Thermodynamically consistent formulation of quasiparticle viscous hydrodynamics

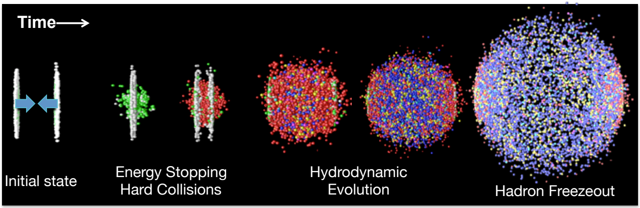
Radoslaw Ryblewski

THE HENRYK NIEWODNICZAŃSKI
INSTITUTE OF NUCLEAR PHYSICS
POLISH ACADEMY OF SCIENCES

details: L. Tinti, A. Jaiswal, R.R., Phys. Rev. **D95** 5, 054007 (2017)

Excited QCD
May 7–13, 2017, Sintra, Portugal

Hydrodynamic description of heavy-ion collisions



- if the **thermal equilibrium** is achieved locally in heavy-ion collisions the **fluid dynamics (*a.k.a.* hydrodynamics)** may be applied
- usually constructed as an **order-by-order expansion around equilibrium** state in powers of thermodynamic gradients
  - 0th order – ideal fluid dynamics
  - 1st order – Navier-Stokes theory (parabolic PDE, acausal, unstable)
  - 2nd order – Israel-Stewart theory (hyperbolic PDE, causal, stability not guaranteed)
- the universal existence of the viscous effects in Nature as well as the increasing precision of flow measurements at the LHC suggest **necessity of using relativistic viscous hydrodynamics (VH)**
- **multiple successes of the VH** description of the space-time evolution of the QGP
- **form of VH equations is not universal** – most often simple **relativistic kinetic theory (KT)** is used

Hydrodynamic modelling ingredients

- to model QGP evolution within VH one needs to incorporate its properties through the **transport coefficients (shear viscosity, bulk viscosity, etc.)** and **equation of state (EOS)**

$$
\begin{aligned}
\dot{\mathcal{E}} &= -\left(\mathcal{E}+\mathcal{P}\right)\theta - \Pi\,\theta + \pi : \sigma, \\
\nabla^\mu \mathcal{P} &= \left(\mathcal{E}+\mathcal{P}\right)\dot{u}^\mu + \Pi\,\dot{u}^\mu - \nabla^\mu \Pi + \Delta^\mu_\alpha \partial_\beta \pi^{\alpha\beta} \\
\dot{\Pi} &= -\frac{\Pi}{\tau_\Pi} - \beta_\Pi \theta - \delta_{\Pi\Pi}\Pi\theta + \lambda_{\Pi\pi}\pi : \sigma \\
\dot{\pi}^{\langle\mu\nu\rangle} &= -\frac{\pi^{\mu\nu}}{\tau_\pi} + \beta_\pi 2\sigma^{\mu\nu} + 2\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma} - \tau_{\pi\pi}\pi_\gamma^{\langle\mu}\sigma^{\nu\rangle\gamma} - \delta_{\pi\pi}\pi^{\mu\nu}\theta + \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu}
\end{aligned}
$$

- much of the research in the field is devoted to the extraction of these properties
- in principle they should be extracted from the experimental data
- usually one **incorporates some results of ab-initio calculations** done within **lattice QCD (lQCD)** framework

  10 PDE for $T, u^\mu, \Pi, \pi^{\mu\nu}$

  $\mathcal{E}(T) = ?$ $\quad\mathcal{P}(T) = ?$ precise lQCD EOS results

  $\beta_\Pi \tau_R = \zeta(T) = ?$ and $\beta_\pi \tau_R = \eta(T) = ?$ some lQCD results, **too large uncertainties !**

  <the rest of transport coefficients>$(T) = ?$ no idea

- in general it is a highly non-trivial task – multidimensional fit, numerically expensive calculations

way out?

Kinetic-theory-wise approach

- one typically resorts to a simple KT to derive the VH **equations of motion (EOM)**
- consider a **system of ideal (non-interacting) uncharged massive particles** of a single species

$$m = \text{const}$$

- the EOS of such a system depends parametrically only on the mass (m) of the particle
- impossible to reproduce exactly the temperature (T) dependence of energy density ($\mathcal{E}$) and pressure ($\mathcal{P}$) given by lQCD

example: Maxwell–Boltzmann distribution

$$f_{\rm eq} = g \exp[-\beta(u \cdot p)] \quad \Rightarrow \quad \mathcal{E}_0 = \frac{gT^4 z^2}{2\pi^2}\left[3K_2(z) + zK_1(z)\right] \qquad \mathcal{P}_0 = \frac{gT^4 z^2}{2\pi^2} K_2(z)$$

$\beta \equiv 1/T; \quad z \equiv m/T$
K_n – Bessel functions

Imposing arbitrary EOS through quasiparticle mass

- to describe an arbitrary EOS consider T-dependent mass
  V. Goloviznin and H. Satz, Z. Phys. C57, 671 (1993)

$$m = \text{const} \rightarrow m = m(T)$$

- physically sound when considering high-T QCD ($m(T) = g_s T$)
  E. Braaten and R. D. Pisarski, Nucl. Phys. B337, 569 (1990)
- in general these quasi-particles do not correspond to any real excitations of the underlying fundamental theory (QCD) – especially close to the crossover region!

Thermodynamic consistency violation

- introducing $m = m(T)$ **violates basic thermodynamic identities**
  M. I. Gorenstein and S.-N. Yang, Phys. Rev. D52, 5206 (1995)
- the thermodynamic relation must be satisfied

$$\mathcal{S}_0 \equiv \frac{d\mathcal{P}_0}{dT} = \frac{\mathcal{E}_0 + \mathcal{P}_0}{T}$$

- within KT one has

$$\mathcal{E}_0 = u \cdot T_{\rm eq} \cdot u$$

$$\mathcal{P}_0 = -\frac{1}{3}\Delta : T_{\rm eq}$$

where

$$T^{\mu\nu}_{\rm eq} = \int dP\, p^\mu p^\nu f_{\rm eq}$$

$p \cdot t \equiv p_\mu t^\nu$
$A : B \equiv A^{\mu\nu} B_{\mu\nu}$
$\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$
$\int dP = \int \frac{d^4p}{(2\pi)^4} 2\,\Theta(p\cdot t)\,(2\pi)\,\delta(p^2 - m^2)$

- example: Maxwell-Boltzmann distribution

$$f_{\rm eq} = g \exp[-\beta(u \cdot p)]$$

$$\mathcal{E}_0 = \frac{gT^4 z^2}{2\pi^2}\Big[3K_2(z) + zK_1(z)\Big]$$

$$\mathcal{P}_0 = \frac{gT^4 z^2}{2\pi^2} K_2(z)$$

- if $m = \text{const} \to m = m(T)$

$$\frac{d\mathcal{P}_0}{dT} = \frac{gT^3 z^2}{2\pi^2}\left[4K_2(z) + zK_1(z) - \frac{dm}{dT} K_1(z)\right]$$

$$= \frac{\mathcal{E}_0 + \mathcal{P}_0}{T} \underbrace{- m\frac{dm}{dT}\int dP f_{\rm eq}}_{\neq 0}$$

Restoring thermodynamic consistency

- restore thermodynamic consistency by introducing additional effective mean field through a **bag function** $B_0(T)$
  M. I. Gorenstein and S.-N. Yang, Phys. Rev. D52, 5206 (1995)

$$\mathcal{E}_0 \to \mathcal{E}_0 + B_0 \qquad \mathcal{P}_0 \to \mathcal{P}_0 - B_0$$

- the thermodynamic relation must be satisfied

$$\mathcal{S}_0 \equiv \frac{d\mathcal{P}_0}{dT} = \frac{\mathcal{E}_0 + \mathcal{P}_0}{T}$$

with

$$\mathcal{E}_0 = u \cdot T_{\rm eq} \cdot u, \quad \mathcal{P}_0 = -\frac{1}{3}\Delta : T_{\rm eq}$$

- $B_0(T)$ may be included in the Lorentz covariant way by modifying the definition of $T^{\mu\nu}_{\rm eq}$
  S. Jeon, Phys. Rev. D52, 3591 (1995)
  S. Jeon and L. G. Yaffe, Phys. Rev. D53, 5799 (1996)
  P. Chakraborty and J. I. Kapusta, Phys. Rev. C83, 014906 (2011)
  P. Romatschke, Phys. Rev. D85, 065012 (2012)
  M. Albright and J. I. Kapusta, Phys. Rev. C93, 014903 (2016)

$$T^{\mu\nu}_{\rm eq} = \int dP\, p^\mu p^\nu\, f_{\rm eq} + B_0(T)\, g^{\mu\nu}$$

- example: Maxwell-Boltzmann distribution

$$f_{\rm eq} = g \exp[-\beta(u \cdot p)]$$

$$\mathcal{E}_0 = \frac{gT^4z^2}{2\pi^2}\left[3K_2(z) + zK_1(z)\right] + B_0$$

$$\mathcal{P}_0 = \frac{gT^4z^2}{2\pi^2}K_2(z) - B_0$$

- if $m = \text{const} \to m = m(T)$

$$\frac{d\mathcal{P}_0}{dT} = \frac{gT^3z^2}{2\pi^2}\left[4K_2(z) + zK_1(z) - \frac{dm}{dT}K_1(z)\right] - \frac{dB_0}{dT}$$

$$= \frac{\mathcal{E}_0 + \mathcal{P}_0}{T} \underbrace{-\left(\frac{dB_0}{dT} + m\frac{dm}{dT}\int dP f_{\rm eq}\right)}_{=0!}$$

- $\partial_\mu T^{\mu\nu}_{\rm eq} = 0$ gives the same condition

Imposing lQCD EOS

- to solve EOM one has to impose EOS
- necessary to fix the T dependence of the thermodynamic quantities
- it is sufficient to define $m(T)$
  P. Romatschke, Phys. Rev. D85, 065012 (2012)
- consider finite-T lQCD EOS at $\mu_B = 0$ by the Wuppertal-Budapest collaboration
  S. Borsanyi et al., JHEP 11, 077 (2010)

- degeneracy factor g is obtained by reproducing correct Stefan–Boltzmann limit requires fixing

$$g = \frac{\pi^4}{180}\left(4(N_c^2 - 1) + 7N_c N_f\right)$$

$N_c = 3,\ N_f = 3$

- $m(T)$ is determined from equilibrium entropy density, $\mathcal{S}_0 = (\mathcal{E}_0 + \mathcal{P}_0)/T$ (independent of B_0) by numerically solving

$$\frac{g}{2\pi^2}\left(\frac{m(T)}{T}\right) K_3\left(\frac{m(T)}{T}\right) = \left.\frac{\mathcal{S}_0(T)}{T^3}\right|_{\mathrm{lQCD}}$$

- $B_0(T)$ is given through the relation expressing thermodynamic consistency

$$\frac{dB_0(T)}{dT} = -\frac{gT^3 z^2}{2\pi^2} K_1(z)\,\frac{dm}{dT}$$

Off-equilibrium mean field

- for the **non-equilibrium case** one can have in general
L. Tinti, A. Jaiswal, R.R., Phys.Rev. D95 (2017) no.5, 054007

$$T^{\mu\nu} = \int dP\, p^\mu p^\nu f \; + B^{\mu\nu}$$

- in equilibrium ($f \to f_{\rm eq}$) we require
$$B^{\mu\nu}|_{\rm eq} = B_0 g^{\mu\nu}$$
- out of equilibrium ($f \to f_{\rm eq} + \delta f$) we split
$$B^{\mu\nu} = B_0\, g^{\mu\nu} + \delta B^{\mu\nu}$$
- $\delta B^{\mu\nu}$ is fixed by requiring $\partial_\mu T^{\mu\nu} = 0$

$$\partial_\mu B^{\mu\nu} + m\partial^\nu m \int dPf$$
$$+ \int dP\, p^\nu \left[(p\cdot\partial)f + m\,(\partial^\rho m)\,\partial_\rho^{(p)}\, f\right] = 0$$

- the effective **Boltzmann equation (BE)** for $m = m(T)$ reads
W. Florkowski, J. Hufner, S. P. Klevansky, and L. Neise, Annals Phys. 245, 445 (1996)
P. Romatschke, Phys. Rev. D85, 065012 (2012)

$$(p\cdot\partial)f + m\,(\partial^\rho m)\,\partial_\rho^{(p)}\, f = \mathcal{C}[f]$$

- we assume the collision kernel in the **relaxation-time approximation (RTA)**
J. Anderson and H. Witting, Physica 74, 466 (1974)

$$\mathcal{C}[f] = -\frac{(u\cdot p)}{\tau_R}\delta f$$

Ansatz for off-equilibrium field

- energy and momentum conservation becomes

$$\partial_\mu \delta B^{\mu\nu} + m\,\partial^\nu m \int dP\,\delta f + \frac{1}{\tau_R} u_\mu \delta B^{\mu\nu} = 0$$

- if $\delta B^{\mu\nu} = 0$ it reduces to

$$-3\,m\,(\partial^\nu m)\,\Pi = 0$$

- for $m = m(T)$ the bulk pressure Π is non-vanishing which means that in general **we must have**

$$\delta B^{\mu\nu} \neq 0$$

- the symmetry of $T^{\mu\nu}$ restricts $\delta B^{\mu\nu}$ to have 10 independent components
- $\partial_\mu T^{\mu\nu} = 0$ leads to only 4 constraints
- we make an ***ansatz* for** $\delta B^{\mu\nu}$ of the form

$$\boxed{\delta B^{\mu\nu} = b_0\, g^{\mu\nu} + u^\mu b^\nu + b^\mu u^\nu}$$

$$u \cdot b = 0$$

- at first order in the gradient expansion one gets $b_0 = 0$ and $b^\mu = 0$
- up to second-order one has

$$\boxed{b_0 = -3\,\tau_R\,\kappa c_s^2\,\Pi\,\theta, \quad b^\mu = 3\,\tau_R\,\kappa\,\Pi\,\dot{u}^\mu}$$

Tensor decomposition of E-M tensor

- $T^{\mu\nu}$ may be tensor decomposed into

$$T^{\mu\nu} = \mathcal{E} u^\mu u^\nu - (\mathcal{P} + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}$$

- four-momentum conservation gives EOM for u^μ and T

$$\dot{\mathcal{E}} = -\,(\mathcal{E} + \mathcal{P})\,\theta - \Pi\,\theta + \pi : \sigma,$$

$$\nabla^\mu \mathcal{P} = (\mathcal{E} + \mathcal{P})\,\dot{u}^\mu + \Pi\,\dot{u}^\mu - \nabla^\mu \Pi + \Delta^\mu_\alpha \partial_\beta \pi^{\alpha\beta}$$

$$\sigma^{\mu\nu} \equiv \Delta^{\mu\nu}_{\alpha\beta} \nabla^\alpha u^\beta$$

$$\Delta^{\mu\nu}_{\alpha\beta} = \tfrac{1}{2}(\Delta^\mu_\alpha \Delta^\nu_\beta + \Delta^\mu_\beta \Delta^\nu_\alpha - \tfrac{2}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta})$$

- **bulk pressure** Π and **shear-stress tensor** $\pi^{\mu\nu}$ are defined as

$$\Pi \equiv -\frac{1}{3}\Delta : \left(T - T_{\rm eq}\right)$$

$$\pi^{\mu\nu} \equiv \Delta^{\mu\nu}_{\alpha\beta}\left(T^{\alpha\beta} - T^{\alpha\beta}_{\rm eq}\right) = \Delta^{\mu\nu}_{\alpha\beta}\, T^{\alpha\beta}$$

- in our case they read

$$\Pi = -\frac{1}{3}\Delta_{\alpha\beta}\int dP p^\alpha p^\beta \delta f - b_0$$

$$\pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta}\int dP p^\alpha p^\beta \delta f$$

- EOM of dissipative quantities are obtained by applying the co-moving derivative $\dot{(\ )} \equiv (u\cdot\partial)$

G. S. Denicol, T. Koide, and D. H. Rischke, Phys. Rev. Lett. 105, 162501 (2010)

$$\dot{\Pi} = -\frac{1}{3}\,(u\cdot\partial)\int dP\,(p\cdot\Delta\cdot p)\,\delta f - \dot{b}_0$$

$$\dot{\pi}^{\langle\mu\nu\rangle} = \Delta^{\mu\nu}_{\alpha\beta}\,(u\cdot\partial)\int dP\, p^{\langle\alpha} p^{\beta\rangle}\,\delta f$$

$$X^{\langle\mu\nu\rangle} \equiv \Delta^{\mu\nu}_{\alpha\beta} X^{\alpha\beta} \qquad \dot{X}^{\langle\mu\nu\rangle} \equiv \Delta^{\mu\nu}_{\alpha\beta} \dot{X}^{\alpha\beta}$$

Evolution equations for dissipative quantities

- explicit second-order EOM for the dissipative quantities are obtained using the **Chapman-Enskog-like iterative solution of the BE** for δf
  A. Jaiswal, Phys.Rev. C87 (2013) no.5, 051901
  A. Jaiswal, R. Ryblewski, and M. Strickland, Phys. Rev. C90, 044908 (2014)

$$\dot{\Pi} = -\frac{\Pi}{\tau_\Pi} - \beta_\Pi \theta - \delta_{\Pi\Pi}\Pi\theta + \lambda_{\Pi\pi}\pi : \sigma$$

$$\dot{\pi}^{\langle\mu\nu\rangle} = -\frac{\pi^{\mu\nu}}{\tau_\pi} + 2\beta_\pi \sigma^{\mu\nu} + 2\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma} - \tau_{\pi\pi}\pi_\gamma^{\langle\mu}\sigma^{\nu\rangle\gamma} - \delta_{\pi\pi}\pi^{\mu\nu}\theta + \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu}$$

- the bulk and shear viscosities are given by $\beta_\Pi \tau_R = \zeta$ and $\beta_\pi \tau_R = \eta$ with

$$\beta_\Pi = \frac{5}{3}\beta\, I_{3,2} - c_s^2\,(\mathcal{E}+\mathcal{P}) + \kappa c_s^2\, m^2 \beta\, I_{1,1}$$

$$\beta_\pi = \beta I_{3,2}$$

- $\kappa \equiv \frac{T}{m}\frac{dm}{dT}$
- in the limit $\kappa \to 0$ these match the constant mass results

$$\delta_{\Pi\Pi} = -\frac{5}{9}\chi - \left(1 - \kappa m^2 \frac{I_{1,1}}{I_{3,1}}\right) c_s^2 + \frac{1}{3}\frac{\beta\kappa c_s^2 m^2}{\beta_\Pi}\Big[\left(1-3c_s^2\right)\left(\beta I_{2,1} - I_{1,1}\right) - \left(1-3\kappa c_s^2\right) m^2 \left(\beta I_{0,1} + I_{-1,1}\right)\Big]$$

$$\lambda_{\Pi\pi} = \frac{\beta}{3\beta_\pi}\left(2I_{3,2} - 7I_{3,3}\right) - \left(1 - \kappa m^2 \frac{I_{1,1}}{I_{3,1}}\right) c_s^2$$

$$\tau_{\pi\pi} = 2 - \frac{4\beta}{\beta_\pi} I_{3,3}$$

$$\delta_{\pi\pi} = \frac{5}{3} - \frac{7}{3}\frac{\beta}{\beta_\pi} I_{3,3} - \frac{\beta}{\beta_\pi}\kappa c_s^2 m^2 \left(I_{1,2} - I_{1,1}\right)$$

$$\lambda_{\pi\Pi} = -\frac{2}{3}\chi$$

where

$$\chi = \frac{\beta}{\beta_\Pi}\Big[\left(1-3c_s^2\right)\left(I_{3,2} - I_{3,1}\right) - \left(1-3\kappa c_s^2\right) m^2 \left(I_{1,2} - I_{1,1}\right)\Big]$$

Longitudinal Bjorken flow solution

- consider transversely homogeneous and purely-longitudinal boost-invariant (Bjorken) expansion
  J. D. Bjorken, Phys. Rev. D27, 140 (1983)

$$u^\mu = \gamma(1,0,0,\tfrac{z}{t})$$

- all quantities are independent of ς and therefore unchanged when performing a Lorentz-boost – evolution only in τ!
- EOM reduce to

$$\dot{\mathcal{E}} = -\frac{1}{\tau}\left(\mathcal{E}+\mathcal{P}+\Pi-\pi_s\right)$$

$$\dot{\Pi} + \frac{\Pi}{\tau_\Pi} = -\frac{\beta_\Pi}{\tau} - \delta_{\Pi\Pi}\frac{\Pi}{\tau} + \lambda_{\Pi\pi}\frac{\pi_s}{\tau}$$

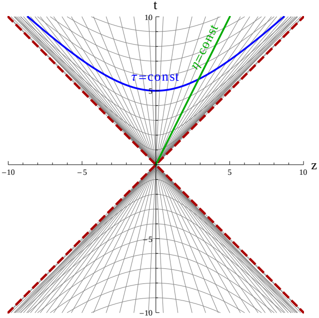
$$\dot{\pi}_s + \frac{\pi_s}{\tau_\pi} = \frac{4}{3}\frac{\beta_\pi}{\tau} - \left(\frac{1}{3}\tau_{\pi\pi} + \delta_{\pi\pi}\right)\frac{\pi_s}{\tau} + \frac{2}{3}\lambda_{\pi\Pi}\frac{\Pi}{\tau}$$

where $\pi_s \equiv -\tau^2\pi^{\varsigma\varsigma}$.

- the relaxation time is

$$\tau_\pi = \tau_\Pi = \tau_R = \frac{\bar{\eta}\mathcal{S}_0}{\beta_\pi}$$

- use Milne coordinates



$$x^\mu = (t,x,y,z)$$
$$\Downarrow$$
$$x^{\mu\prime} = (\tau,x,y,\varsigma)$$

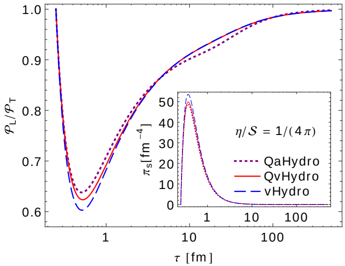
$$\begin{aligned} t &= \tau\cosh\varsigma \\ z &= \tau\sinh\varsigma \\ &\Downarrow \\ \tau &= \sqrt{t^2-z^2} \\ \varsigma &= \tanh^{-1}\left(\frac{z}{t}\right) \end{aligned}$$

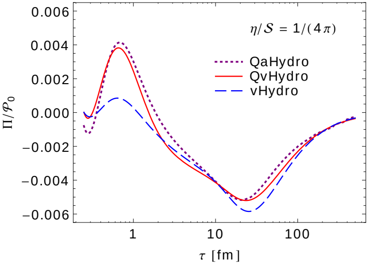
- in $x^{\mu\prime}$ the fluid becomes static, $u^{\mu\prime} = (1,0,0,0)$

Longitudinal Bjorken flow

- study the evolution of viscous QCD matter by numerically solving EOM supplemented with the lQCD EOS
- we choose LHC initial conditions $T(\tau_i) = 0.6$ GeV, $\pi_s(\tau_i) = 0$, $\Pi(\tau_i) = 0$ at $\tau_i = 0.25$ fm
- in addition $\tau_f = 500$ fm, $\eta/\mathcal{S} = 1/(4\pi)$
- we compare:
  - quasiparticle second-order VH (QvHydro, this work)
  - quasiparticle anisotropic hydrodynamics (QaHydro)
  - standard second-order VH (vHydro)

$$\mathcal{P}_L = \mathcal{P}_0 + \Pi - \pi_s$$
$$\mathcal{P}_T = \mathcal{P}_0 + \Pi + \pi_s/2$$





Summary

- we have presented a **first derivation of the second-order VH** for a system of quasiparticles of a single species **from an effective BE**
- we devised a **thermodynamically-consistent framework** to formulate second-order EOM for Π and $\pi^{\mu\nu}$ **for quasiparticles with T-dependent masses**
- the presented formulation is **capable of accommodating an arbitrary EOS** within the framework of KT
- **we studied the effect** of this new formulation **in the case of Bjorken expansion** of viscous QCD medium

Thank you for your attention!

Backup slides

Imposing lattice QCD equation of state - details

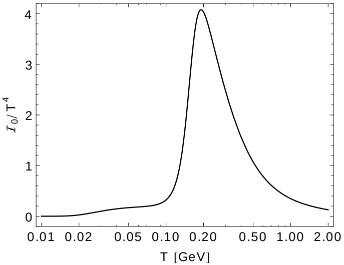
- For numerical convenience, we use analytic fit to the lQCD results for the interaction measure (trace anomaly)

$$\frac{\mathcal{I}_0(T)}{T^4} = \exp\left[-\left(\frac{h_1}{\hat{T}}+\frac{h_2}{\hat{T}^2}\right)\right] \times\left[\frac{h_0}{1+h_3\hat{T}^2}+\frac{f_0\left[\tanh(f_1\hat{T}+f_2)+1\right]}{1+g_1\hat{T}+g_2\hat{T}^2}\right]$$

with $\hat{T} \equiv T/(0.2\ \mathrm{GeV})$ and $h_0 = 0.1396$, $h_1 = -0.18$, $h_2 = 0.035$, $f_0 = 2.76$, $f_1 = 6.79$, $f_2 = -5.29$, $g_1 = -0.47$, $g_2 = 1.04$, and $h_3 = 0.01$

$$\frac{\mathcal{P}_0(T)}{T^4} = \int_0^T \frac{dT}{T}\frac{\mathcal{I}_0(T)}{T^4}$$

$$\frac{\mathcal{E}_0(T)}{T^4} = 3\frac{\mathcal{P}_0(T)}{T^4}+\frac{\mathcal{I}_0(T)}{T^4}$$



- $B_0(T)$ is given through the relation expressing thermodynamic consistency

$$\frac{dB_0(T)}{dT} = -\frac{gT^3z^2}{2\pi^2}K_1(z)\frac{dm}{dT}$$

The above equation can be solved numerically using the boundary condition $B_0 = 0$ at $T \simeq 0$.

Imposing lattice QCD equation of state in standard approach

- in the standard formulations of the second-order viscous fluid dynamics approaches the particle mass is treated as a constant parameter, $m =$ const.
- In this case there is however no unambiguous prescription how to impose the lattice QCD equation of state in the dynamical equations.
- The usual methodology, which we call here *standard*, is to extract the $m(T)$ dependence by matching squared speed of sound measured on the lattice using the following equation

$$c_s^2(T)\Big|_{\mathrm{lQCD}} = c_s^2\left(\frac{m(T)}{T}\right)\Bigg|_{\text{ideal gas (m=const.)}}$$

where on the right hand side we use the expression for c_s^2 valid in the case of ideal Boltzmann gas of particles with constant mass.

- Using $m(T)$ extracted in this way one may also express all the second-order transport coefficients and the relaxation times $\tau_R = \bar{\eta} I_{3,1}/(I_{3,2}T)$.
- This method is however not applicable to the energy density and pressure and the first-order terms, which require the knowledge of the degeneracy factor g. For that reason for these quantities we use directly values from the lattice QCD calculations.