Scale Invariant Resummed Thermal Perturbative Expansion

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Context: QCD phase diagram/ Quark Gluon Plasma

Complete QCD phase diagram far from being confirmed:

$T \neq 0, \mu = 0$ well-established from lattice: no sharp phase transition, continuous crossover at $T_c \simeq 154 \pm 9$ MeV (Aoki et al '06).

Goal: more analytical approximations, ultimately in regions not much accessible on the lattice: large density (chemical potential) due to the (in)famous "sign problem"

Introduction/Motivations

Context: (unconventional) resummation of perturbative expansions

Very general: relevant both at $T=0$ or $T\neq 0$ (and finite density)
$\rightarrow$ addresses well-known problems of unstable +badly scale-dependent thermal perturbative expansions:
Illustrate here $T\neq 0$ σ model, + (preliminary) QCD (pure glue)

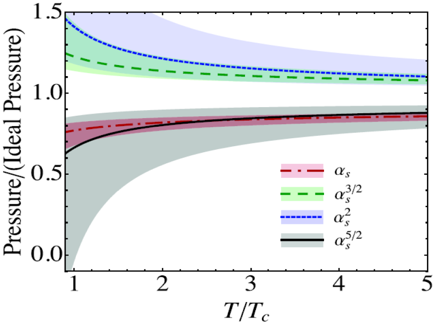
NB Previous results ($T=0$):
estimate with our RGOPT approach the order parameter $F_\pi(m_q=0)/\Lambda^{\rm QCD}_{\overline{MS}}$:
$F_\pi \simeq 92.2\text{MeV} \rightarrow F_\pi(m_q=0) \rightarrow \Lambda^{n_f=3}_{\overline{MS}} \rightarrow \alpha_S^{\overline{MS}}(\mu=m_Z)$.

N^3LO: $F_\pi^{m_q=0}/\Lambda^{n_f=3}_{\overline{MS}} \simeq 0.25 \pm .01 \rightarrow \alpha_S(m_Z) \simeq 0.1174 \pm .001 \pm .001$
(JLK, A.Neveu, PRD88 (2013))
(compares well with latest (2016) α_S lattice and world average values [PDG2016])

Also applied to $\langle \bar{q}q \rangle$ at N^3LO (using spectral density of Dirac operator):
$\langle \bar{q}q \rangle^{1/3}_{m_q=0}(2\,\mathrm{GeV}) \simeq -(0.84 \pm 0.01)\Lambda_{\overline{MS}}$ (JLK, A.Neveu, PRD 92 (2015))
(compares well with latest most precise lattice value.)

$T \neq 0$: perturbative Pressure (QCD or $\lambda\phi^4$)

Long-standing problem: poorly convergent and very scale-dependent (ordinary) perturbative expansion



QCD (pure glue) pressure at successive (standard) perturbation orders
shaded regions: scale-dependence for $\pi T < \mu < 4\pi T$
(illustration from Andersen, Strickland, Su '10)

(Variationally) Optimized Perturbation (OPT)

Trick ($T = 0$): add and subtract a mass, consider $m\,\delta$ as interaction:

$$\mathcal{L}_{QCD}(g, m) \to \mathcal{L}_{QCD}(\delta\, g, m(1-\delta)) \quad \text{(e.g. in QCD } g \equiv 4\pi\alpha_S\text{)}$$

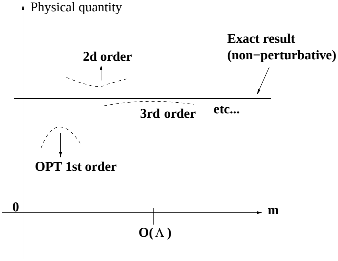
where $0 < \delta < 1$ interpolates between $\mathcal{L}_{free}$ and *massless* $\mathcal{L}_{int}$;
e.g. (quark) mass $m_q \to m$: arbitrary trial parameter

• Take any standard (renormalized) QCD pert. series, expand in δ *after:*
$m_q \to m\,(1-\delta)$; $\quad g \to \delta\, g$
then take $\delta \to 1$ (to recover original massless theory):

BUT a m-dependence remains at any finite δ^k-order:
fixed typically by stationarity prescription: optimization (OPT):
$\frac{\partial}{\partial m}$(physical quantity) $= 0$ for $m = \bar{m}_{opt}(\alpha_S) \neq 0$:
•$T = 0$: exhibits *dimensional transmutation*: $\bar{m}_{opt}(g) \sim \mu\, e^{-const./g}$

•At $T \neq 0$, same idea dubbed "screened perturbation" (SPT), or "hard thermal loop (HTLpt) resummation", etc.
But does this 'cheap trick' always work? and why?

Expected behaviour (Ideally...)



But not quite what happens... except in simple models:
•Convergence proof of this procedure for $D = 1$ $g\phi^4$ oscillator (cancels large pert. order factorial divergences!) Guida et al '95

particular case of 'order-dependent mapping' Seznec, Zinn-Justin '79

•But in QFT: multi-loop calculations (specially $T \neq 0$) (very) difficult beyond first order:
→ what about convergence? not much apparent in fact

•Main pb at higher order: OPT: $\partial_m(...) = 0$ has multi-solutions (some complex!), how to choose right one, if no nonperturbative "insight"??

RG compatible OPT ($\equiv$ RGOPT)

Our main additional ingredient to OPT (JLK, A. Neveu 2010):

Consider a physical quantity (i.e. perturbatively RG invariant)
(in present context, will be the pressure $P(m, g, T)$):

in addition to OPT Eq: $\frac{\partial}{\partial\, m} P^{(k)}(m, g, \delta = 1)|_{m \equiv \tilde{m}} \equiv 0$,
Require (δ-modified!) series at order δ^k to satisfy a standard (perturbative) Renormalization Group (RG) equation:

$$\mathrm{RG}\left(P^{(k)}(m, g, \delta = 1)\right) = 0$$

with standard RG operator ($g = 4\pi\alpha_S$ for QCD):

$$\mathrm{RG} \equiv \mu \frac{d}{d\,\mu} = \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} - \gamma_m(g)\, m \frac{\partial}{\partial m}$$

$\beta(g) \equiv -b_0 g^2 - b_1 g^3 + \cdots\,, \;\; \gamma_m(g) \equiv \gamma_0 g + \gamma_1 g^2 + \cdots$

$\rightarrow$ Additional nontrivial constraint (even if started from RG invariant standard perturbation)

RG compatible OPT (RGOPT)

$\rightarrow$ Combined with OPT, RG Eq. reduces to massless form:

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}\right]P^{(k)}(m,g,\delta=1)=0$$

Note: using OPT AND RG completely fix $m\equiv\bar{m}$ and $g\equiv\bar{g}$.

But $\Lambda_{\overline{\rm MS}}(g)$ satisfies by def.:
$[\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}]\,\Lambda_{\overline{\rm MS}}\equiv 0$ consistently at a given pert. order for $\beta(g)$.

Thus equivalent to:

$$\frac{\partial}{\partial\, m}\left(\frac{P^k(m,g,\delta=1)}{\Lambda_{\overline{\rm MS}}(g)}\right)=0\ ;\quad \frac{\partial}{\partial\, g}\left(\frac{P^k(m,g,\delta=1)}{\Lambda_{\overline{\rm MS}}(g)}\right)=0 \text{ for } \bar{m},\bar{g}$$

Optimal $\bar{m},\bar{g}=4\pi\bar{\alpha}_S$ unphysical: final (physical) result from $P(\bar{m},\bar{g},T)$

At $T=0$ reproduces at first order exact nonperturbative results in simpler models [e.g. Gross-Neveu model]

OPT + RG = RGOPT main new features

•Standard OPT: embarrassing freedom (a priori) in interpolating form:
e.g. why not $m \to m\,(1-\delta)^a$?
Most previous works: linear case $a = 1$ for simplicity
but generally (we have shown) $a = 1$ spoils RG invariance!

•OPT,RG Eqs: many solutions at increasing δ^k-orders

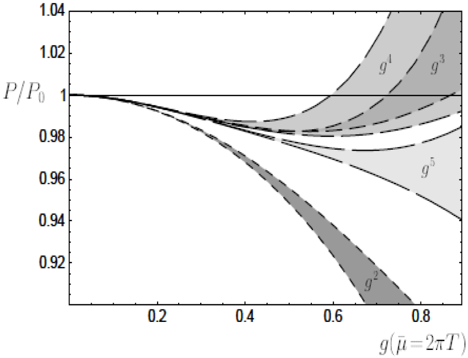
$\to$ Our approach restores RG, +requires OPT, RG sol. to match standard perturbation (e.g. Asymptotic Freedom for QCD ($T = 0$)):
$\alpha_S \to 0$, $\mu \to \infty$: $\bar{g} = 4\pi\bar{\alpha}_S \sim \frac{1}{2b_0 \ln \frac{\mu}{\bar{m}}} + \cdots$

$\to$ At arbitrary order, AF-compatible RG + OPT branch, often unique, *only* appear for a critical universal a:
$m \to m\,(1-\delta)^{\frac{\gamma_0}{b_0}}$ (e.g. $\frac{\gamma_0}{b_0}(\mathrm{QCD}, n_f = 3) = \frac{4}{9}$)

$\to$ Goes beyond simple "add and subtract" trick
$+$ It removes spurious solutions incompatible with AF
$-$ But does not always avoid complex solutions
(if those (perturbative artifacts) occur, are possibly cured by renormalization scheme change [JLK, Neveu '13])

Problems of thermal perturbation: $g\phi^4$ model

$g\phi^4$ pressure at successive pert. orders: poorly convergent and badly scale dependent (just like for QCD)



Problems of thermal perturbation (QCD and generic)

Main culprit: mix up of *hard* $p \sim T$ and *soft* $p \sim \alpha_S T$ modes.

Thermal 'Debye' screening mass $m_D^2 \sim \alpha_S T^2$ gives IR cutoff,
BUT $\Rightarrow$ perturbative expansion in $\sqrt{\alpha}_S$ in QCD
$\rightarrow$ advocated slower convergence

Yet many interesting QGP physics features happen at not that large $\alpha_S(\gtrsim 2\pi T_c) \simeq .5$ or lower values.

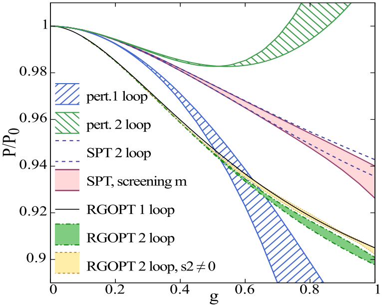
Many efforts to improve this (review e.g. Blaizot, Iancu, Rebhan '03):

Screened PT (SPT) (Karsh et al '97), $\sim$ Hard Thermal Loop (HTL) resummation (Andersen, Braaten, Strickland '99); Functional RG, 2-particle irreducible (2PI) formalism (Blaizot, Iancu, Rebhan '01; Berges, Borsanyi, Reinosa, J. Serreau '05)

RGOPT $T \neq 0$: essentially treats thermal mass 'RG consistently':
$\rightarrow$ it induces an anomalous dimension, treated RG-consistently.

(NB some qualitative connections with 2PI results, also with recent "massive scheme" approach (Blaizot, Wschebor '14)

Previous $T \neq 0$: two-loop RGOPT($g\phi^4$) vs standard PT and SPT



[JLK, M.B Pinto, PRL 116 (2016) [1507.03508]; PRD92 (2015)]

•Definite scale-dependence improvement (a factor $\sim$ 3) w.r.t. SPT [J.O. Andersen et al '01]

•Improvement should be more drastic at 3-loops, where SPT scale dependence strongly increases.

How this is obtained: details next for the nonlinear σ model

One step closer to QGP: $O(N)$ nonlinear σ model (NLSM)

[G. Ferreri, JLK, M.B. Pinto, R.0 Ramos, to appear on arXiv very soon]

(1+1)D NLSM shares many properties with QCD: asymptotic freedom, mass gap, $T \neq 0$ pressure, trace anomaly have QCD-similar shape
Other nonperturbative $T \neq 0$ results available for comparison
(lattice [Giacosa et al '12], $1/N$ expansion [Andersen et al '04], others)

$$\mathcal{L}_0 = \frac{1}{2}(\partial\pi_i)^2 + \frac{g(\pi_i\partial\pi_i)^2}{2(1-g\pi_i^2)} - \frac{m^2}{g}\left[\left(1-g\pi_i^2\right)^{1/2} - 1\right]$$

two-loop pressure from:

•Advantage w.r.t. QCD: exact T-dependence at 2-loops:

$$P_{\rm pert.2loop} = -\frac{(N-1)}{2}\left[I_0^{\rm r}(m,T) + \frac{(N-3)}{4} m^2 g I_1^{\rm r}(m,T)^2\right] + \mathcal{E}_0,$$

$$I_0(m,T) = T \sum\!\!\!\!\!\!\!\int_{n,\mathbf{p}} \ln\left[(2\pi n T)^2 + \mathbf{p}^2 + m^2\right] = \frac{1}{2\pi}\left(m^2(1-\ln\frac{m}{M}) + 4T^2 J_0(\frac{m}{T})\right)$$

$$J_0(x) = \int_0^\infty dz \ln\left(1 - e^{-\sqrt{z^2+x^2}}\right), \quad I_1(m,T) = \partial I_0(m,T)/\partial m^2$$

First crucial step: standard perturbative RG invariance

$\mathcal{E}_0$ in P_{2-loop}: *finite* (T-independent) vacuum energy contribution:

$\mathcal{E}_0(g,m) = -m^2\left(\frac{s_0}{g} + s_1 + s_2 g + \cdots\right)$ such that $M\frac{d}{d\,M}\mathcal{E}_0$ cancels the remnant M dependence:

$s_0 = \frac{(N-1)}{4\pi(b_0-2\gamma_0)} = 1\ ,\ \ s_1 = (b_1 - 2\gamma_1)\frac{s_0}{2\gamma_0} = 0$ (NB: accident of NLSM)

•Next step: $m^2 \to m^2(1-\delta)^a$; $g \to \delta g$;
expand in δ; then $\delta \to 1$:

•RG only consistent for $a = 2\gamma_0/b_0 = (N-3)/(N-2)/2$ for NLSM ($\neq 1$ as in SPT/HTLpt)

•Another practical bonus: non-trivial OPT mass gap $\tilde{m}(g,T)$ already at one-loop

•Aim: illustrate in NLSM the scale dependence (and other) improvements wrt former SPT$\sim$ HTLpt

One-loop RGOPT ($\mathcal{O}(\delta^0)$) for NLSM pressure

Exact (arbitrary T) OPT "thermal mass gap" $\bar{m}$ from $\partial_m P(m) = 0$:

$$\ln \frac{\bar{m}}{\mu} = -\frac{1}{b_0\, g(\mu)} - 2 J_1(\frac{\bar{m}}{T}), \qquad (b_0^{\rm nlsm} = \frac{N-2}{2\pi})$$

or more explicitly, for $T = 0$: $\bar{m} = \mu e^{-\frac{1}{b_0\, g(\mu)}} = \Lambda_{\overline{\bf MS}}^{1-{\rm loop}}$
and for $T \gg m$:

$$\frac{\bar{m}}{T} = \frac{\pi\, b_0\, g}{1 - b_0\, g\, L_T} \simeq \pi\, b_0\, g(\mu) + \mathcal{O}(g^2), \quad (L_T \equiv \ln \frac{\mu\, e^{\gamma_E}}{4\pi T})$$

$$P_{1L}^{\rm RGOPT} = -\frac{(N-1)}{\pi} T^2 \left[J_0(\bar{x}) + \frac{\bar{x}^2}{8}\, (1 + 4 J_1(\bar{x})\,) \right], \quad (\bar{x} \equiv \bar{m}/T)$$

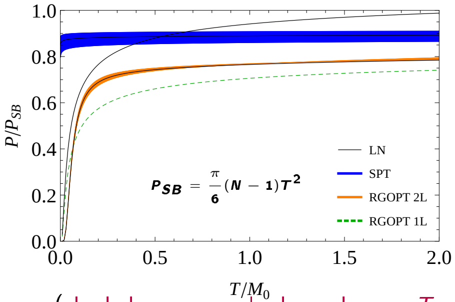
- Standard one-loop running: $g^{-1}(\mu) = g^{-1}(M_0) + b_0 \ln \frac{\mu}{M_0}$

$\Rightarrow \bar{m}, P(\bar{m})$ **are explicitly 'exactly' (one-loop) scale-invariant**

•+ It reproduces exact (all orders) known large N results
(Andersen et al '04)

RGOPT NLSM mass and pressure: two-loop order

$P/P_{SB}(N=4, g(M_0)=1)$ vs standard perturbation (PT), large N (LN), and SPT $\equiv$ ignoring RG-induced subtraction; $m^2 \to m^2(1-\delta)$:



(shaded range: scale-dependence $\pi T < \mu \equiv M < 4\pi T$)

$\to$ A moderate scale-dependence reappears, from unperfectly matched 2-loop $T=0$ standard running coupling.

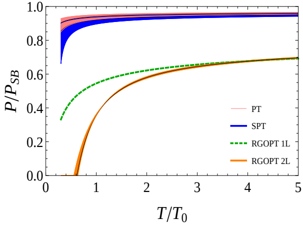
NB for 2-loop NLSM, alternative $\bar{g}(\mu)$ from combining RG+OPT accidentally gives $g=0$... (traced to 2-loop subtraction $s_1=0$)

(not expected in other models, and nontrivial NLSM $\bar{g}(\mu)$ appears at 3-loop)

Generically: RGOPT at $\mathcal{O}(g^k) \to \bar{m}(\mu)$ appears at $\mathcal{O}(g^{k+1})$ for *any* $\bar{m}$, but $\bar{m} \sim gT \to P \simeq \bar{m}^2/g + \cdots$ has leading μ-dependence at $\mathcal{O}(g^{k+2})$.

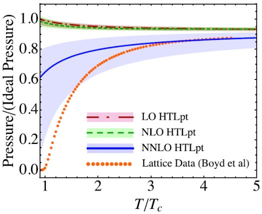
High T: pressure shape more comparable to QCD HTLpt

$$\frac{P_{PT}^{2-\text{loop}}}{P_{SB}} \simeq \frac{P_{SPT}^{2-\text{loop}}}{P_{SB}} = 1 - \frac{3}{2}\frac{(N-3)}{8\pi} g(\mu) + \mathcal{O}(g^2)$$



(NB: RGOPT 1,2L reach SB limit for $T \to \infty$ but more slowly than PT)

HTLpt (beyond 1-loop only $T \gg m$ approximation): QCD (pure glue)
[Andersen, Strickland, Su '10]:

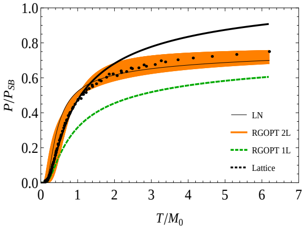


RGOPT(NLSM) lattice comparison

•NLSM $T \neq$ lattice simulations: (apparently) only available for $N = 3$ [E. Seel, D. Smith, S. Lottini, F. Giacosa '12]

•Remind: at 2-loop NLSM combined RG +OPT Eqs. gives no nontrivial $\bar{g}(\bar{m})$ by accident (traced to $s_1 = 0$), yet one remarkable value: $g(M_0) = 2\pi \Rightarrow \bar{m}(g) = M_0$ (NB similar feature in Gross-Neveu model)

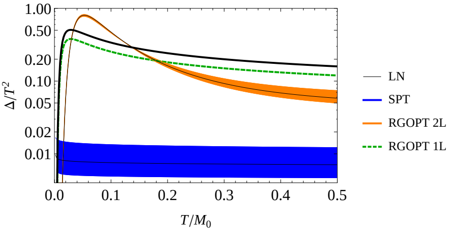
•Drawback: for such large coupling , 2-loop RGOPT remnant scale dependence becomes much more sizable.
(at 3-loop order $\bar{g}(m)$ would likely be more reasonable)



shaded regions: scale-dependence $\pi T < \mu = M < 4\pi T$
(Notice: $N = 3$ lattice pressure appears very close to large-N for low $T \lesssim M_0$)

NLSM interaction measure (trace anomaly)

NB $\Delta_{\text{2D NLSM}} \equiv \mathcal{E} - P = S\,T - 2P \equiv T^3 \partial_T(\frac{P}{T^2})$



$N = 4, g(M_0) = 1$ (shaded regions: scale-dependence $\pi T < \mu = M < 4\pi T$)

•2-loop SPT Δ small, monotonic behaviour + sizeable scale dependence.

•RGOPT shape 'qualitatively' comparable to QCD, showing a peak
(but no spontaneous sym breaking/phase transition in 2D NLSM
(Mermin-Wagner-Coleman theorem)

Thermal (pure glue) QCD: hard thermal loop formalism

QCD generalization of OPT = HTLpt [Andersen, Braaten, Strickland '99]: same "OPT" trick operates on a gluon "mass" term [Braaten-Pisarski '90]:

$$\mathcal{L}_{QCD}(\text{gauge}) - \frac{m_D^2}{2} Tr\left[G_{\mu\alpha} \langle \frac{y^\alpha y^\beta}{(y.D)^2} \rangle_y G^\mu_\beta \right], \quad D^\mu = \partial^\mu - ig\, A^\mu, \quad y^\mu = (1, \hat{\mathbf{y}})$$

(effective, gauge-invariant):

describes screening mass $m_D^2 \sim \alpha_S T^2$, but also many more 'hard thermal loop' contributions [modifies vertices and gluon propagators in highly nontrivial way]

Other gluon "mass prescriptions" exist [e.g. Reinosa et al '15] but HTLpt nice advantage: calculations up to 3-loop α_S^2 (NNLO) [Andersen et al '99-'15]: highly nontrivial, available analytically as m_D/T expansions, neglecting consistently higher orders [e.g. $m_D^4 \alpha_S = \mathcal{O}(\alpha_S^3)$].

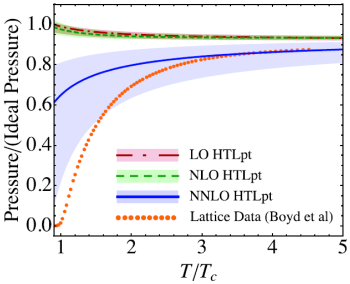
e.g. $P^{HTLpt}_{1\text{-loop},\overline{MS}} =$

$$P_{\text{ideal}} \left[1 - \frac{15}{2}\hat{m}_D^2 + 30\hat{m}_D^3 + \frac{45}{4}\hat{m}_D^4 (\ln \frac{\mu}{4\pi T} + \gamma_E - \frac{7}{2} + \frac{\pi^2}{3}) \right]$$

$\hat{m}_D \equiv \frac{m}{2\pi T}$, $P_{\text{ideal}} = (N_c^2 - 1)\pi^2 \frac{T^4}{45}$

standard HTLpt results: $m_D^2 \to m_D^2(1-\delta)$

(pure glue) [Andersen, Strickland, Su '10]



Reasonable agreement with lattice occurs at NNLO (3-loop), down to $T \sim 2-3\,T_c$, for low scale $\mu \sim \pi T - 2\pi T$.

Unfortunately, exhibits a scale dependence worsening at higher order (generic trend also when introducing quarks and chemical potential].

Moreover HTLpt (frequent) mass prescription $\tilde{m} \to m_D^{pert}(\alpha_S)$ [to avoid complex optimized solutions]: may miss more "nonperturbative" information.

RGOPT adaptation of HTLpt

Our main changes: RG-induced subtractions, + take $m_D^2(1-\delta)^{\frac{\gamma_0}{b_0}}$, where gluon 'mass' anomalous dimension defined (as it should) from its (known) counterterm.

RGOPT scale dependence should improve at higher orders from basically consistent RG invariance:
both from subtraction terms (prior to interpolation), and from above interpolation maintaining RG invariance.

Yet HTLpt scale dependence moderate at 2-loop:
$\rightarrow$ Because the (leading order) missing subtraction, of $\mathcal{O}(m_D^4 s_0/\alpha_S)$, acts formally like a (3-loop order) α_S^2 term:
$\rightarrow$ explains why scale dependence plainly resurfaces at 3-loops.

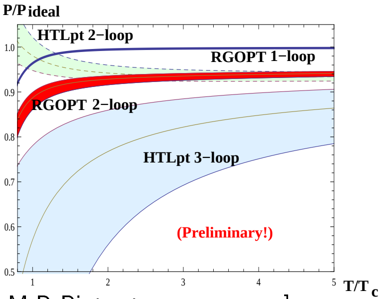
Preliminary RGO(HTL) (1- and 2-loop, pure glue) results

One-loop: obtain exactly scale-invariant pressure (like for ϕ^4 and NLSM):

$$\frac{P}{P_{ideal}}(G) = 1 - \frac{5}{4}G - 15G^2(2G+1) + \frac{10}{3}\sqrt{3}\,[G(1+3G)]^{3/2} + \cdots$$

where $G(b_0\alpha_S(\mu_0))$ a reference coupling;

2-loops: moderate RGOPT scale-dependence similar to ϕ^4, NLSM case:



[JLK, M.B Pinto, to appear soon]

Scale dependence improvement should be more drastic at 3-loops, but genuine low $T \sim T_c$ pressure shape needs determining higher order subtraction terms (requires more involved new calculations of 2-loop integrals) to get terms of $\mathcal{O}(m_D^4 \alpha_S \ln\mu)$, neglected in standard HTLpt since formally $\mathcal{O}(\alpha_S^3)$ [work in progress]

Summary and Outlook

•OPT gives a simple procedure to resum perturbative expansions, using only perturbative information.

•Our RGOPT version includes 2 major differences w.r.t. previous OPT/SPT/HTLpt... approaches:

1) OPT+ RG minimizations fix optimized $\tilde{m}$ *and possibly* $\tilde{g} = 4\pi\tilde{\alpha}_S$

2) Requiring AF-compatible solutions uniquely fixes the basic interpolation $m \to m(1-\delta)^{\gamma_0/b_0}$: discards spurious solutions and accelerates convergence.
($T = 0$: $\mathcal{O}(10\%)$ accuracy at 1-2-loops, empirical stability exhibited at 3-loop)

Applied to $T \neq 0$: exhibits improved stability + scale independence (with respect to standard PT, but also wrt SPT $\sim$ HTLpt)

•Paves the way to extend reliability of such methods to full QCD thermodynamics, (work in progress, start with $T \neq 0$ pure gluodynamics) specially for exploring also finite density