get,

$$\beta \simeq -0.02n. \tag{6.15}$$

The tensor fluctuations power spectrum in $f(T)$ gravity, is given by the standard expression,

$$\mathcal{P}_t = \frac{2}{\pi^2 M_p^2} \left. \frac{a^2 H^2}{z_t^2} \right|_{\lambda=\lambda_H}, \tag{6.16}$$

where the parameter z_t is equal to [73, 74],

$$z_t = a \exp\left(\int \frac{\gamma}{2} dt \right), \qquad \gamma = \frac{\dot{T} f_{TT}}{f_T}. \tag{6.17}$$

All quantities in the right hand side of Eq. (6.16), should evaluated at the horizon crossing $\lambda = \lambda_H$, where $\lambda_H = H^{-1}$ is the Hubble horizon. In the general case, the freezing out moment of the scalar fluctuations is determined at the sound horizon crossing time instance, which is different from the freezing out of the tensor fluctuations which occur when the Hubble horizon crossing occurs. However, this difference is negligible if we restrict ourselves to the lowest order slow-roll parameters [76]. In Ref. [73], it has been proposed that if the following holds true,

$$\delta \equiv \frac{|\gamma|}{2H} \ll 1, \tag{6.18}$$

the tensor fluctuations power spectrum in the $f(T)$ gravity reduces to the standard inflationary model where $z_t \approx a$. In order to check the validity of this condition in the present model, one may rewrite the parameter δ as follows,

$$\delta = \frac{\epsilon_1}{2}|1 - c_s^{-2}|. \tag{6.19}$$

Using (6.5) and (6.7), δ becomes,

$$\delta = \left|\frac{\beta(n-1)}{n}\right| = 0.02|n-1|, \tag{6.20}$$

where the last quantity in the above equation is evaluated by using (6.15). It is clear that the parameter $\delta = 0$ in the TEGR limit ($n = 1$), while $\delta < O(1)$ when $-49 < n < 51$.

Therefore, we find that the condition $\delta \ll 1$ is valid in our case, and thus, the power spectrum of the tensor fluctuations is,

$$\mathcal{P}_t = \left.\frac{2H^2}{\pi^2 M_p^2}\right|_{\lambda=\lambda_H} = \left.\frac{2}{\pi^2 M_p^2}(\beta t - nt_i)^{-2} n^2\right|_{\lambda=\lambda_H}, \tag{6.21}$$

At the horizon exit $\lambda = \lambda_H$, we determine the time of the horizon exit,

$$t_t = \frac{1}{\beta}\left(nt_i + (-1)^{\frac{\beta}{n+\beta}}\left[\frac{a_i n}{k}\right]^{\frac{\beta}{n+\beta}}\right). \tag{6.22}$$

Inserting (6.22) in (6.21), we write the tensor power spectrum in terms of the comoving wavenumber k as follows,

$$\mathcal{P}_t(k) = \frac{2}{\pi^2 M_p^2}(-1)^{\frac{2\beta}{n+\beta}}\left(\frac{kn}{a_i}\right)^{\frac{2\beta}{n+\beta}}. \tag{6.23}$$

By comparing the above expression, with (6.10), we get the spectral index of the tensor power spectrum,

$$n_t = \frac{2\beta}{n+\beta}. \tag{6.24}$$

Using (6.15), we find that $n_t \simeq -0.04$. It is worth to mention that this value does not depend on the index n. Thus, the scale-dependance of the tensor fluctuations power spectrum (6.10) can be measured by using the spectral index,

$$n_t \equiv \frac{d \ln \mathcal{P}_t}{d \ln k} = -2\epsilon_1. \tag{6.25}$$

This observable is not measured accurately up to date, however, using (6.11) and (6.21), the scalar-to-tensor ratio in $f(T)$ is given by,

$$r \equiv \frac{\mathcal{P}_t}{\mathcal{P}_s} = 16 c_s^3 \epsilon_1 = -8 c_s^3 n_t. \tag{6.26}$$

Remarkably, there is no way to put an upper limit on the parameter ϵ_1 from the above relation, without constraining the speed of sound. However, it reduces to the standard consistency relation by setting $c_s = 1$. From (6.7) and (6.24), we find,

$$r = -\frac{16\beta}{(2n-1)^{3/2}(n+\beta)} = \frac{0.32}{(2n-1)^{3/2}}, \tag{6.27}$$

where the last quantity in the above equation is evaluated by using (6.15). It is clear that the TEGR limit produces a large tensor-to-scalar ratio $r = 0.32$. However, Eq. (6.27) shows that for $n \gtrsim 1.5$, the model fulfills the upper bound of the Planck data $r < 0.10$. Finally, we summarize some numerical values of the model parameters, for different choices of the parameter n in Table 2.

Table 2: Model parameters

n	β	c_s	n_s	δ	n_t	r
1.5	−0.029	0.707	0.96	0.0098	−0.04	0.11
2	−0.039	0.577	0.96	0.0196	−0.04	0.062
4	−0.078	0.378	0.96	0.0588	−0.04	0.017
6	−0.118	0.302	0.96	0.098	−0.04	0.008
Eq. No.	(6.15)	(6.7)	(6.14)	(6.20)	(6.24)	(6.27)

As it can be seen in Table 2, the compatibility with the Planck data occurs for a wide range of the free parameters of the model.

[1] We note that the torsion scalar dependence in Eq. (6.6), as is given in [45, 73, 74], has been replaced by the Hubble parameter which is more appropriate for our analysis.

Concluding Remarks

In this paper we investigated the implications of a constant-roll condition on $f(T)$ gravity inflation. We assumed that the theory is described by an inflaton minimally coupled to an $f(T)$ teleparallel gravity, and we examined in detail the implications of the constant-roll condition in the cosmological evolution. Our approach enabled us to introduce a reconstruction technique, in the context of which it is possible by fixing the Hubble evolution, to find both the constant-roll scalar potential and also the $f(T)$ gravity which may generate such evolution. Also, by fixing the $f(T)$ gravity, by employing the reconstruction technique we developed, we were able to find both the Hubble rate corresponding to it and also the scalar potential. Also we calculated the power spectrum of primordial scalar curvature perturbations and also the power spectrum of primordial tensor perturbations, and we investigated the implications of the constant-roll condition on the spectral index and the scalar-to-tensor ratio. As we showed, the resulting observational indices can be compatible with the observational data, and we examined the parameter space in order to find which values allow the compatibility with current observational data.

As a general conclusion, by taking into account the results of the present study but also of previous studies of $F(R)$ gravity, the constant-roll condition can provide an appealing theoretical framework, in the context of which a viable theory of inflation is obtained, which is compatible with the current observational data. What now remains, is to investigate the

Concluding Remarks

implications of the constant-roll scenario on Gauss-Bonnet $F(G)$ theories, and also other modified gravity theories such as mimetic gravity or $F(R, T)$ gravity.

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Thank You