Spin 1 low lying meson spectra and the subtle link to the spin 0 mesons

Jorge Morais

Center for Physics of the University of Coimbra

May 8th 2017

Based on:
Morais, J., Hiller, B. and Osipov, A.A.
Phys. Rev. D 95, 074033 (2017)

Excited QCD, Sintra

Index

Effective Spin 0 Model - Introduction

- **Effective Model of QCD** (low energy)

Effective Spin 0 Model - Introduction

- **Effective Model of QCD** (low energy)
  - Multiquark interactions based on **chiral symmetry** (χ-**S** for short).

Effective Spin 0 Model - Introduction

- **Effective Model of QCD** (low energy)
  - Multiquark interactions based on **chiral symmetry** (χ-**S** for short).
  - Scalar and pseudoscalar bilinears: $s_a = \bar{q}\lambda_a q$, $p_a = \bar{q}i\gamma_5\lambda_a\gamma_5 q$;
    $U(3)$-**valued field:** $\Sigma = \frac{1}{2}(s_a - ip_a)\lambda_a$.

Effective Spin 0 Model - Introduction

- **Effective Model of QCD** (low energy)
  - Multiquark interactions based on **chiral symmetry** (χ-**S** for short).
  - Scalar and pseudoscalar bilinears: $s_a = \bar{q}\lambda_a q$, $p_a = \bar{q}i\gamma_5\lambda_a\gamma_5 q$;
    $U(3)$-**valued field:** $\Sigma = \frac{1}{2}(s_a - ip_a)\lambda_a$.
  - **Natural expansion parameter:** Λ (explicit regularization parameter - the model is non-renormalizable).

Effective Spin 0 Model - Introduction

- **Effective Model of QCD** (low energy)
  - Multiquark interactions based on **chiral symmetry** (χ-**S** for short).
  - Scalar and pseudoscalar bilinears: $s_a = \bar{q}\lambda_a q$, $p_a = \bar{q}i\gamma_5\lambda_a\gamma_5 q$;
    $U(3)$-**valued field:** $\Sigma = \frac{1}{2}(s_a - ip_a)\lambda_a$.
  - **Natural expansion parameter:** Λ (explicit regularization parameter - the model is non-renormalizable).
- **Multiquark interactions:** 4q (NJL),6q ('t Hooft),8q (relevant at 4D)

$$\mathcal{L}_{int} = \frac{\bar{G}}{\Lambda^2}\text{tr}\left(\Sigma^\dagger\Sigma\right) + \frac{\bar{\kappa}}{\Lambda^5}\left(\det\Sigma + \det\Sigma^\dagger\right) + \frac{\bar{g}_1}{\Lambda^8}\left[\text{tr}\left(\Sigma^\dagger\Sigma\right)\right]^2 + \frac{\bar{g}_2}{\Lambda^8}\text{tr}\left(\Sigma^\dagger\Sigma\Sigma^\dagger\Sigma\right)$$

Effective Spin 0 Model - Introduction

- **Effective Model of QCD** (low energy)
  - Multiquark interactions based on **chiral symmetry** (χ-**S** for short).
  - Scalar and pseudoscalar bilinears: $s_a = \bar{q}\lambda_a q$, $p_a = \bar{q}i\gamma_5\lambda_a\gamma_5 q$; $U(3)$-**valued field:** $\Sigma = \frac{1}{2}(s_a - ip_a)\lambda_a$.
  - **Natural expansion parameter:** Λ (explicit regularization parameter - the model is non-renormalizable).
- **Multiquark interactions:** 4q (NJL),6q ('t Hooft),8q (relevant at 4D)

$$\mathcal{L}_{int} = \frac{\bar{G}}{\Lambda^2}\mathrm{tr}\left(\Sigma^\dagger\Sigma\right) + \frac{\bar{\kappa}}{\Lambda^5}\left(\det\Sigma + \det\Sigma^\dagger\right) + \frac{\bar{g}_1}{\Lambda^8}\left[\mathrm{tr}\left(\Sigma^\dagger\Sigma\right)\right]^2 + \frac{\bar{g}_2}{\Lambda^8}\mathrm{tr}\left(\Sigma^\dagger\Sigma\Sigma^\dagger\Sigma\right)$$

- Current quark masses $m \Longrightarrow$ **Explicit symmetry breaking (ESB)** $\Longrightarrow$ Introduce **external source** χ (assumed to transform as Σ) $\Longrightarrow$ Generalize effective terms by suitable replacements $\Sigma \to \chi$.

Effective Spin 0 Model - ESB Terms

- Possible **source-dependent** terms (by symmetry considerations):

Effective Spin 0 Model - ESB Terms

- Possible **source-dependent** terms (by symmetry considerations):

Effective Spin 0 Model - ESB Terms

- Possible **source-dependent** terms (by symmetry considerations):

$$\mathcal{L}_0 = -\text{tr}\left(\Sigma^\dagger\chi + \chi^\dagger\Sigma\right)$$

$$\mathcal{L}_1 = -\frac{\bar{\kappa}_1}{\Lambda}\epsilon_{ijk}\epsilon_{lmn}\Sigma_{il}\chi_{jm}\chi_{kn} + \text{h.c.}$$

$$\mathcal{L}_2 = \frac{\bar{\kappa}_2}{\Lambda^3}\epsilon_{ijk}\epsilon_{lmn}\Sigma_{il}\Sigma_{jm}\chi_{kn} + \text{h.c.}$$

$$\mathcal{L}_3 = \frac{\bar{g}_3}{\Lambda^6}\text{tr}\left(\Sigma^\dagger\Sigma\Sigma^\dagger\chi\right) + \text{h.c.}$$

$$\mathcal{L}_4 = \frac{\bar{g}_4}{\Lambda^6}\text{tr}\left(\Sigma^\dagger\Sigma\right)\text{tr}\left(\Sigma^\dagger\chi\right) + \text{h.c.}$$

$$\mathcal{L}_5 = \frac{\bar{g}_5}{\Lambda^4}\text{tr}\left(\Sigma^\dagger\chi\Sigma^\dagger\chi\right) + \text{h.c.}$$

Effective Spin 0 Model - ESB Terms

- Possible **source-dependent** terms (by symmetry considerations):

$$\mathcal{L}_0 = -\mathrm{tr}\left(\Sigma^\dagger\chi + \chi^\dagger\Sigma\right)$$

$$\mathcal{L}_1 = -\frac{\bar{\kappa}_1}{\Lambda}\epsilon_{ijk}\epsilon_{lmn}\Sigma_{il}\chi_{jm}\chi_{kn} + \mathrm{h.c.}$$

$$\mathcal{L}_2 = \frac{\bar{\kappa}_2}{\Lambda^3}\epsilon_{ijk}\epsilon_{lmn}\Sigma_{il}\Sigma_{jm}\chi_{kn} + \mathrm{h.c.}$$

$$\mathcal{L}_3 = \frac{\bar{g}_3}{\Lambda^6}\mathrm{tr}\left(\Sigma^\dagger\Sigma\Sigma^\dagger\chi\right) + \mathrm{h.c.}$$

$$\mathcal{L}_4 = \frac{\bar{g}_4}{\Lambda^6}\mathrm{tr}\left(\Sigma^\dagger\Sigma\right)\mathrm{tr}\left(\Sigma^\dagger\chi\right) + \mathrm{h.c.}$$

$$\mathcal{L}_5 = \frac{\bar{g}_5}{\Lambda^4}\mathrm{tr}\left(\Sigma^\dagger\chi\Sigma^\dagger\chi\right) + \mathrm{h.c.}$$

$$\mathcal{L}_6 = \frac{\bar{g}_6}{\Lambda^4}\mathrm{tr}\left(\Sigma^\dagger\Sigma\chi^\dagger\chi + \Sigma\Sigma^\dagger\chi\chi^\dagger\right) + \mathrm{h.c.}$$

$$\mathcal{L}_7 = \frac{\bar{g}_7}{\Lambda^4}\left[\mathrm{tr}\left(\Sigma^\dagger\chi\right) + \mathrm{h.c.}\right]^2$$

$$\mathcal{L}_8 = \frac{\bar{g}_8}{\Lambda^4}\left[\mathrm{tr}\left(\Sigma^\dagger\chi\right) - \mathrm{h.c.}\right]^2$$

$$\mathcal{L}_9 = -\frac{\bar{g}_9}{\Lambda^2}\mathrm{tr}\left(\Sigma^\dagger\chi\chi^\dagger\chi\right) + \mathrm{h.c.}$$

$$\mathcal{L}_{10} = -\frac{\bar{g}_{10}}{\Lambda^2}\mathrm{tr}\left(\Sigma^\dagger\chi\right)\mathrm{tr}\left(\chi^\dagger\chi\right) + \mathrm{h.c.}$$

Effective Spin 0 Model - ESB Terms

- Possible **source-dependent** terms (by symmetry considerations):

$$\mathcal{L}_0 = -\text{tr}\left(\Sigma^\dagger\chi + \chi^\dagger\Sigma\right)$$

$$\mathcal{L}_1 = -\frac{\bar{\kappa}_1}{\Lambda}\epsilon_{ijk}\epsilon_{lmn}\Sigma_{il}\chi_{jm}\chi_{kn} + \text{h.c.}$$

$$\mathcal{L}_2 = \frac{\bar{\kappa}_2}{\Lambda^3}\epsilon_{ijk}\epsilon_{lmn}\Sigma_{il}\Sigma_{jm}\chi_{kn} + \text{h.c.}$$

$$\mathcal{L}_3 = \frac{\bar{g}_3}{\Lambda^6}\text{tr}\left(\Sigma^\dagger\Sigma\Sigma^\dagger\chi\right) + \text{h.c.}$$

$$\mathcal{L}_4 = \frac{\bar{g}_4}{\Lambda^6}\text{tr}\left(\Sigma^\dagger\Sigma\right)\text{tr}\left(\Sigma^\dagger\chi\right) + \text{h.c.}$$

$$\mathcal{L}_5 = \frac{\bar{g}_5}{\Lambda^4}\text{tr}\left(\Sigma^\dagger\chi\Sigma^\dagger\chi\right) + \text{h.c.}$$

$$\mathcal{L}_6 = \frac{\bar{g}_6}{\Lambda^4}\text{tr}\left(\Sigma^\dagger\Sigma\chi^\dagger\chi + \Sigma\Sigma^\dagger\chi\chi^\dagger\right) + \text{h.c.}$$

$$\mathcal{L}_7 = \frac{\bar{g}_7}{\Lambda^4}\left[\text{tr}\left(\Sigma^\dagger\chi\right) + \text{h.c.}\right]^2$$

$$\mathcal{L}_8 = \frac{\bar{g}_8}{\Lambda^4}\left[\text{tr}\left(\Sigma^\dagger\chi\right) - \text{h.c.}\right]^2$$

$$\mathcal{L}_9 = -\frac{\bar{g}_9}{\Lambda^2}\text{tr}\left(\Sigma^\dagger\chi\chi^\dagger\chi\right) + \text{h.c.}$$

$$\mathcal{L}_{10} = -\frac{\bar{g}_{10}}{\Lambda^2}\text{tr}\left(\Sigma^\dagger\chi\right)\text{tr}\left(\chi^\dagger\chi\right) + \text{h.c.}$$

- Use $\chi \to m/2$ to make **Dirac mass term** $\mathcal{L}_0 = \bar{q}mq$.

Extension to Spin 1 - Motivation

- The spin 0 model:

Extension to Spin 1 - Motivation

- The spin 0 model:
  - is fitted to reproduce the **low-lying meson spectra** with great accuracy;

Extension to Spin 1 - Motivation

- The spin 0 model:
  - is fitted to reproduce the **low-lying meson spectra** with great accuracy;
  - gives good predictions for **meson strong decays** and **pseudoscalar radiative decays**.

Extension to Spin 1 - Motivation

- The spin 0 model:
  - is fitted to reproduce the **low-lying meson spectra** with great accuracy;
  - gives good predictions for **meson strong decays** and **pseudoscalar radiative decays**.
  - has been applied to study the QCD phase diagram and EoS.

Extension to Spin 1 - Motivation

- The spin 0 model:
  - is fitted to reproduce the **low-lying meson spectra** with great accuracy;
  - gives good predictions for **meson strong decays** and **pseudoscalar radiative decays**.
  - has been applied to study the QCD phase diagram and EoS.
- Motivations for including spin 1 states:

Extension to Spin 1 - Motivation

- The spin 0 model:
  - is fitted to reproduce the **low-lying meson spectra** with great accuracy;
  - gives good predictions for **meson strong decays** and **pseudoscalar radiative decays**.
  - has been applied to study the QCD phase diagram and EoS.
- Motivations for including spin 1 states:
  - Study decay processes of spin 1 mesons.

Extension to Spin 1 - Motivation

- The spin 0 model:
  - is fitted to reproduce the **low-lying meson spectra** with great accuracy;
  - gives good predictions for **meson strong decays** and **pseudoscalar radiative decays**.
  - has been applied to study the QCD phase diagram and EoS.
- Motivations for including spin 1 states:
  - Study decay processes of spin 1 mesons.
  - May provide insights regarding the VMD hypothesis.

Extension to Spin 1 - Motivation

- The spin 0 model:
  - is fitted to reproduce the **low-lying meson spectra** with great accuracy;
  - gives good predictions for **meson strong decays** and **pseudoscalar radiative decays**.
  - has been applied to study the QCD phase diagram and EoS.
- Motivations for including spin 1 states:
  - Study decay processes of spin 1 mesons.
  - May provide insights regarding the VMD hypothesis.
  - Vector mesons have important impact in the phase diagram and the EoS.

Extension to Spin 1 - Motivation

- The spin 0 model:
  - is fitted to reproduce the **low-lying meson spectra** with great accuracy;
  - gives good predictions for **meson strong decays** and **pseudoscalar radiative decays**.
  - has been applied to study the QCD phase diagram and EoS.
- Motivations for including spin 1 states:
  - Study decay processes of spin 1 mesons.
  - May provide insights regarding the VMD hypothesis.
  - Vector mesons have important impact in the phase diagram and the EoS.
- Spin 0 Version: [Osipov,Hiller,Blin 2013(1), (arXiv:1206.1920 [hep-ph])]; [Osipov,Hiller,Blin 2013(2), (arXiv:1309.2497 [hep-ph])]

Extension to Spin 1 - Motivation

- The spin 0 model:
  - is fitted to reproduce the **low-lying meson spectra** with great accuracy;
  - gives good predictions for **meson strong decays** and **pseudoscalar radiative decays**.
  - has been applied to study the QCD phase diagram and EoS.
- Motivations for including spin 1 states:
  - Study decay processes of spin 1 mesons.
  - May provide insights regarding the VMD hypothesis.
  - Vector mesons have important impact in the phase diagram and the EoS.
- Spin 0 Version: [Osipov,Hiller,Blin 2013(1), (arXiv:1206.1920 [hep-ph])]; [Osipov,Hiller,Blin 2013(2), (arXiv:1309.2497 [hep-ph])]
- Extension to Spin 1 Modes: [Morais,Hiller,Osipov 2017, (arXiv:1702.06894 [hep-ph])]

Extension to Spin 1 - Multiquark Terms (I)

- Vector and axial vector bilinears: $v_a^\mu = \bar{q}\lambda_a\gamma^\mu q$, $a_a^\mu = \bar{q}\lambda_a\gamma^\mu\gamma_5 q$; $U(3)$-**valued fields:** $R^\mu, L^\mu = \frac{1}{2}\left(v_a^\mu \pm a_a^\mu\right)\lambda_a$.

Extension to Spin 1 - Multiquark Terms (I)

- Vector and axial vector bilinears: $v_a^\mu = \bar{q}\lambda_a\gamma^\mu q$, $a_a^\mu = \bar{q}\lambda_a\gamma^\mu\gamma_5 q$; $U(3)$-**valued fields:** $R^\mu, L^\mu = \frac{1}{2}\left(v_a^\mu \pm a_a^\mu\right)\lambda_a$.
- **Source-independent** terms (vector):

$$\mathcal{L}'_{int} = \frac{\bar{w}_1}{\Lambda^2}\mathrm{tr}\left(R^\mu R_\mu + L^\mu L_\mu\right) + \frac{\bar{w}_2}{\Lambda^8}\left[\mathrm{tr}\left(R^\mu R_\mu + L^\mu L_\mu\right)\right]^2 + \frac{\bar{w}_3}{\Lambda^8}\left[\mathrm{tr}\left(R^\mu R_\mu - L^\mu L_\mu\right)\right]^2 + \frac{\bar{w}_4}{\Lambda^8}\mathrm{tr}\left(R^\mu R^\nu R_\mu R_\nu + L^\mu L^\nu L_\mu L_\nu\right) + \frac{\bar{w}_5}{\Lambda^8}\mathrm{tr}\left(R^\mu R_\mu R^\nu R_\nu + L^\mu L_\mu L^\nu L_\nu\right)$$

Extension to Spin 1 - Multiquark Terms (I)

- Vector and axial vector bilinears: $v_a^\mu = \bar{q}\lambda_a\gamma^\mu q$, $a_a^\mu = \bar{q}\lambda_a\gamma^\mu\gamma_5 q$; $U(3)$-**valued fields:** $R^\mu, L^\mu = \frac{1}{2}\left(v_a^\mu \pm a_a^\mu\right)\lambda_a$.
- **Source-independent** terms (vector):

$$\mathcal{L}'_{int} = \frac{\bar{w}_1}{\Lambda^2}\mathrm{tr}\left(R^\mu R_\mu + L^\mu L_\mu\right) + \frac{\bar{w}_2}{\Lambda^8}\left[\mathrm{tr}\left(R^\mu R_\mu + L^\mu L_\mu\right)\right]^2 + \frac{\bar{w}_3}{\Lambda^8}\left[\mathrm{tr}\left(R^\mu R_\mu - L^\mu L_\mu\right)\right]^2 + \frac{\bar{w}_4}{\Lambda^8}\mathrm{tr}\left(R^\mu R^\nu R_\mu R_\nu + L^\mu L^\nu L_\mu L_\nu\right) + \frac{\bar{w}_5}{\Lambda^8}\mathrm{tr}\left(R^\mu R_\mu R^\nu R_\nu + L^\mu L_\mu L^\nu L_\nu\right)$$

- **Source-independent** terms (scalar-vector):

$$\mathcal{L}''_{int} = \frac{\bar{w}_6}{\Lambda^8}\mathrm{tr}\left[R^\mu R_\mu + L^\mu L_\mu\right]\mathrm{tr}\left[\Sigma^\dagger\Sigma\right] + \frac{\bar{w}_7}{\Lambda^8}\mathrm{tr}\left[\Sigma^\dagger R^\mu \Sigma L_\mu\right] + \frac{\bar{w}_8}{\Lambda^8}\mathrm{tr}\left[\Sigma^\dagger\Sigma L^\mu L_\mu + \Sigma\Sigma^\dagger R^\mu R_\mu\right]$$

Extension to Spin 1 - Multiquark Terms (II)

- **Source-dependent** terms:

$$\mathcal{L}'_1 = \frac{\bar{w}_9}{\Lambda^6} \mathrm{tr}\left(R^\mu R_\mu + L^\mu L_\mu\right) \mathrm{tr}\left(\Sigma^\dagger \chi + \Sigma \chi^\dagger\right)$$

$$\mathcal{L}'_2 = \frac{\bar{w}_{10}}{\Lambda^6} \mathrm{tr}\left(\chi^\dagger R^\mu \Sigma L_\mu + \Sigma^\dagger R^\mu \chi L_\mu\right)$$

$$\mathcal{L}'_3 = \frac{\bar{w}_{11}}{\Lambda^6} \mathrm{tr}\left(\left(\Sigma^\dagger \chi + \chi^\dagger \Sigma\right) L^\mu L_\mu + \left(\Sigma \chi^\dagger + \chi \Sigma^\dagger\right) R^\mu R_\mu\right)$$

$$\mathcal{L}'_4 = \frac{\bar{w}_{12}}{\Lambda^4} \mathrm{tr}\left(\chi^\dagger R^\mu \chi L_\mu\right)$$

$$\mathcal{L}'_5 = \frac{\bar{w}_{13}}{\Lambda^4} \mathrm{tr}\left(\chi^\dagger \chi L^\mu L_\mu + \chi \chi^\dagger R^\mu R_\mu\right)$$

Extension to Spin 1 - Multiquark Terms (II)

- **Source-dependent** terms:

$$\mathcal{L}'_1 = \frac{\bar{w}_9}{\Lambda^6}\text{tr}\left(R^\mu R_\mu + L^\mu L_\mu\right)\text{tr}\left(\Sigma^\dagger\chi + \Sigma\chi^\dagger\right)$$

$$\mathcal{L}'_2 = \frac{\bar{w}_{10}}{\Lambda^6}\text{tr}\left(\chi^\dagger R^\mu \Sigma L_\mu + \Sigma^\dagger R^\mu \chi L_\mu\right)$$

$$\mathcal{L}'_3 = \frac{\bar{w}_{11}}{\Lambda^6}\text{tr}\left(\left(\Sigma^\dagger\chi + \chi^\dagger\Sigma\right) L^\mu L_\mu + \left(\Sigma\chi^\dagger + \chi\Sigma^\dagger\right) R^\mu R_\mu\right)$$

$$\mathcal{L}'_4 = \frac{\bar{w}_{12}}{\Lambda^4}\text{tr}\left(\chi^\dagger R^\mu \chi L_\mu\right)$$

$$\mathcal{L}'_5 = \frac{\bar{w}_{13}}{\Lambda^4}\text{tr}\left(\chi^\dagger\chi L^\mu L_\mu + \chi\chi^\dagger R^\mu R_\mu\right)$$

- **13 new parameters**, but not all will contribute to the vacuum properties of the model.

Extension to Spin 1 - Functional Bosonization

- Quark degrees of freedom are integrated out (in a functional integral formalism) in favour of mesonic ones.

Extension to Spin 1 - Functional Bosonization

- Quark degrees of freedom are integrated out (in a functional integral formalism) in favour of mesonic ones.
- We perform the shift $\sigma \longrightarrow \sigma + M$ to describe the system in the Nambu-Goldstone phase.

Extension to Spin 1 - Functional Bosonization

- Quark degrees of freedom are integrated out (in a functional integral formalism) in favour of mesonic ones.
- We perform the shift $\sigma \longrightarrow \sigma + M$ to describe the system in the Nambu-Goldstone phase.
- Multiquark interactions are rewritten in terms of auxiliary s_a, p_a, v_a^μ, a_a^μ.

Extension to Spin 1 - Functional Bosonization

- Quark degrees of freedom are integrated out (in a functional integral formalism) in favour of mesonic ones.
- We perform the shift $\sigma \longrightarrow \sigma + M$ to describe the system in the Nambu-Goldstone phase.
- Multiquark interactions are rewritten in terms of auxiliary s_a, p_a, v_a^μ, a_a^μ.
- This part of the functional integral is evaluated in a stationary phase approximation using the asymptotic expansions

$$
\begin{aligned}
s_a^{st} &= h_a + h_{ab}^{(1)}\sigma_b + h_{abc}^{(1)}\sigma_b\sigma_c + h_{abc}^{(2)}\phi_b\phi_c + H_{abc}^{\mu\nu\,(1)}V_{b\mu}V_{c\nu} \\
&\quad + H_{abc}^{\mu\nu\,(2)}A_{b\mu}A_{c\nu} + \dots \\
p_a^{st} &= h_{ab}^{(2)}\phi_b + h_{abc}^{(3)}\phi_b\sigma_c + H_{abc}^{\mu\nu\,(3)}V_{b\mu}A_{c\nu} + \dots \\
v_a^{\mu st} &= H_{ab}^{\mu\nu(1)}V_{b\nu} + H_{abc}^{\mu\nu\,(4)}\sigma_b V_{c\nu} + H_{abc}^{\mu\nu\,(5)}\phi_b A_{c\nu} + \dots \\
a_a^{\mu st} &= H_{ab}^{\mu\nu(2)}A_{b\nu} + H_{abc}^{\mu\nu\,(6)}\phi_b V_{c\nu} + H_{abc}^{\mu\nu\,(7)}\sigma_b A_{c\nu} + \dots
\end{aligned}
$$

Extension to Spin 1 - Functional Bosonization

- Quark degrees of freedom are integrated out (in a functional integral formalism) in favour of mesonic ones.
- We perform the shift $\sigma \longrightarrow \sigma + M$ to describe the system in the Nambu-Goldstone phase.
- Multiquark interactions are rewritten in terms of auxiliary s_a, p_a, v_a^μ, a_a^μ.
- This part of the functional integral is evaluated in a stationary phase approximation using the asymptotic expansions

$$
\begin{aligned}
s_a^{st} &= h_a + h_{ab}^{(1)}\sigma_b + h_{abc}^{(1)}\sigma_b\sigma_c + h_{abc}^{(2)}\phi_b\phi_c + H_{abc}^{\mu\nu\,(1)}V_{b\mu}V_{c\nu} \\
&\quad + H_{abc}^{\mu\nu\,(2)}A_{b\mu}A_{c\nu} + \dots \\
p_a^{st} &= h_{ab}^{(2)}\phi_b + h_{abc}^{(3)}\phi_b\sigma_c + H_{abc}^{\mu\nu\,(3)}V_{b\mu}A_{c\nu} + \dots \\
v_a^{\mu st} &= H_{ab}^{\mu\nu(1)}V_{b\nu} + H_{abc}^{\mu\nu\,(4)}\sigma_b V_{c\nu} + H_{abc}^{\mu\nu\,(5)}\phi_b A_{c\nu} + \dots \\
a_a^{\mu st} &= H_{ab}^{\mu\nu(2)}A_{b\nu} + H_{abc}^{\mu\nu\,(6)}\phi_b V_{c\nu} + H_{abc}^{\mu\nu\,(7)}\sigma_b A_{c\nu} + \dots
\end{aligned}
$$

- The remaining quark determinant is evaluated in a generalized heat kernel expansion approach.

Extension to Spin 1 - Mixing

- The **quadratic part of the bosonized Lagrangian** includes mixing terms of the form

$$\mathrm{tr}_F(i\,[V^\mu, M]\,\partial_\mu\sigma - \{A^\mu, M\}\,\partial_\mu\phi).$$

Extension to Spin 1 - Mixing

- The **quadratic part of the bosonized Lagrangian** includes mixing terms of the form

$$\mathrm{tr}_F(i\,[V^\mu, M]\,\partial_\mu\sigma - \{A^\mu, M\}\,\partial_\mu\phi).$$

- To deal with $V-\sigma$ and $A-\phi$ mixing we introduce the shifts:

$$V_{a\mu} \to V_{a\mu} + k_a X_{a\mu}, \quad A_{a\mu} \to A_{a\mu} + k'_a Y_{a\mu},$$
$$X_{a\mu} = 2f_{abc} M_b \partial_\mu \sigma_c, \quad Y_{a\mu} = 2d_{abc} M_b \partial_\mu \phi_c,$$

Extension to Spin 1 - Mixing

- The **quadratic part of the bosonized Lagrangian** includes mixing terms of the form

$$\mathrm{tr}_F(i\,[V^\mu, M]\,\partial_\mu\sigma - \{A^\mu, M\}\,\partial_\mu\phi).$$

- To deal with $V-\sigma$ and $A-\phi$ mixing we introduce the shifts:

$$V_{a\mu} \to V_{a\mu} + k_a X_{a\mu}, \quad A_{a\mu} \to A_{a\mu} + k'_a Y_{a\mu},$$
$$X_{a\mu} = 2f_{abc} M_b \partial_\mu \sigma_c, \quad Y_{a\mu} = 2d_{abc} M_b \partial_\mu \phi_c,$$

- Due to **8q interactions** and **ESB**, mass diagonalization conditions lead to different k_as and k'_as.

Extension to Spin 1 - Mixing

- The **quadratic part of the bosonized Lagrangian** includes mixing terms of the form

$$\mathrm{tr}_F(i\,[V^\mu, M]\,\partial_\mu\sigma - \{A^\mu, M\}\,\partial_\mu\phi).$$

- To deal with $V-\sigma$ and $A-\phi$ mixing we introduce the shifts:

$$V_{a\mu} \to V_{a\mu} + k_a X_{a\mu}, \quad A_{a\mu} \to A_{a\mu} + k'_a Y_{a\mu},$$
$$X_{a\mu} = 2f_{abc} M_b \partial_\mu \sigma_c, \quad Y_{a\mu} = 2d_{abc} M_b \partial_\mu \phi_c,$$

- Due to **8q interactions** and **ESB**, mass diagonalization conditions lead to different k_as and k'_as.
- The shifts provide new contributions to spin 0 kinetic terms $\Longrightarrow$ New (unsymmetric) field renormalizations: $(\xi_a^{\sigma,\phi} \equiv (M_i \mp M_j)^2 \,/ H_a^{(1,2)})$

$$\sigma_i \to \varrho\sigma_i \;,\; \sigma_a \to \varrho\sqrt{1+\frac{\xi_a^\sigma}{\varrho^2}}\sigma_a \;,\; \phi_i \to \varrho\sqrt{1+\frac{\xi_i^\phi}{\varrho^2}}\phi_i \;,\; \phi_a \to \varrho\sqrt{1+\frac{\xi_a^\phi}{\varrho^2}}\phi_a$$

Extension to Spin 1 - Vector Masses

- Spin 1 fields are renormalized as previously

$$V_a^\mu \to \sqrt{\frac{3}{2}}\varrho V_a^\mu \quad , \quad A_a^\mu \to \sqrt{\frac{3}{2}}\varrho A_a^\mu$$

Extension to Spin 1 - Vector Masses

- Spin 1 fields are renormalized as previously

$$V_a^\mu \to \sqrt{\frac{3}{2}}\varrho V_a^\mu \quad , \quad A_a^\mu \to \sqrt{\frac{3}{2}}\varrho A_a^\mu$$

- **Vector** and **axial vector** meson masses:

$$M_\rho^2 = M_\omega^2 = \frac{3}{2}\varrho^2 H_{11}^{(1)} \quad , \quad M_{a_1}^2 = M_{f_1}^2 = \frac{3}{2}\varrho^2 H_{11}^{(2)} + 6M_u^2$$

$$M_{K^*}^2 = \frac{3}{2}\left[\varrho^2 H_{44}^{(1)} + (M_u - M_s)^2\right] \quad , \quad M_{K_1}^2 = \frac{3}{2}\left[\varrho^2 H_{44}^{(2)} + (M_u + M_s)^2\right]$$

$$M_\varphi^2 = 3\varrho^2 H_{ss}^{(1)} \quad , \quad M_{f_1'}^2 = 3\varrho^2 H_{ss}^{(2)} + 6M_s^2$$

Extension to Spin 1 - Vector Masses

- Spin 1 fields are renormalized as previously

$$V_a^\mu \to \sqrt{\frac{3}{2}}\varrho V_a^\mu \quad , \quad A_a^\mu \to \sqrt{\frac{3}{2}}\varrho A_a^\mu$$

- **Vector** and **axial vector** meson masses:

$$M_\rho^2 = M_\omega^2 = \frac{3}{2}\varrho^2 H_{11}^{(1)} \quad , \quad M_{a_1}^2 = M_{f_1}^2 = \frac{3}{2}\varrho^2 H_{11}^{(2)} + 6M_u^2$$

$$M_{K^*}^2 = \frac{3}{2}\left[\varrho^2 H_{44}^{(1)} + (M_u - M_s)^2\right] \quad , \quad M_{K_1}^2 = \frac{3}{2}\left[\varrho^2 H_{44}^{(2)} + (M_u + M_s)^2\right]$$

$$M_\varphi^2 = 3\varrho^2 H_{ss}^{(1)} \quad , \quad M_{f_1'}^2 = 3\varrho^2 H_{ss}^{(2)} + 6M_s^2$$

- In the **isospin approximation** ($m_u = m_d \neq m_s$) we still need to deal with $\sigma_0 - \sigma_8$ and $\phi_0 - \phi_8$ (or $\sigma_{ns} - \sigma_s$ and $\phi_{ns} - \phi_s$) mixing $\Rightarrow$ Introduce **mixing angles** θ_σ and θ_ϕ (or ψ_σ and ψ_ϕ).

Extension to Spin 1 - Meson Masses (II)

- **Scalar** meson masses:

$$M_\rho^2 = M_\omega^2 = \frac{3}{2}\varrho^2 H_{11}^{(1)}$$

$$M_{K^*}^2 = \frac{3}{2}\left[\varrho^2 H_{44}^{(1)} + (M_u - M_s)^2\right]$$

$$M_\varphi^2 = 3\varrho^2 H_{ss}^{(1)}$$

$$M_{a_0}^2 = \frac{2}{3} M_\rho^2 \frac{1}{H_{11}^{(1)}} \left(\frac{h_u}{M_u} - h_{11}^{(1)}\right) + 4M_u^2$$

$$M_\kappa^2 = \frac{2}{3} M_{K^*}^2 \frac{1}{H_{44}^{(1)}} \left(\frac{h_u - h_s}{M_u - M_s} - h_{44}^{(1)}\right)$$

$$M_{f_0}^2 = \frac{1}{1 - \tan^2\psi_\sigma}\left[\frac{M_\omega^2}{3H_{uu}^{(1)}}\left(\frac{h_u}{M_u} - 2h_{uu}^{(1)} - 2h_{ud}^{(1)}\right) + 4M_u^2\right] + \frac{1}{1 - \cot^2\psi_\sigma}\left[\frac{M_\varphi^2}{3H_{ss}^{(1)}}\left(\frac{h_s}{M_s} - 2h_{ss}^{(1)}\right) + 4M_s^2\right]$$

$$M_\sigma^2 = \frac{1}{1 - \cot^2\psi_\sigma}\left[\frac{M_\omega^2}{3H_{uu}^{(1)}}\left(\frac{h_u}{M_u} - 2h_{uu}^{(1)} - 2h_{ud}^{(1)}\right) + 4M_u^2\right] + \frac{1}{1 - \tan^2\psi_\sigma}\left[\frac{M_\varphi^2}{3H_{ss}^{(1)}}\left(\frac{h_s}{M_s} - 2h_{ss}^{(1)}\right) + 4M_s^2\right]$$

Extension to Spin 1 - Meson Masses (II)

- **Pseudoscalar** meson masses:

$$M_{a_1}^2 = M_{f_1}^2 = \frac{3}{2}\varrho^2 H_{11}^{(2)} + 6M_u^2$$

$$M_{K_1}^2 = \frac{3}{2}\left[\varrho^2 H_{44}^{(2)} + (M_u + M_s)^2\right]$$

$$M_{f_1'}^2 = 3\varrho^2 H_{ss}^{(2)} + 6M_s^2$$

$$M_\pi^2 = \frac{2}{3} M_{a_1}^2 \frac{1}{H_{11}^{(2)}} \left(\frac{h_u}{M_u} - h_{11}^{(2)} \right)$$

$$M_K^2 = \frac{2}{3} M_{K_1}^2 \frac{1}{H_{44}^{(2)}} \left(\frac{h_u + h_s}{M_u + M_s} - h_{44}^{(2)} \right)$$

$$M_\eta^2 = \frac{1}{1 - \tan^2\psi_\phi} \frac{M_{f_1}^2}{3H_{uu}^{(2)}} \left(\frac{h_u}{M_u} - 2h_{uu}^{(2)} - 2h_{ud}^{(2)} \right) + \frac{1}{1 - \cot^2\psi_\phi} \frac{M_{f_1'}^2}{3H_{ss}^{(2)}} \left(\frac{h_s}{M_s} - 2h_{ss}^{(2)} \right)$$

$$M_{\eta'}^2 = \frac{1}{1 - \cot^2\psi_\phi} \frac{M_{f_1}^2}{3H_{uu}^{(2)}} \left(\frac{h_u}{M_u} - 2h_{uu}^{(2)} - 2h_{ud}^{(2)} \right) + \frac{1}{1 - \tan^2\psi_\phi} \frac{M_{f_1'}^2}{3H_{ss}^{(2)}} \left(\frac{h_s}{M_s} - 2h_{ss}^{(2)} \right)$$

Extension to Spin 1 - Weak Decay Constants

- Using the PCAC hypothesis, we may compute the π and K weak decay constants:

$$f_\pi = \frac{M_u}{\sqrt{\varrho^2 + \frac{4M_u^2}{H_{11}^{(2)}}}} \equiv \frac{M_u}{g_\pi}$$

$$f_K = \frac{M_u + M_s}{2\sqrt{\varrho^2 + \frac{(M_u+M_s)^2}{H_{44}^{(2)}}}} \equiv \frac{M_u + M_s}{2g_K}.$$

Extension to Spin 1 - Weak Decay Constants

- Using the PCAC hypothesis, we may compute the π and K weak decay constants:

$$f_\pi = \frac{M_u}{\sqrt{\varrho^2 + \frac{4M_u^2}{H_{11}^{(2)}}}} \equiv \frac{M_u}{g_\pi}$$

$$f_K = \frac{M_u + M_s}{2\sqrt{\varrho^2 + \frac{(M_u+M_s)^2}{H_{44}^{(2)}}}} \equiv \frac{M_u + M_s}{2g_K}.$$

- These express the well-known **Goldberger-Treiman relations**.

Model Fitting - Analytic Results (I)

- Parameters w_7, w_{10} and w_{12} appear in coefficients $H^{(1,2)}$ in the same form, but with opposite signs.

Model Fitting - Analytic Results (I)

- Parameters w_7, w_{10} and w_{12} appear in coefficients $H^{(1,2)}$ in the same form, but with opposite signs.
- This leads to the relations:

$$w_7 h_u^2 + 2w_{10} m_u h_u + w_{12} m_u^2 = 3\varrho^2 \left(\frac{1}{M_{a_1}^2 - 6M_u^2} - \frac{1}{M_\rho^2} \right)$$

$$w_7 h_s^2 + 2w_{10} m_s h_s + w_{12} m_s^2 = 3\varrho^2 \left(\frac{1}{M_{f_1}^2 - 6M_s^2} - \frac{1}{M_\varphi^2} \right)$$

$$w_7 h_u h_s + w_{10} \left(m_u h_s + m_s h_u \right) + w_{12} m_u m_s$$
$$= 3\varrho^2 \left[\frac{1}{M_{K_1}^2 - \frac{3}{2}\left(M_u + M_s\right)^2} - \frac{1}{M_{K^*}^2 - \frac{3}{2}\left(M_u - M_s\right)^2} \right]$$

Model Fitting - Analytic Results (I)

- Parameters w_7, w_{10} and w_{12} appear in coefficients $H^{(1,2)}$ in the same form, but with opposite signs.
- This leads to the relations:

$$w_7 h_u^2 + 2w_{10} m_u h_u + w_{12} m_u^2 = 3\varrho^2 \left(\frac{1}{M_{a_1}^2 - 6M_u^2} - \frac{1}{M_\rho^2} \right)$$

$$w_7 h_s^2 + 2w_{10} m_s h_s + w_{12} m_s^2 = 3\varrho^2 \left(\frac{1}{M_{f_1}^2 - 6M_s^2} - \frac{1}{M_\varphi^2} \right)$$

$$w_7 h_u h_s + w_{10} \left(m_u h_s + m_s h_u \right) + w_{12} m_u m_s$$
$$= 3\varrho^2 \left[\frac{1}{M_{K_1}^2 - \frac{3}{2}\left(M_u + M_s\right)^2} - \frac{1}{M_{K^*}^2 - \frac{3}{2}\left(M_u - M_s\right)^2} \right]$$

- These three parameters are **tightly constrained** by the spin 1 spectra $\Rightarrow$ Crucial for vector - axial vector mass differences.

Model Fitting - Analytic Results (II)

- The model predicts a relation between M_u and M_s depending solely on the spin-1 meson masses:

$$\frac{2}{M_{K^*}^2 - \frac{3}{2}\left(M_u - M_s\right)^2} + \frac{2}{M_{K_1}^2 - \frac{3}{2}\left(M_u + M_s\right)^2} =$$
$$\frac{1}{M_\rho^2} + \frac{1}{M_\varphi^2} + \frac{1}{M_{a_1}^2 - 6M_u^2} + \frac{1}{M_{f_1}^2 - 6M_s^2}.$$

Model Fitting - Analytic Results (II)

- The model predicts a relation between M_u and M_s depending solely on the spin-1 meson masses:

$$\frac{2}{M_{K^*}^2 - \frac{3}{2}\left(M_u - M_s\right)^2} + \frac{2}{M_{K_1}^2 - \frac{3}{2}\left(M_u + M_s\right)^2} = \frac{1}{M_\rho^2} + \frac{1}{M_\varphi^2} + \frac{1}{M_{a_1}^2 - 6M_u^2} + \frac{1}{M_{f_1}^2 - 6M_s^2}.$$

- We may also write

$$M_{a_1}^2 = \frac{6M_u^4}{M_u^2 - \varrho^2 f_\pi^2}, \qquad M_{K_1}^2 = \frac{\frac{3}{2}\left(M_u + M_s\right)^4}{\left(M_u + M_s\right)^2 - 4\varrho^2 f_K^2},$$

Model Fitting - Analytic Results (II)

- The model predicts a relation between M_u and M_s depending solely on the spin-1 meson masses:

$$\frac{2}{M_{K^*}^2 - \frac{3}{2}(M_u - M_s)^2} + \frac{2}{M_{K_1}^2 - \frac{3}{2}(M_u + M_s)^2} = \frac{1}{M_\rho^2} + \frac{1}{M_\varphi^2} + \frac{1}{M_{a_1}^2 - 6M_u^2} + \frac{1}{M_{f_1}^2 - 6M_s^2}.$$

- We may also write

$$M_{a_1}^2 = \frac{6M_u^4}{M_u^2 - \varrho^2 f_\pi^2}, \qquad M_{K_1}^2 = \frac{\frac{3}{2}(M_u + M_s)^4}{(M_u + M_s)^2 - 4\varrho^2 f_K^2},$$

- These relations completely determine M_u, M_s and Λ.

Model Fitting - Results

- **Empirical Input:**

M_π	M_K	M_η	$M_{\eta'}$	M_σ	M_κ	M_{a_0}	M_{f_0}	M_ρ	M_{K^*}
138	496	548	958	500	850	980	980	778	893

M_φ	M_{a_1}	M_{K_1}	M_{f_1}	m_u	m_s	f_π	f_K	θ_ϕ
1019	1270	1274	1426	4	100	92	111	-15°

Model Fitting - Results

- **Empirical Input:**

M_π	M_K	M_η	$M_{\eta'}$	M_σ	M_κ	M_{a_0}	M_{f_0}	M_ρ	M_{K^*}
138	496	548	958	500	850	980	980	778	893

M_φ	M_{a_1}	M_{K_1}	M_{f_1}	m_u	m_s	f_π	f_K	θ_ϕ
1019	1270	1274	1426	4	100	92	111	-15°

- **Results of the Fit:** (boldface = externally fixed)

G	κ	g_1	g_2	κ_2	g_3	g_4	g_5	g_6
2.54	-2.66	15.3	-35.2	0.143	-148	36.1	-21.9	-115

g_7	g_8	θ_σ	Λ	M_u	M_s	$\mathbf{w_1}$	$\mathbf{w_6}$	w_7
-32.6	-21.8	25.1°	1633	244	508	-10	0	-1903

w_8	$\mathbf{w_9}$	w_{10}	w_{11}	w_{12}	$\mathbf{w_{13}}$
2505	0	-2540	1425	-1523	0

Model Fitting - Naturalness

- In order to remove the several scales present within the model, we consider the general form of each term in the Lagrangian as

$$\mathcal{L} = \bar{c} \left(\frac{\pi}{f}\right)^A \left(\frac{q}{f\sqrt{\Lambda}}\right)^B \left(\frac{M}{\Lambda}\right)^C \left(\frac{\partial}{\Lambda}\right)^D \left(\frac{\chi}{M}\right)^E f^2\Lambda^2$$

Model Fitting - Naturalness

- In order to remove the several scales present within the model, we consider the general form of each term in the Lagrangian as

$$\mathcal{L} = \bar{c}\left(\frac{\pi}{f}\right)^{A}\left(\frac{q}{f\sqrt{\Lambda}}\right)^{B}\left(\frac{M}{\Lambda}\right)^{C}\left(\frac{\partial}{\Lambda}\right)^{D}\left(\frac{\chi}{M}\right)^{E} f^{2}\Lambda^{2}$$

- This gives a scale factor for each coupling:

$$\bar{c} = Sc, \qquad S = \frac{f^{A+B-2}\Lambda^{\frac{B}{2}+C+D-2}}{M^{C-E}}$$

Model Fitting - Naturalness

- In order to remove the several scales present within the model, we consider the general form of each term in the Lagrangian as

$$\mathcal{L} = \bar{c} \left(\frac{\pi}{f}\right)^A \left(\frac{q}{f\sqrt{\Lambda}}\right)^B \left(\frac{M}{\Lambda}\right)^C \left(\frac{\partial}{\Lambda}\right)^D \left(\frac{\chi}{M}\right)^E f^2\Lambda^2$$

- This gives a scale factor for each coupling:

$$\bar{c} = Sc, \qquad S = \frac{f^{A+B-2}\Lambda^{\frac{B}{2}+C+D-2}}{M^{C-E}}$$

- Results in natural units:

c	G	κ	g_1	g_2	κ_2	g_3	g_4	g_5
S	$\frac{f^2\Lambda^2}{M^2}$	$\frac{f^4\Lambda^4}{M^3}$	$\frac{f^6\Lambda^6}{M^4}$	$\frac{f^6\Lambda^6}{M^4}$	$\frac{f^2\Lambda^2}{M}$	$\frac{f^4\Lambda^4}{M^2}$	$\frac{f^4\Lambda^4}{M^2}$	$f^2\Lambda^2$
$\bar{c}$	1.0	-0.1	0.05	-0.1	0.01	-1.3	0.3	-0.5

g_6	g_7	g_8	w_1	w_7	w_8	w_{10}	w_{11}	w_{12}
$f^2\Lambda^2$	$f^2\Lambda^2$	$f^2\Lambda^2$	f^2	$\frac{f^6\Lambda^4}{M^2}$	$\frac{f^6\Lambda^4}{M^2}$	$f^4\Lambda^2$	$f^4\Lambda^2$	f^2M^2
-2.6	-0.7	-0.5	-0.1	-0.1	0.1	-0.5	0.3	-0.8

Closing Remarks

- The spin 1 extended model may be fitted to **reproduce the 4 low-lying meson spectra**.

Closing Remarks

- The spin 1 extended model may be fitted to **reproduce the 4 low-lying meson spectra**.
- The parameters are **severely constrained by symmetry requirements**.

Closing Remarks

- The spin 1 extended model may be fitted to **reproduce the 4 low-lying meson spectra**.
- The parameters are **severely constrained by symmetry requirements**.
- $V - \sigma$ and $A - \phi$ mixing result in subtle **relations between spin 1 and spin 0 masses**.

Closing Remarks

- The spin 1 extended model may be fitted to **reproduce the 4 low-lying meson spectra**.
- The parameters are **severely constrained by symmetry requirements**.
- $V - \sigma$ and $A - \phi$ mixing result in subtle **relations between spin 1 and spin 0 masses**.
- Parameters $\mathbf{w_2}$-$\mathbf{w_5}$ are expected to contribute at finite T or μ.

Closing Remarks

- The spin 1 extended model may be fitted to **reproduce the 4 low-lying meson spectra**.
- The parameters are **severely constrained by symmetry requirements**.
- $V - \sigma$ and $A - \phi$ mixing result in subtle **relations between spin 1 and spin 0 masses**.
- Parameters $\mathbf{w_2}$-$\mathbf{w_5}$ are expected to contribute at finite T or μ.
- Parameter sets which are **degenerate** for the vacuum yield different results when going over to finite T or μ.

References

Morais, J., Hiller, B. and Osipov, A.A. (2017)
Masses of the lowest spin-0 and spin-1 meson nonets: explicit symmetry breaking effects.
Phys. Rev. D 95, 074033.

Osipov, A.A., Hiller, B. and Blin, A.H. (2013)
Light quark masses in multi-quark interactions.
Eur. Phys. J. A 49(1).

Osipov, A.A., Hiller, B. and Blin, A.H. (2013)
Effective multiquark interactions with explicit breaking of chiral symmetry.
Phys. Rev. D 88, 054032.

Basics of NJL Model

- **NJL Lagangian:**

$$\mathcal{L}_{NJL} = i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi + G\left[\left(\bar{\psi}\psi\right)^2 + \left(\bar{\psi}i\gamma_5\vec{\tau}\psi\right)^2\right]$$

- Built in analogy with BCS theory of superconductivity (model for nucleons based on SBχS).
- Self-consistent mean field approach $\Longrightarrow$ **Mass gap equation** (dynamically generated mass M).

$$M = 8iGN_cN_fM\lim_{\epsilon\to 0}\int\frac{d^4p}{(2\pi)^4}\frac{1}{p^2 - M^2 + i\epsilon}\rho\left(p,\Lambda\right)$$

- Admits solutions with $M \neq 0$ for some values of $G \Longrightarrow$ DB of χ-symmetry $\Longrightarrow$ Finite condensate $\langle\bar{\psi}\psi\rangle \neq 0$.
- May include small quark current mass term $-\bar{\psi}m\psi$ that explicitly breaks the χ-symmetry.

3 Flavour NJL Model

- Three flavours $\Longrightarrow$ Axial anomaly $\Longrightarrow$ **'t Hooft term** (N_c suppresed):

$$\mathcal{L}_H = \kappa \left[\det \bar{q} P_R q + \det \bar{q} P_L q\right]$$

- Functional Integral Formalism:

$$Z = \int \mathcal{D}q\mathcal{D}\bar{q} e^{i \int d^4x \mathcal{L}}$$

- **Functional Bosonization with SPA and HKE** $\Longrightarrow$ Effective Potential $\mathcal{V} \Longrightarrow$ Generally unbounded from below $\Longrightarrow$ **Unstable (or at most metastable) vacuum.**

- Eight quark terms stabilize the vacuum (same order in N_c counting as the 't Hooft term).

Effective Model - Philosophy

- **Effective multi-quark vertices** stem from the *frozen* (integrated out) gluon degrees of freedom $\Longrightarrow$ Must respect all **QCD symmetries +** χ**-symmetry** (working assumption).
- Three quark flavours $(u, d, s) \Longrightarrow$ Must include 't Hooft term to account for the axial anomaly $\Longrightarrow$ Must include 8q terms to assure boundedness of $\mathcal{V}$.
- **General idea:** multi-quark vertices form a **hierarchy of effective quark interactions in** N_c **counting** $\Longrightarrow$ Consistency in model building demands that all terms up to the desired order should be included.

Effective Model - Philosophy

- **Current quark masses arise from interaction with an external source**, explicitly breaking the χ-symmetry $\Longrightarrow$ This symmetry breaking pattern should be included to the desired order as well.

- Within this philosophy, the usual **2 flavour NJL model with a current quark mass term consists of the leading order** effective model.

- With 3 flavours, in order to take into account the axial anomaly, one must go **beyond the leading order in a consistent way**.

Effective Model - Multiquark Terms

- Scalar and pseudoscalar bilinears: $s_a = \bar{q}\lambda_a q$, $p_a = \bar{q}i\gamma_5\lambda_a\gamma_5 q$; $U(3)$-**valued field:** $\Sigma = \frac{1}{2}(s_a - ip_a)\lambda_a$.
- **External source** χ which is assumed to transform as Σ.
- **Natural expansion parameter:** Λ (explicit regularization parameter - the model in non-renormalizable).
- General (non-derivative) term in the Lagrangian:

$$\mathcal{L}_i \sim \frac{\bar{g}_i}{\Lambda^\gamma}\Sigma^\alpha\chi^\beta$$

- **Dimensional analysis:** $3\alpha + \beta - \gamma = 4$.
- **Contribution to** $\mathcal{V}$ **at** $\Lambda \to \infty$**:** $2\alpha - \gamma \geq 0$.
- **General condition:** $4 - \alpha - \beta \geq 0$.

Effective Model - Lagrangian

- **Dirac term:**

$$\mathcal{L}_D = i\bar{q}\gamma^\mu \partial_\mu q$$

- **Source-independent** terms (by symmetry considerations):

$$\mathcal{L}_{int} = \frac{\bar{G}}{\Lambda^2}\mathrm{tr}\left(\Sigma^\dagger \Sigma\right) + \frac{\bar{\kappa}}{\Lambda^5}\left(\det \Sigma + \det \Sigma^\dagger\right) + \frac{\bar{g}_1}{\Lambda^8}\left[\mathrm{tr}\left(\Sigma^\dagger \Sigma\right)\right]^2 + \frac{\bar{g}_2}{\Lambda^8}\mathrm{tr}\left(\Sigma^\dagger \Sigma \Sigma^\dagger \Sigma\right)$$

- **Source-dependent** terms by replacing Σ fields for χ fields.

Effective Model - Lagrangian

- Possible **source-dependent** terms (by symmetry considerations):

$$\mathcal{L}_0 = -\mathrm{tr}\left(\Sigma^\dagger\chi + \chi^\dagger\Sigma\right)$$

$$\mathcal{L}_1 = -\frac{\bar{\kappa}_1}{\Lambda}\epsilon_{ijk}\epsilon_{lmn}\Sigma_{il}\chi_{jm}\chi_{kn} + \mathrm{h.c.}$$

$$\mathcal{L}_2 = \frac{\bar{\kappa}_2}{\Lambda^3}\epsilon_{ijk}\epsilon_{lmn}\Sigma_{il}\Sigma_{jm}\chi_{kn} + \mathrm{h.c.}$$

$$\mathcal{L}_3 = \frac{\bar{g}_3}{\Lambda^6}\mathrm{tr}\left(\Sigma^\dagger\Sigma\Sigma^\dagger\chi\right) + \mathrm{h.c.}$$

$$\mathcal{L}_4 = \frac{\bar{g}_4}{\Lambda^6}\mathrm{tr}\left(\Sigma^\dagger\Sigma\right)\mathrm{tr}\left(\Sigma^\dagger\chi\right) + \mathrm{h.c.}$$

$$\mathcal{L}_5 = \frac{\bar{g}_5}{\Lambda^4}\mathrm{tr}\left(\Sigma^\dagger\chi\Sigma^\dagger\chi\right) + \mathrm{h.c.}$$

$$\mathcal{L}_6 = \frac{\bar{g}_6}{\Lambda^4}\mathrm{tr}\left(\Sigma^\dagger\Sigma\chi^\dagger\chi + \Sigma\Sigma^\dagger\chi\chi^\dagger\right) + \mathrm{h.c.}$$

$$\mathcal{L}_7 = \frac{\bar{g}_7}{\Lambda^4}\left[\mathrm{tr}\left(\Sigma^\dagger\chi\right) + \mathrm{h.c.}\right]^2$$

$$\mathcal{L}_8 = \frac{\bar{g}_8}{\Lambda^4}\left[\mathrm{tr}\left(\Sigma^\dagger\chi\right) - \mathrm{h.c.}\right]^2$$

$$\mathcal{L}_9 = -\frac{\bar{g}_9}{\Lambda^2}\mathrm{tr}\left(\Sigma^\dagger\chi\chi^\dagger\chi\right) + \mathrm{h.c.}$$

$$\mathcal{L}_{10} = -\frac{\bar{g}_{10}}{\Lambda^2}\mathrm{tr}\left(\Sigma^\dagger\chi\right)\mathrm{tr}\left(\chi^\dagger\chi\right) + \mathrm{h.c.}$$

Effective Model - Lagrangian

- **Kaplan-Manohar ambiguity:** can set $\bar{\kappa}_1 = \bar{g}_9 = \bar{g}_{10} = 0$.
- Set $\chi = m/2$, $m = \mathrm{diag}\,(m_u, m_d, m_s)$.
- **Total Lagrangian:**

$$\mathcal{L} = \bar{q}\,(i\gamma^\mu \partial_\mu - m)\,q + \mathcal{L}_{int} + \sum_{i=2}^{8} L_i$$

- **Functional Bosonization:** Quark degrees of freedom are integrated out (in a functional integral formalism) in favour of mesonic ones.
- Bosonized Lagrangian + Mass gap equations + Stationary phase conditions are solved together to fit the low-lying scalar and pseudoscalar meson spectrum.

Effective Model - Fitting

- Thus, the model is fitted using only the linear and quadratic terms of the bosonized Lagrangian $\Longrightarrow$ Further validation is provided by using the fitted parameter values to compute effective meson-meson interactions from higher order terms in the Lagrangian.
- **Frailty of the model:** large number of parameters.
- **Merits of the model:** excellent meson spectrum fit + good results for meson strong decays.
- The model reinforces the ideas that chiral symmetry considerations are crucial in the description of low-energy strong phenomenology.
- It also shows that the explicit symmetry breaking pattern is essential for the best results in the meson spectrum fitting.

Model Extension - Multiquark Interactions

- **Quark bilinears**:

$$
\begin{aligned}
\text{Scalar:} \quad s_a &= \bar{q}\lambda_a q \\
\text{Pseudoscalar:} \quad p_a &= \bar{q} i\gamma_5 \lambda_a \gamma_5 q \\
\text{Vector:} \quad v_a^\mu &= \bar{q}\gamma^\mu \lambda_a q \\
\text{Axial Vector:} \quad a_a^\mu &= \bar{q}\gamma^\mu \gamma_5 \lambda_a q
\end{aligned}
$$

- $U(3)$**-valued fields:**

$$
\begin{aligned}
\Sigma &= \bar{q}_L \lambda_a q_R \lambda_a = \frac{1}{2}\left(s_a - i p_a\right)\lambda_a \\
\Sigma^\dagger &= \bar{q}_R \lambda_a q_L \lambda_a = \frac{1}{2}\left(s_a + i p_a\right)\lambda_a \\
R^\mu &= \bar{q}_R \lambda_a \gamma^\mu q_R \lambda_a = \frac{1}{2}\left(v_a^\mu + a_a^\mu\right)\lambda_a \\
L^\mu &= \bar{q}_L \lambda_a \gamma^\mu q_L \lambda_a = \frac{1}{2}\left(v_a^\mu - a_a^\mu\right)\lambda_a
\end{aligned}
$$

- **Chiral and Parity Transformations:**

$$
\begin{aligned}
\Sigma &\to V_R \Sigma V_L^\dagger \quad , \quad \Sigma^\dagger \to V_L \Sigma^\dagger V_R^\dagger \qquad\qquad \Sigma \leftrightarrow \Sigma^\dagger \\
L^\mu &\to V_L L^\mu V_L^\dagger \quad , \quad R^\mu \to V_R R^\mu V_R^\dagger \qquad\qquad L^\mu \leftrightarrow R^\mu
\end{aligned}
$$

Model Extension - Multiquark Interactions

- **External source** χ which is assumed to transform as Σ.
- **Natural expansion parameter:** Λ (explicit regularization parameter - the model in non-renormalizable).
- General (non-derivative) term in the Lagrangian:

$$\mathcal{L}_i \sim \frac{\bar{g}_i}{\Lambda^\gamma} \Sigma^{\alpha_1} (L, R)^{\alpha_2} \chi^\beta$$

- **Dimensional analysis:** $3(\alpha_1 + \alpha_2) + \beta - \gamma = 4$.
- **Contibution to $\mathcal{V}$ at $\Lambda \to \infty$:** $2(\alpha_1 + \alpha_2) - \gamma \geq 0$.
- **General condition:** $4 - (\alpha_1 + \alpha_2) - \beta \geq 0$.

Model Extension - New Terms

- **Source-independent** terms (vector):

$$\mathcal{L}'_{int} = \frac{\bar{w}_1}{\Lambda^2}\mathrm{tr}\left(R^\mu R_\mu + L^\mu L_\mu\right) + \frac{\bar{w}_2}{\Lambda^8}\left[\mathrm{tr}\left(R^\mu R_\mu + L^\mu L_\mu\right)\right]^2 + \frac{\bar{w}_3}{\Lambda^8}\left[\mathrm{tr}\left(R^\mu R_\mu - L^\mu L_\mu\right)\right]^2 + \frac{\bar{w}_4}{\Lambda^8}\mathrm{tr}\left(R^\mu R^\nu R_\mu R_\nu + L^\mu L^\nu L_\mu L_\nu\right) + \frac{\bar{w}_5}{\Lambda^8}\mathrm{tr}\left(R^\mu R_\mu R^\nu R_\nu + L^\mu L_\mu L^\nu L_\nu\right)$$

- **Source-independent** terms (scalar-vector):

$$\mathcal{L}''_{int} = \frac{\bar{w}_6}{\Lambda^8}\mathrm{tr}\left[R^\mu R_\mu + L^\mu L_\mu\right]\mathrm{tr}\left[\Sigma^\dagger\Sigma\right] + \frac{\bar{w}_7}{\Lambda^8}\mathrm{tr}\left[\Sigma^\dagger R^\mu \Sigma L_\mu\right] + \frac{\bar{w}_8}{\Lambda^8}\mathrm{tr}\left[\Sigma^\dagger\Sigma L^\mu L_\mu + \Sigma\Sigma^\dagger R^\mu R_\mu\right]$$

Model Extension - New Terms

- **Source-dependent** terms:

$$\mathcal{L}'_1 = \frac{\bar{w}_9}{\Lambda^6}\mathrm{tr}\left(R^\mu R_\mu + L^\mu L_\mu\right)\mathrm{tr}\left(\Sigma^\dagger\chi + \Sigma\chi^\dagger\right)$$

$$\mathcal{L}'_2 = \frac{\bar{w}_{10}}{\Lambda^6}\mathrm{tr}\left(\chi^\dagger R^\mu \Sigma L_\mu + \Sigma^\dagger R^\mu \chi L_\mu\right)$$

$$\mathcal{L}'_3 = \frac{\bar{w}_{11}}{\Lambda^6}\mathrm{tr}\left(\left(\Sigma^\dagger\chi + \chi^\dagger\Sigma\right)L^\mu L_\mu + \left(\Sigma\chi^\dagger + \chi\Sigma^\dagger\right)R^\mu R_\mu\right)$$

$$\mathcal{L}'_4 = \frac{\bar{w}_{12}}{\Lambda^4}\mathrm{tr}\left(\chi^\dagger R^\mu \chi L_\mu\right)$$

$$\mathcal{L}'_5 = \frac{\bar{w}_{13}}{\Lambda^4}\mathrm{tr}\left(\chi^\dagger\chi L^\mu L_\mu + \chi\chi^\dagger R^\mu R_\mu\right)$$

Model Extension - Functional Integral

- **Total Lagrangian:**

$$\mathcal{L} = \bar{q}\left(i\gamma^\mu \partial_\mu - m\right) q + \mathcal{L}_{int}$$

- All multi-quark terms (written is terms of s, p, v, a) are included in $\mathcal{L}_{int}$.
- **Functional Integral:**

$$Z = \int \mathcal{D}q\mathcal{D}\bar{q} \quad e^{i\int d^4x \mathcal{L}}$$

- **Multiply by a functional unit:**

$$1 = \int \mathcal{D}s_a \mathcal{D}p_a \mathcal{D}v_a^\mu \mathcal{D}a_a^\mu \delta\left(s_a - \bar{q}\lambda_a q\right) \delta\left(p_a - i\bar{q}\lambda_a\gamma_5 q\right) \times \delta\left(v_a^\mu - \bar{q}\lambda_a\gamma^\mu q\right) \delta\left(a_a^\mu - \bar{q}\lambda_a\gamma^\mu\gamma_5 q\right)$$

$$= \int \mathcal{D}\sigma_a \mathcal{D}\phi_a \mathcal{D}V_{a\mu} \mathcal{D}A_{a\mu} \int \mathcal{D}s_a \mathcal{D}p_a \mathcal{D}v_a^\mu \mathcal{D}a_a^\mu \times e^{i\int d^4x\left[\sigma_a(s_a - \bar{q}\lambda_a q) + \phi_a(p_a - i\bar{q}\lambda_a\gamma_5 q) + V_{a\mu}\left(v_a^\mu - \bar{q}\lambda_a\gamma^\mu q\right) + A_{a\mu}\left(a_a^\mu - \bar{q}\lambda_a\gamma^\mu\gamma_5 q\right)\right]}$$

Model Extension - Functional Integral

- **Functional Integral:** (new)

$$Z = \int \mathcal{D}\sigma_a \mathcal{D}\phi_a \mathcal{D}V_{a\mu}\mathcal{D}A_{a\mu} \int \mathcal{D}s_a \mathcal{D}p_a \mathcal{D}v_a^\mu \mathcal{D}a_a^\mu$$
$$\times\; e^{i\int d^4x \left[\mathcal{L}_{int} + s_a(\sigma_a - m_a) + p_a\phi_a + v_a^\mu V_{a\mu} + a_a^\mu A_{a\mu}\right]}$$
$$\times \int \mathcal{D}q\mathcal{D}\bar{q} \quad e^{i\int d^4x \bar{q}(i\gamma^\mu\partial_\mu - \sigma - i\gamma_5\phi - \gamma^\mu V_\mu - \gamma^\mu\gamma_5 A_\mu)q}$$

- s, p, v, a are auxiliary fields (they do not propagate) used to linearize multi-quark interactions.
- σ, ϕ, V, A are the dynamical meson fields (kinetic terms come from the quark determinant).
- Expect finite condensate: shift $\sigma \to \sigma + M$ so that the new field has zero VEV.
- Also shift $V^\mu \to V^\mu + \eta\delta_0^\mu$ (time component of a vector behaves as a scalar; contributes to the chemical potential in a medium).

Model Extension - Functional Integral

- **Functional Integral:**

$$Z = \int \mathcal{D}\sigma_a \mathcal{D}\phi_a \mathcal{D}V_{a\mu} \mathcal{D}A_{a\mu} \int \mathcal{D}s_a \mathcal{D}p_a \mathcal{D}v_a^\mu \mathcal{D}a_a^\mu$$
$$\times\, e^{i\int d^4x \left[\mathcal{L}_{int} + s_a(\sigma_a+\Delta_a) + p_a\phi_a + v_a^\mu (V_{a\mu}+\eta_a\delta_{\mu 0}) + a_a^\mu A_{a\mu}\right]}$$
$$\times \int \mathcal{D}q\mathcal{D}\bar{q} \quad e^{i\int d^4x \bar{q}\left(i\gamma^\mu\partial_\mu - M - \gamma^0\eta - \sigma - i\gamma_5\phi - \gamma^\mu V_\mu - \gamma^\mu\gamma_5 A_\mu\right)q}$$
$$= \int \mathcal{D}\sigma_a \mathcal{D}\phi_a \mathcal{D}V_{a\mu} \mathcal{D}A_{a\mu} e^{i\int d^4x (\mathcal{L}_{SPA}+\mathcal{L}_{HK})}$$

- $\mathcal{L}_{SPA}$ comes from the integration in s, p, v, a using a **stationary phase approximation** (SPA).

- $\mathcal{L}_{HK}$ comes from the integration in $q, \bar{q}$ using a **proper time heat kernel expansion** (HK).

Model Extension - Stationary Phase Integral

- **Stationary Phase Expansions:**

$$
\begin{aligned}
s_a^{st} &= h_a + h_{ab}^{(1)}\sigma_b + h_{abc}^{(1)}\sigma_b\sigma_c + h_{abc}^{(2)}\phi_b\phi_c + H_{abc}^{\mu\nu\,(1)} V_{b\mu} V_{c\nu} \\
&\quad + H_{abc}^{\mu\nu\,(2)} A_{b\mu} A_{c\nu} + \dots \\
p_a^{st} &= h_{ab}^{(2)}\phi_b + h_{abc}^{(3)}\phi_b\sigma_c + H_{abc}^{\mu\nu\,(3)} V_{b\mu} A_{c\nu} + \dots \\
v_a^{\mu st} &= K_a\delta_{\mu 0} + H_{ab}^{\mu\nu(1)} V_{b\nu} + H_{abc}^{\mu\nu\,(4)}\sigma_b V_{c\nu} + H_{abc}^{\mu\nu\,(5)}\phi_b A_{c\nu} + \dots \\
a_a^{\mu st} &= H_{ab}^{\mu\nu(2)} A_{b\nu} + H_{abc}^{\mu\nu\,(6)}\phi_b V_{c\nu} + H_{abc}^{\mu\nu\,(7)}\sigma_b A_{c\nu} + \dots
\end{aligned}
$$

- **SPA Conditions:**

$$
\frac{\partial \bar{\mathcal{L}}_{int}}{\partial \sigma_a} = \frac{\partial \bar{\mathcal{L}}_{int}}{\partial \phi_a} = \frac{\partial \bar{\mathcal{L}}_{int}}{\partial V_a^\mu} = \frac{\partial \bar{\mathcal{L}}_{int}}{\partial A_a^\mu} = 0
$$

$$
\bar{\mathcal{L}}_{int} = \mathcal{L}_{int} + s_a\left(\sigma_a + \Delta_a\right) + p_a\phi_a + v_a^\mu\left(V_{a\mu} + \eta_a\delta_{\mu 0}\right) + a_a^\mu A_{a\mu}
$$

Model Extension - Stationary Phase Integral

- **SPA Coefficient Conditions:** (lowest order)

$$\Delta_i + \frac{h_i}{4}\left[4G + 2g_1 h^2 + 2g_4 mh + 2w_6 K^2 + (w_7 + 2w_8) K_i^2\right] + \frac{g_2}{2} h_i^3$$
$$+ \frac{m_i}{4}\left[3g_3 h_i^2 + g_4 h^2 + 2(g_5 + g_6) m_i h_i + 4g_7 mh + 2w_9 K^2\right.$$
$$\left. + (w_{11} + 2w_{12}) K_i^2\right] + \frac{\kappa}{4} t_{ijk} h_j h_k + \kappa_2 t_{ijk} h_j m_k = 0$$

$$\eta_i + \frac{K_i}{4}\left[8w_1 + 8w_2 K^2 + 4(w_4 + w_5) K_i^2 + 2w_6 h^2 + (w_7 + 2w_8) h_i^2\right.$$
$$\left. + 4w_9 mh + 2(w_{10} + 2w_{11}) m_i h_i + (w_{12} + 2w_{13}) m_i^2\right] = 0$$

- Basis with $i, j, k \in (u, d, s)$.

Model Extension - Stationary Phase Integral

- **Transformation between** $0, 3, 8$ **and** u, d, s **basis:**

$$e_{ai} = \frac{(\lambda_a)_{ii}}{2} \equiv \frac{1}{2\sqrt{3}} \begin{pmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ \sqrt{3} & -\sqrt{3} & 0 \\ 1 & 1 & -2 \end{pmatrix} \quad , \quad h_a = e_{ai} h_i \quad , \cdots$$

- **Final Lagrangian:** (up to cubic terms)

$$\begin{aligned} \mathcal{L}_{SPA} = & \, h_a \sigma_a + K_a V_a^0 \\ & + \frac{1}{2} \left(h_{ab}^{(1)} \sigma_a \sigma_b + h_{ab}^{(2)} \phi_a \phi_b + H_{ab}^{\mu\nu(1)} V_{a\mu} V_{b\nu} + H_{ab}^{\mu\nu(2)} A_{a\mu} A_{b\nu} \right) \\ & + \sigma_a \left(\frac{1}{3} h_{abc}^{(1)} \sigma_b \sigma_c + h_{abc}^{(2)} \phi_b \phi_c + H_{abc}^{\mu\nu\,(1)} V_{b\mu} V_{c\nu} + H_{abc}^{\mu\nu\,(2)} A_{b\mu} A_{c\nu} \right) \\ & + H_{abc}^{\mu\nu\,(3)} \phi_a V_{b\mu} A_{c\nu} + \ldots \end{aligned}$$

- SPA conditions determine h, H coefficients.

Model Extension - Quark Integral

- **Quark Determinant:**

$$\int \mathcal{D}q\mathcal{D}\bar{q} \quad e^{i\int d^4x\bar{q}\left(i\gamma^\mu\partial_\mu - M - \gamma^0\eta - \sigma - i\gamma_5\phi - \gamma^\mu V_\mu - \gamma^\mu\gamma_5 A_\mu\right)q}$$
$$= \det\left(i\gamma^\mu\partial_\mu - M - \gamma^0\eta - \sigma - i\gamma_5\phi - \gamma^\mu V_\mu - \gamma^\mu\gamma_5 A_\mu\right)$$
$$= e^{\ln\det\left(i\gamma^\mu\partial_\mu - M - \gamma^0\eta - \sigma - i\gamma_5\phi - \gamma^\mu V_\mu - \gamma^\mu\gamma_5 A_\mu\right)}$$
$$= e^{\tilde{\mathrm{Tr}}\ln\left(i\gamma^\mu\partial_\mu - M - \gamma^0\eta - \sigma - i\gamma_5\phi - \gamma^\mu V_\mu - \gamma^\mu\gamma_5 A_\mu\right)} = e^{iW_q}$$

- **Quark Effective Action:**

$$W_q = -i\tilde{\mathrm{Tr}}\ln\left(i\gamma^\mu\partial_\mu - M - \gamma^0\eta - \sigma - i\gamma_5\phi - \gamma^\mu V_\mu - \gamma^\mu\gamma_5 A_\mu\right)$$
$$= -i\mathrm{Tr}\int d^4x \left\langle x\left|\ln\left(i\gamma^\mu\partial_\mu - M - \gamma^0\eta - \sigma - i\gamma_5\phi\right.\right.\right.$$
$$\left.\left.\left.-\gamma^\mu V_\mu - \gamma^\mu\gamma_5 A_\mu\right)\right| x\right\rangle$$

Model Extension - Quark Integral

- **Wick Rotation:**

$$x^0 \to -ix_4 \quad \gamma^0 \to -i\gamma_4 \quad V^0 \to -iV_4 \quad A^0 \to -iA_4 \quad \eta \to -i\eta'$$
$$x^i \to x_i \quad \gamma^i \to \gamma_i \quad V^i \to V_i \quad A^i \to A_i \quad \gamma_5 \to -\gamma_5$$

- **Euclidean Gamma Matrices:**

$$\{\gamma_\alpha, \gamma_\beta\} = -2\delta_{\alpha\beta} \qquad \gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$$
$$\{\gamma_\alpha, \gamma_5\} = 0 \qquad \gamma_\alpha^\dagger = -\gamma_\alpha$$
$$(\gamma_\alpha)^2 = -(\gamma_5)^2 = -1 \qquad \gamma_5^\dagger = \gamma_5$$

- **Euclidean Effective Action:**

$$W_E = i\mathrm{Tr}\int d^4x_E \, \langle x_E \left|\ln\left(D_E\right)\right| x_E\rangle$$

Model Extension - Quark Integral

- **Real Effective Action:**

$$W_E = \frac{1}{2}\mathrm{Tr}\int d^4x_E \left\langle x_E \left| \ln\left(D_E^\dagger D_E\right) \right| x_E \right\rangle$$

- Define $V'_\alpha = V_\alpha + \delta_{\alpha 4}\eta$ and

$$\nabla'_\alpha \sigma = \partial_\alpha \sigma - \{A_\alpha, \phi\} + i\left[V'_\alpha, \sigma + M\right]$$

$$\nabla'_\alpha \phi = \partial_\alpha \phi + \{A_\alpha, \sigma + M\} - i\left[V'_\alpha, \phi\right]$$

$$V_{\alpha\beta} = \partial_\alpha V'_\beta - \partial_\beta V'_\alpha - i\left[V'_\alpha, V'_\beta\right] - i\left[A_\alpha, A_\beta\right]$$

$$A_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha - i\left[V'_\alpha, A_\beta\right] - i\left[A_\alpha, V'_\beta\right]$$

$$d_\alpha = \partial_\alpha - i\Gamma_\alpha \qquad \Gamma_\alpha = V'_\alpha - \gamma_5 A_\alpha$$

$$\Gamma_{\alpha\beta} = V_{\alpha\beta} - \gamma_5 A_{\alpha\beta} = i\left[d_\alpha, d_\beta\right]$$

Model Extension - Quark Integral

- Thus,

$$D_E^\dagger D_E = -d_\alpha^2 + i\gamma_\alpha \nabla'_\alpha (\sigma - i\gamma_5\phi) - \frac{i}{4}[\gamma_\alpha, \gamma_\beta]\,\Gamma_{\alpha\beta} + \sigma^2 + \phi^2 + M^2 + \{\sigma, M\} + i\gamma_5[\phi, \sigma + M]$$

- Use integral (asymptotic) representation for the logarithm

$$\begin{aligned} W_E &= -\frac{1}{2}\mathrm{Tr}\int d^4x \int_0^\infty \frac{d\tau}{\tau}\left\langle x \left| e^{-\tau D_E^\dagger D_E} \right| x \right\rangle \\ &= -\frac{1}{2}\mathrm{Tr}\int d^4x \int \frac{d^4p}{(2\pi)^4}\int_0^\infty \frac{d\tau}{\tau}\langle x| e^{-\tau D_E^\dagger D_E}|p\rangle\langle p|x\rangle \\ &= -\frac{1}{2}\mathrm{Tr}\int d^4x \int \frac{d^4p}{(2\pi)^4}\int_0^\infty \frac{d\tau}{\tau}\langle p|x\rangle\, e^{-\tau D_E^\dagger D_E}\langle x|p\rangle \\ &= -\frac{1}{2}\mathrm{Tr}\int d^4x \int \frac{d^4p}{(2\pi)^4}\int_0^\infty \frac{d\tau}{\tau} e^{-ipx} e^{-\tau D_E^\dagger D_E} e^{ipx} \end{aligned}$$

Model Extension - Quark Integral

$$W_E = -\frac{1}{2}\mathrm{Tr}\int d^4x \int \frac{d^4p}{(2\pi)^4}\int_0^\infty \frac{d\tau}{\tau} e^{-ipx} e^{-\tau D_E^\dagger D_E} e^{ipx}$$

$$= -\frac{1}{2}\mathrm{Tr}\int d^4x \int \frac{d^4p}{(2\pi)^4}\int_0^\infty \frac{d\tau}{\tau} e^{-\tau\left(D_E^\dagger D_E + p^2 - 2ip_\alpha d_\alpha\right)}$$

$$= -\frac{1}{2}\mathrm{Tr}\int d^4x \int \frac{d^4p}{(2\pi)^4} e^{-p^2}\int_0^\infty \frac{d\tau}{\tau^3} e^{-\tau\left(D_E^\dagger D_E - \frac{2ip_\alpha d_\alpha}{\sqrt{\tau}}\right)}$$

$$= -\int \frac{d^4x}{32\pi^4}\int d^4p \; e^{-p^2}\int_0^\infty \frac{d\tau}{\tau^3}\mathrm{Tr}\left\{e^{-\tau\left(M^2+\Delta\right)}\right\}$$

- In the last step we have rescaled $p_\alpha \to p_\alpha/\sqrt{\tau}$, and

$$\Delta = -d_\alpha^2 - \frac{2ip_\alpha d_\alpha}{\sqrt{\tau}} + Y \quad , \quad Y = i\gamma_\alpha \nabla'_\alpha(\sigma - i\gamma_5\phi) - \frac{i}{4}[\gamma_\alpha, \gamma_\beta]\Gamma_{\alpha\beta}$$

$$+ \sigma^2 + \phi^2 + \{\sigma, M\} + i\gamma_5[\phi, \sigma + M]$$

Model Extension - Quark Integral

- Use the operator expansion

$$e^{-\tau\left(M^2+\Delta\right)} = e^{-\tau M^2}\left[1+\sum_{n=1}^{\infty}(-1)^n f_n(\tau,\Delta)\right]$$

$$f_n(\tau,\Delta) = \int_0^{\tau} ds_1 \ldots \int_0^{s_{n-1}} ds_n \Delta(s_1)\ldots\Delta(s_n)$$

$$\Delta(s) = e^{sM^2}\Delta e^{-sM^2}$$

- General n-th order term

$$\mathrm{Tr}\left\{e^{-\tau\left(M^2+\Delta\right)}\right\} \overset{n\text{-th term}}{\equiv} (-1)^n \int_0^{\tau} ds_1 \ldots \int_0^{s_{n-1}} ds_n \times \mathrm{Tr}\left\{e^{-\tau M^2}\Delta(s_1)\ldots\Delta(s_n)\right\}$$

Model Extension - Quark Integral

- Define

$$E_u = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad E_d = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad E_s = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathit{\Delta}_i = E_i \mathit{\Delta} \qquad , \qquad \Delta_{ij} = M_i^2 - M_j^2$$

- General n-th order term:

$$\mathrm{Tr}\left\{ e^{-\tau\left(M^2+\mathit{\Delta}\right)} \right\} \overset{n\text{-th term}}{\equiv} \sum_{i_1,\ldots,i_n} (-1)^n \, e^{-\tau M_{i_1}^2}$$

$$\times \int_0^\tau ds_1 e^{s_1 \Delta_{i_1 i_2}} \ldots \int_0^{s_{n-1}} ds_n e^{s_n \Delta_{i_n i_1}} \, \mathrm{Tr}\left\{ \mathit{\Delta}_{i_1} \ldots \mathit{\Delta}_{i_n} \right\}$$

Model Extension - Quark Integral

- The general n-th term of the expansion may then be decomposed into a combinatorial factor c and a trace of products of Δ_i, i.e.

$$\mathrm{Tr}\left\{e^{-\tau\left(M^2+\Delta\right)}\right\} \overset{n\text{-th term}}{\equiv} \sum_{i_1,\ldots,i_n} c_{i_1\ldots i_n} \mathrm{Tr}\left\{\Delta_{i_1}\ldots\Delta_{i_n}\right\}$$

- Combinatorial factors

$$c_{i_1\ldots i_n} = (-1)^n e^{-\tau M_{i_1}^2} \int_0^{\tau} ds_1 e^{s_1 \Delta_{i_1 i_2}} \ldots \int_0^{s_{n-1}} ds_n e^{s_n \Delta_{i_n i_1}}$$

- From the cyclical nature of the trace we may write these factors in symmetrical form:

$$c_i = -\tau e^{-\tau M_i^2} \qquad c_{ij} = \frac{\tau}{2}\left[\frac{e^{-\tau M_j^2} - e^{-\tau M_i^2}}{\Delta_{ij}}\right]$$

$$c_{ijk} = -\frac{\tau}{3}\left[\frac{e^{-\tau M_i^2}}{\Delta_{ji}\Delta_{ki}} + \frac{e^{-\tau M_j^2}}{\Delta_{ij}\Delta_{kj}} + \frac{e^{-\tau M_k^2}}{\Delta_{ik}\Delta_{jk}}\right]$$

$$\ldots$$

Model Extension - Quark Integral

- Including the τ integration and a UV regulating function $\rho\left(\tau\Lambda^2\right)$, we may define the integrals

$$J_n\left(M_i^2\right) = \int_0^\infty \frac{d\tau}{\tau^{2-n}} \rho\left(\tau\Lambda^2\right) e^{-\tau M_i^2}$$

- These correspond to quark loop integrals and obey the identities

$$\tau^n J_a\left(M_i^2\right) = J_{a+n}\left(M_i^2\right)$$

$$J_a\left(M_j^2\right) - J_a\left(M_i^2\right) = \sum_{n=1}^{\infty} \frac{\Delta_{ij}^n}{2^n n!}\left[J_{a+n}\left(M_i^2\right) - (-1)^n J_{a+n}\left(M_j^2\right)\right]$$

- So, we extend the definition of the combinatorial factors c to include the τ integral in the form of J integrals.

Model Extension - Quark Integral

- New combinatorial factors

$$c_i \equiv -J_0\left(M_i^2\right)$$

$$c_{ij} \equiv \frac{1}{2}\left[\frac{J_0\left(M_j^2\right) - J_0\left(M_i^2\right)}{\Delta_{ij}}\right]$$

$$c_{ijk} \equiv -\frac{1}{3}\left[\frac{J_0\left(M_i^2\right)}{\Delta_{ji}\Delta_{ki}} + \frac{J_0\left(M_j^2\right)}{\Delta_{ij}\Delta_{kj}} + \frac{J_0\left(M_k^2\right)}{\Delta_{ik}\Delta_{jk}}\right]$$

$$c_{ijkl} \equiv \frac{1}{4}\left[\frac{J_0\left(M_i^2\right)}{\Delta_{ji}\Delta_{ki}\Delta_{li}} + \frac{J_0\left(M_j^2\right)}{\Delta_{ij}\Delta_{kj}\Delta_{lj}} + \frac{J_0\left(M_k^2\right)}{\Delta_{ik}\Delta_{jk}\Delta_{lk}} + \frac{J_0\left(M_l^2\right)}{\Delta_{il}\Delta_{jl}\Delta_{kl}}\right]$$

Model Extension - Quark Integral

- Using the J recurrence relation, we may rewrite

$$c_i \equiv -J_0\left(M_i^2\right)$$
$$c_{ij} \equiv \frac{J_1\left(M_i^2\right)+J_1\left(M_j^2\right)}{4}+\mathcal{O}\left(J_3\right)$$
$$c_{ijk} \equiv -\frac{J_2\left(M_i^2\right)+J_2\left(M_j^2\right)+J_2\left(M_k^2\right)}{18}+\mathcal{O}\left(J_3\right)$$
$$c_{ijkl} \equiv \frac{J_3\left(M_i^2\right)+J_3\left(M_j^2\right)+J_3\left(M_k^2\right)+J_3\left(M_l^2\right)}{96}+\mathcal{O}\left(J_4\right)$$

- For the p integrals use ($\delta_{\alpha_1\ldots\alpha_n}$ is the generalized Kronecker delta)

$$\int d^4p\ \ e^{-p^2} p_{\alpha_1}\ldots p_{\alpha_n} = \begin{cases} 2^{-n}\pi^2\delta_{\alpha_1\ldots\alpha_n} & ,\ n \text{ par} \\ 0 & ,\ n \text{ mpar} \end{cases}$$

Model Extension - Quark Integral

- We combine the p integrations with the traces to get

$$n = 1 : \pi^2 \, \mathrm{Tr} \left\{ \left(-d_\alpha^2 + Y\right)_i \right\}$$

$$n = 2 : \pi^2 \, \mathrm{Tr} \left\{ \left(-d_\alpha^2 + Y\right)_i \left(-d_\beta^2 + Y\right)_j - \frac{2}{\tau} (d_\alpha)_i (d_\alpha)_j \right\}$$

$$\begin{aligned} n = 3 : \pi^2 \, \mathrm{Tr} \Big\{ & \left(-d_\alpha^2 + Y\right)_i \left(-d_\beta^2 + Y\right)_j \left(-d_\gamma^2 + Y\right)_k \\ & - \frac{6}{\tau} \left(-d_\alpha^2 + Y\right)_i (d_\beta)_j (d_\beta)_k \Big\} \end{aligned}$$

$$\begin{aligned} n = 4 : \pi^2 \, \mathrm{Tr} \Big\{ & \left(-d_\alpha^2 + Y\right)_i \left(-d_\beta^2 + Y\right)_j \left(-d_\gamma^2 + Y\right)_k \left(-d_\delta^2 + Y\right)_l \\ & - \frac{4}{\tau} \Big[2 \left(-d_\alpha^2 + Y\right)_i \left(-d_\beta^2 + Y\right)_j (d_\gamma)_k (d_\gamma)_l \\ & + \left(-d_\alpha^2 + Y\right)_i (d_\gamma)_j \left(-d_\beta^2 + Y\right)_k (d_\gamma)_l \Big] \\ & + \frac{4}{\tau^2} \Big[2 (d_\alpha)_i (d_\alpha)_j (d_\beta)_k (d_\beta)_l + (d_\alpha)_i (d_\beta)_j (d_\alpha)_k (d_\beta)_l \Big] \Big\} \end{aligned}$$

Model Extension - Quark Integral

- Truncation of the expansion in $n \Longrightarrow$ violation of the chiral symmetry pattern of the original Lagrangian.
- Reorganize the sum in factors of chirally invariant integrals

$$I_n = \frac{1}{3}\sum_{i=1}^{3} J_n\left(M_i^2\right)$$

- **Heat Kernel Expansion** for the quark determinant:

$$W_E = -\int \frac{d^4x_E}{32\pi^2}\sum_{n=0}^{\infty} I_{n-1}\mathrm{Tr}\left\{b_n\right\}$$

$$b_0 = 1 \quad , \quad b_1 = -Y$$

$$b_2 = \frac{Y^2}{2} + \frac{\Delta_{12}}{2}\lambda_3 Y + \frac{\Delta_{13}+\Delta_{23}}{2\sqrt{3}}\lambda_8 Y - \frac{\Gamma_{\alpha\beta}^2}{12} \quad , \cdots$$

Model Extension - Gap Equations

- The total bosonized Lagrangian is given by a SPA contribution plus a HK contribution:

$$\mathcal{L}_{bos} = \mathcal{L}_{SPA} - \frac{1}{32\pi^2} \sum_{n=0}^{\infty} I_{n-1} Tr\{b_n\}$$

- **Gap Equation** by demanding that the σ tadpole terms vanish:

$$h_i + \frac{N_c}{6\pi^2} M_i \left[3I_0 - \left(3M_i^2 - M^2\right) I_1\right] = 0$$

- There are also conditions for the V^0 tadpole terms, which simplify to $\eta_i = 0$.

- Together with the SPA conditions, this allows us to set $K_i = 0$ (this contribution will be important in a medium only).

Model Extension - Quadratic Terms

- The SPA quadratic terms have the simple form (mass terms)

$$\mathcal{L}_{SPA}^{quad} = \frac{1}{2}\left(h_{ab}^{(1)}\sigma_a\sigma_b + h_{ab}^{(2)}\phi_a\phi_b + H_{ab}^{\mu\nu(1)}V_{a\mu}V_{b\nu} + H_{ab}^{\mu\nu(2)}A_{a\mu}A_{b\nu}\right)$$

- The HK contributions are more complicated, and consist of **kinetic, mass and mixing terms**.

$$\mathcal{L}_{HK}^{quad} = \mathcal{L}_{HK}^{kin} + \mathcal{L}_{HK}^{mass} + \mathcal{L}_{HK}^{mix}$$

- **Kinetic Terms:**

$$\mathcal{L}_{HK}^{kin} = \frac{N_c I_1}{16\pi^2}\mathrm{tr}_F\left\{ (\partial^\mu\sigma)(\partial_\mu\sigma) + (\partial^\mu\phi)(\partial_\mu\phi) - \frac{1}{3}\left(F_{(V)}^{\mu\nu}F_{\mu\nu}^{(V)} + F_{(A)}^{\mu\nu}F_{\mu\nu}^{(A)}\right)\right\}$$

Model Extension - Quadratic Terms

- **Mass Terms:**

$$\mathcal{L}_{HK}^{mass} = \frac{N_c I_0}{8\pi^2}\mathrm{tr}_F\left\{\sigma^2+\phi^2\right\} + \frac{N_c I_1}{16\pi^2}\mathrm{tr}_F\Big\{-\{\sigma, M\}^2 + [\phi, M]^2 - \left(\Delta_{ud}\lambda_3 + \frac{\Delta_{us}+\Delta_{ds}}{\sqrt{3}}\lambda_8\right)\left(\sigma^2+\phi^2\right) - [V^\mu, M][V_\mu, M] + \{A^\mu, M\}\{A_\mu, M\}\Big\}$$

- **Mixing Terms:**

$$\mathcal{L}_{HK}^{mix} = \frac{N_c I_1}{8\pi^2}\mathrm{tr}_F\left\{-i\,[V^\mu, M]\,\partial_\mu\sigma - \{A^\mu, M\}\,\partial_\mu\phi\right\}$$

- There is also a surface term (involving vector and axial vector fields only) which may generally be set to zero.

Model Extension - Quadratic Terms

- In order to eliminate $\sigma - V$ and $\phi - A$ mixing, we may then use the shifts

$$V_{a\mu} \longrightarrow \bar{V}_{a\mu} + 2k_a f_{abc} M_b \partial_\mu \sigma_c$$
$$A_{a\mu} \longrightarrow \bar{A}_{a\mu} + 2k'_a d_{abc} M_b \partial_\mu \phi_c$$

- k_a,k'_a are constants to be adjusted in order to eliminate the mixing terms in the Lagrangian.

Model Extension - Quadratic Terms

- New SPA quadratic Lagrangian:

$$\mathcal{L}_{SPA}^{quad} = \frac{1}{2}\left(h_{ab}^{(1)}\sigma_a\sigma_b + h_{ab}^{(2)}\phi_a\phi_b + H_{ab}^{\mu\nu(1)}\bar{V}_{a\mu}\bar{V}_{b\nu} + H_{ab}^{\mu\nu(2)}\bar{A}_{a\mu}\bar{A}_{b\nu}\right)$$
$$+ 2\left(H_{ab}^{\mu\nu(1)}\bar{V}_{a\mu}k_b f_{bcd}M_c\partial_\nu\sigma_d + H_{ab}^{\mu\nu(2)}\bar{A}_{a\mu}k'_b d_{bcd}M_c\partial_\nu\phi_d\right.$$
$$+ H_{ab}^{\mu\nu(1)}f_{acd}f_{bc'd'}k_a k_b M_c M_{c'}\partial_\mu\sigma_d\partial_\nu\sigma_{d'}$$
$$\left.+H_{ab}^{\mu\nu(2)}d_{acd}d_{bc'd'}k'_a k'_b M_c M_{c'}\partial_\mu\phi_d\partial_\nu\phi_{d'}\right)$$

- New contributions to $\sigma - V$ and $\phi - A$ mixing arise, as well as new contributions to σ and ϕ kinetic terms.

Model Extension - Quadratic Terms

- New HK quadratic Lagrangian:

$$\mathcal{L}_{HK}^{quad} = \frac{N_c I_1}{16\pi^2}\text{tr}_F \left\{ (\partial^\mu \sigma)(\partial_\mu \sigma) + (\partial^\mu \phi)(\partial_\mu \phi) - \frac{1}{3}\left(\bar{F}_{(V)}^{\mu\nu} \bar{F}_{\mu\nu}^{(V)} + \bar{F}_{(A)}^{\mu\nu} \bar{F}_{\mu\nu}^{(A)} \right) \right.$$

$$\left. + \frac{2I_0}{I_1}(\sigma^2 + \phi^2) - \{\sigma, M\}^2 + [\phi, M]^2 - \left(\Delta_{ud}\lambda_3 + \frac{\Delta_{us} + \Delta_{ds}}{\sqrt{3}}\lambda_8 \right)(\sigma^2 + \phi^2) \right\}$$

$$+ \frac{N_c I_1}{2\pi^2} \left\{ f_{ace} f_{bde} \left(\bar{V}_a^\mu + 2k_a f_{amn} M_m \partial^\mu \sigma_n \right) \left(\bar{V}_{b\mu} + 2k_b f_{bm'n'} M_{m'} \partial_\mu \sigma_{n'} \right) M_c M_d \right.$$

$$+ d_{ace} d_{bde} \left(\bar{A}_a^\mu + 2k'_a d_{amn} M_m \partial^\mu \phi_n \right) \left(\bar{A}_{b\mu} + 2k'_b d_{bm'n'} M_{m'} \partial_\mu \phi_{n'} \right) M_c M_d$$

$$- f_{abc} \left(\bar{V}_a^\mu + 2k_a f_{ade} M_d \partial^\mu \sigma_e \right) \partial_\mu \sigma_b M_c$$

$$\left. - d_{abc} \left(\bar{A}_a^\mu + 2k'_a d_{ade} M_d \partial^\mu \phi_e \right) \partial_\mu \phi_b M_c \right\}$$

- The first two lines correspond to the previous quadratic HK contributions, while the rest are new mixing, mass and kinetic contributions.

Model Extension - Quadratic Terms

- We collect all mixing terms as $(H_{ad}^{\mu\nu(1,2)} = H_{ad}{}^{(1,2)} g^{\mu\nu})$:

$$\frac{N_c I_1}{2\pi^2} \bar{V}_{a\mu}\partial_\nu \sigma_b M_c \left\{ 4k_d f_{cbd} \left(\frac{\pi^2 H_{ad}^{\mu\nu(1)}}{N_c I_1} + f_{ame} f_{dne} M_m M_n g^{\mu\nu} \right) \right.$$
$$\left. - f_{abc} g^{\mu\nu} \right\}$$
$$+ \frac{N_c I_1}{2\pi^2} \bar{A}_{a\mu}\partial_\nu \phi_b M_c \left\{ 4k'_d d_{cbd} \left(\frac{\pi^2 H_{ad}^{\mu\nu(2)}}{N_c I_1} + d_{ame} d_{dne} M_m M_n g^{\mu\nu} \right) \right.$$
$$\left. - d_{abc} g^{\mu\nu} \right\}$$

- These result in conditions for the k_d, k'_d for each combination of a, b:

$$M_c k_d f_{bcd} \left[\frac{\pi^2 H_{ad}{}^{(1)}}{N_c I_1} + f_{ame} f_{dne} M_m M_n \right] = -\frac{M_c f_{abc}}{4}$$

$$M_c k'_d d_{bcd} \left[\frac{\pi^2 H_{ad}{}^{(2)}}{N_c I_1} + d_{ame} d_{dne} M_m M_n \right] = \frac{M_c d_{abc}}{4}$$

Model Extension - Quadratic Terms

- **Vector coefficients:**

$$-\frac{1}{k_{1,2}} = \frac{4\pi^2 {H_{11}}^{(1)}}{N_c I_1} + (M_u - M_d)^2 = \frac{4\pi^2 {H_{22}}^{(1)}}{N_c I_1} + (M_u - M_d)^2$$

$$-\frac{1}{k_{4,5}} = \frac{4\pi^2 {H_{44}}^{(1)}}{N_c I_1} + (M_u - M_s)^2 = \frac{4\pi^2 {H_{55}}^{(1)}}{N_c I_1} + (M_u - M_s)^2$$

$$-\frac{1}{k_{6,7}} = \frac{4\pi^2 {H_{66}}^{(1)}}{N_c I_1} + (M_d - M_s)^2 = \frac{4\pi^2 {H_{77}}^{(1)}}{N_c I_1} + (M_d - M_s)^2$$

- The coefficients $k_{0,3,8}$ (or $k_{u,d,s}$, with $k_i = 2\sum_a k_a e_{ai}^2$) are arbitrary; there is no mixture between the neutral scalars and vectors.

Model Extension - Quadratic Terms

- **Axial vector coefficients:**

$$\frac{1}{k'_{1,2}} = \frac{4\pi^2 {H_{11}}^{(2)}}{N_c I_1} + (M_u + M_d)^2 = \frac{4\pi^2 {H_{22}}^{(2)}}{N_c I_1} + (M_u + M_d)^2$$

$$\frac{1}{k'_{4,5}} = \frac{4\pi^2 {H_{44}}^{(2)}}{N_c I_1} + (M_u + M_s)^2 = \frac{4\pi^2 {H_{55}}^{(2)}}{N_c I_1} + (M_u + M_s)^2$$

$$\frac{1}{k'_{6,7}} = \frac{4\pi^2 {H_{66}}^{(2)}}{N_c I_1} + (M_d + M_s)^2 = \frac{4\pi^2 {H_{77}}^{(2)}}{N_c I_1} + (M_d + M_s)^2$$

$$\frac{1}{4k'_{u,d,s}} = \frac{2\pi^2 {H_{uu,dd,ss}}^{(2)}}{N_c I_1} + M_{u,d,s}^2$$

Model Extension - Quadratic Terms

- σ, ϕ **Kinetic Terms:**

$$
\begin{aligned}
&\frac{N_c I_1}{4\pi^2}\partial^\mu\sigma_a\partial_\mu\sigma_b\left\{\frac{\delta_{ab}}{2}+2k_e f_{ace}f_{bde}M_c M_d\right\}\\
&+\frac{N_c I_1}{4\pi^2}\partial^\mu\phi_a\partial_\mu\phi_b\left\{\frac{\delta_{ab}}{2}-2k'_e d_{ace}d_{bde}M_c M_d\right\}\\
=&\frac{N_c I_1}{16\pi^2}\sum_i\partial^\mu\sigma_i\partial_\mu\sigma_i-\sum_{a\neq 0,3,8}\frac{k_a}{2}H_{aa}{}^{(1)}\partial^\mu\sigma_a\partial_\mu\sigma_a\\
&+\sum_i\frac{k'_i}{2}H_{ii}{}^{(2)}\partial^\mu\phi_i\partial_\mu\phi_i+\sum_{a\neq 0,3,8}\frac{k'_a}{2}H_{aa}{}^{(2)}\partial^\mu\phi_a\partial_\mu\phi_a
\end{aligned}
$$

- Renormalize the fields in order to get the usual kinetic terms for the meson fields.

Model Extension - Quadratic Terms

- σ, ϕ **Renormalizations:**

$$\sigma_i \longrightarrow= \varrho\sigma_i \quad , \quad \sigma_a \longrightarrow \varrho\sqrt{1+\frac{\xi_a^\sigma}{\varrho^2}}\sigma_a$$

$$\phi_i \longrightarrow \varrho\sqrt{1+\frac{\xi_i^\phi}{\varrho^2}}\phi_i \quad , \quad \phi_a \longrightarrow \varrho\sqrt{1+\frac{\xi_a^\phi}{\varrho^2}}\phi_a$$

$$\xi_{1,2}^{\sigma,\phi} = \frac{(M_u \mp M_d)^2}{{H_{11,22}}^{(1,2)}} \quad , \quad \xi_{4,5}^{\sigma,\phi} = \frac{(M_u \mp M_s)^2}{{H_{44,55}}^{(1,2)}} \quad , \quad \xi_{6,7}^{\sigma,\phi} = \frac{(M_d \mp M_s)^2}{{H_{66,77}}^{(1,2)}}$$

$$\xi_i^\phi = \frac{2M_i^2}{{H_{ii}}^{(2)}} \quad , \quad \varrho = \sqrt{\frac{4\pi^2}{N_c I_1}}$$

Model Extension - Quadratic Terms

- V, A **Kinetic Terms:**

$$-\frac{N_c I_1}{48\pi^2}\text{tr}_F\left\{\frac{1}{3}\left(\bar{F}^{\mu\nu}_{(V)}\bar{F}^{(V)}_{\mu\nu}+\bar{F}^{\mu\nu}_{(A)}\bar{F}^{(A)}_{\mu\nu}\right)\right\}$$

- V, A **Renormalizations:**

$$V^\mu_a \longrightarrow \sqrt{\frac{3}{2}}\varrho V^\mu_a \quad , \quad A^\mu_a \longrightarrow \sqrt{\frac{3}{2}}\varrho A^\mu_a$$

- Mixing terms eliminated + Renormalization constants for the field components determined $\Longrightarrow$ Find the proper **mass terms** in order to **fit the meson spectrum**.

Model Extension - Mass Terms

- Identification of **scalar and pseudoscalar meson fields:**

$$\sigma = \begin{pmatrix} \sigma_u & a_0^+ & \kappa^+ \\ a_0^- & \sigma_d & \kappa^0 \\ \kappa^- & \bar{\kappa}^0 & \sigma_s \end{pmatrix} \quad , \quad \phi = \begin{pmatrix} \phi_u & \pi^+ & K^+ \\ \pi^- & \phi_d & K^0 \\ K^- & \bar{K}^0 & \phi_s \end{pmatrix}$$

$$\begin{cases} \sigma_u = \sigma_3 + \frac{\sqrt{2}\sigma_0 + \sigma_8}{\sqrt{3}} = \sigma_3 + f_{ns} \\ \sigma_d = -\sigma_3 + \frac{\sqrt{2}\sigma_0 + \sigma_8}{\sqrt{3}} = -\sigma_3 + f_{ns} \\ \sigma_s = \frac{\sqrt{2}\sigma_0 - 2\sigma_8}{\sqrt{3}} = \sqrt{2} f_s \end{cases}$$

$$\begin{cases} \phi_u = \phi_3 + \frac{\sqrt{2}\phi_0 + \phi_8}{\sqrt{3}} = \phi_3 + \eta_{ns} \\ \phi_d = -\phi_3 + \frac{\sqrt{2}\phi_0 + \phi_8}{\sqrt{3}} = -\phi_3 + \eta_{ns} \\ \phi_s = \frac{\sqrt{2}\phi_0 - 2\phi_8}{\sqrt{3}} = \sqrt{2} \eta_s \end{cases}$$

Model Extension - Mass Terms

- **Charged Pseudoscalar Mesons** (isospin limit):

$$\pi^+\pi^-\left(\varrho^2+\frac{4M_{u,d}^2}{H_{11,22}{}^{(2)}}\right)\left[h_{11,22}{}^{(2)}-\frac{h_{u,d}}{M_{u,d}}\right]$$

$$+K^+K^-\left(\varrho^2+\frac{(M_{u,d}+M_s)^2}{H_{44,55}{}^{(2)}}\right)\left[h_{44,55}{}^{(2)}+\frac{N_c I_0}{2\pi^2}\right.$$

$$\left.-\frac{N_c I_1}{6\pi^2}\left(M_{u,d}-M_s\right)\left(M_{u,d}-2M_s\right)\right]$$

$$+K^0\bar{K}^0\left(\varrho^2+\frac{(M_{u,d}+M_s)^2}{H_{66,77}{}^{(2)}}\right)\left[h_{66,77}{}^{(2)}+\frac{N_c I_0}{2\pi^2}\right.$$

$$\left.-\frac{N_c I_1}{6\pi^2}\left(M_{u,d}-M_s\right)\left(M_{u,d}-2M_s\right)\right]$$

Model Extension - Mass Terms

- **Neutral Pseudoscalar Mesons** (isospin limit):

$$\frac{1}{2}\left(\pi^0\right)^2\left(\varrho^2+\frac{2M_{u,d}^2}{H_{uu,dd}{}^{(2)}}\right)\left[2h_{uu,dd}{}^{(2)}-\frac{h_{u,d}}{M_{u,d}}-2h_{ud}{}^{(2)}\right]$$

$$+\frac{1}{2}\eta_{ns}^2\left(\varrho^2+\frac{2M_{u,d}^2}{H_{uu,dd}{}^{(2)}}\right)\left[2h_{uu,dd}{}^{(2)}-\frac{h_{u,d}}{M_{u,d}}+2h_{ud}{}^{(2)}\right]$$

$$+\frac{1}{2}\eta_s^2\left(\varrho^2+\frac{2M_s^2}{H_{ss}{}^{(2)}}\right)\left[2h_{ss}{}^{(2)}-\frac{h_s}{M_s}\right]$$

$$+2\eta_{ns}\eta_s\sqrt{2\left(\varrho^2+\frac{2M_{u,d}^2}{H_{uu,dd}{}^{(2)}}\right)\left(\varrho^2+\frac{2M_s^2}{H_{ss}{}^{(2)}}\right)}h_{us,ds}{}^{(2)}$$

- Introduce a mixing angle ψ_ϕ as

$$\begin{pmatrix}\eta\\ \eta'\end{pmatrix}=\begin{pmatrix}\cos\psi_\phi & -\sin\psi_\phi\\ \sin\psi_\phi & \cos\psi_\phi\end{pmatrix}\begin{pmatrix}\eta_{ns}\\ \eta_s\end{pmatrix}$$

- With this, the η, η' mass terms become

$$\frac{1}{2}\eta^2 \left\{ \cos^2\psi_\phi \tilde{M}^2_{\eta_{ns}} + \sin^2\psi_\phi \tilde{M}^2_{\eta_s} \right.$$
$$\left. -4\sin\psi_\phi\cos\psi_\phi \sqrt{2\left(\varrho^2 + \frac{2M^2_{u,d}}{H_{uu,dd}{}^{(2)}}\right)\left(\varrho^2 + \frac{2M^2_s}{H_{ss}{}^{(2)}}\right) h_{us,ds}{}^{(2)}} \right\}$$
$$+\frac{1}{2}\eta'^2 \left\{ \sin^2\psi_\phi \tilde{M}^2_{\eta_{ns}} + \cos^2\psi_\phi \tilde{M}^2_{\eta_s} \right.$$
$$\left. +4\sin\psi_\phi\cos\psi_\phi \sqrt{2\left(\varrho^2 + \frac{2M^2_{u,d}}{H_{uu,dd}{}^{(2)}}\right)\left(\varrho^2 + \frac{2M^2_s}{H_{ss}{}^{(2)}}\right) h_{us,ds}{}^{(2)}} \right\}$$

Model Extension - Mass Terms

- **Diagonalization condition:**

$$\tan(2\psi_\phi) + \frac{4\sqrt{2\left(\varrho^2 + \frac{2M_{u,d}^2}{H_{uu,dd}{}^{(2)}}\right)\left(\varrho^2 + \frac{2M_s^2}{H_{ss}{}^{(2)}}\right)}\, h_{us,ds}{}^{(2)}}{\tilde{M}_{\eta_{ns}}^2 - \tilde{M}_{\eta_s}^2} = 0$$

- The reasoning is exactly the same for the scalar and vector sectors.
- There is also a scalar mixing angle ψ_σ, with a corresponding diagonalization condition, but no vector mixing angle.
- In the isospin limit, there are 4 distinct pseudoscalar (π, K, η, η') and scalar $(a_0, \kappa, f_0, f_0'(\sigma))$ masses, but only 3 vector (ρ, K^*, φ) masses.

Model Extension - Weak Decay Constants

- **PCAC Hypothesis:**

$$\langle 0 | J_a^\mu(x) | \phi_b(p) \rangle = -i f_{ab} p^\mu e^{ix\cdot p}$$

- J_a^μ are axial currents computed from the variation of the Lagrangian $\delta\mathcal{L}$ due to an axial transformation of parameter β_a as

$$J_a^\mu = \frac{\partial \delta\mathcal{L}}{\partial(\partial_\mu \beta_a)} = \sum_C \frac{\partial \mathcal{L}}{\partial(\partial_\nu C)} \frac{\partial(\partial_\nu \delta C)}{\partial(\partial_\mu \beta_a)}$$

- Here, the sum in C extends over all field components of the Lagrangian.
- The f_{ab} are constant factors known as **weak decay constants**.
- With this, we may compute the π and K meson weak decay constants from the Lagrangian as:

$$f_\pi = \frac{M_{u,d}}{\sqrt{\varrho^2 + \xi_{1,2}^\phi}} \qquad , \qquad f_K = \frac{M_{u,d} + M_s}{2\sqrt{\varrho^2 + \xi_{4,5}^\phi}}$$

Model Extension - Fitting

- **Parameters and quantities to be fixed** (isospin limit): 20 from previous version of the model (m_u, m_s, M_u, M_s, Λ, h_u, h_s, G, g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8, κ, κ_2, θ_σ, θ_ϕ), plus 9 of the new parameters (w_1, w_6, w_7, w_8, w_9, w_{10}, w_{11}, w_{12}, w_{13}).
- **Conditions for fitting:** 2 SPA Conditions + 2 Gap Equations + 4 Pseudoscalar Masses + 4 Scalar Masses + 3 Vector Masses + 3 Axial Vector Masses + 2 Mixing Angle Conditions + 2 Weak Decay Constants = **22 conditions** vs. **29 parameters**
- Current quark masses are set at $m_u = 4$ MeV and $m_s = 100$ MeV; pseudoscalar mixing angle is set at $\theta_{ps} = -15^\circ$.
- Externally fix 4 more parameters.
  - w_1, w_6 and w_9 enters all expressions inthe combination $w_1 + w_6 h^2 + w_9 mh \Rightarrow$ Put $w_6 = w_9 = 0$ and externally set w_1.
  - We also chose $w_{13} = 0$ and then showed that altering this would amount only in a refitting of w_8 and w_{11}.

- **SPA Coefficients:**

$$
\begin{aligned}
-\left(H_{11,22}{}^{(1)}\right)^{-1} &= 2w_1 + \frac{1}{2}w_6 h^2 + \frac{1}{4}\left(w_7 + 2w_8\right) h_u^2 + w_9 m h \\
&+ \frac{1}{2}\left(w_{11} + 2w_{12}\right) h_u m_u + \frac{1}{4}\left(w_{14} + 2w_{15}\right) m_u^2 \\
-\left(H_{ss}{}^{(1)}\right)^{-1} &= 4w_1 + w_6 h^2 + \frac{1}{2}\left(w_7 + 2w_8\right) h_s^2 + 2w_9 m h \\
&+ \left(w_{11} + 2w_{12}\right) h_s m_s + \frac{1}{2}\left(w_{14} + 2w_{15}\right) m_s^2 \\
-\left(H_{44,55}{}^{(1)}\right)^{-1} &= 2w_1 + \frac{1}{2}w_6 h^2 + \frac{1}{4}w_7 h_u h_s + \frac{1}{4}w_8\left(h_u^2 + h_s^2\right) + w_9 m h \\
&+ \frac{1}{4}w_{11}\left(h_u m_s + h_s m_u\right) + \frac{1}{2}w_{12}\left(h_u m_u + h_s m_s\right) \\
&+ \frac{1}{4}w_{14} m_u m_s + \frac{1}{4}w_{15}\left(m_u^2 + m_s^2\right)
\end{aligned}
$$

Model Extension - Fitting

- w_1, w_6 and w_9 contribute in the same way for all vector masses $\Longrightarrow$ Set $w_6 = w_9 = 0$ and try to fit w_1.
- We can identify pairs of parameters which contribute similarly: w_7, w_8; w_{11}, w_{12}; and w_{14}, w_{15} $\Longrightarrow$ For each pair, set one of the parameters to 0 and try to fit the other.
- SPA Coefficient:

$$-\left({H_{11,22}}^{(2)}\right)^{-1} = 2w_1 + \frac{1}{2}w_6 h^2 + \frac{1}{4}\left(-w_7 + 2w_8\right) h_u^2 + w_9 mh + \frac{1}{2}\left(-w_{11} + 2w_{12}\right) h_u m_u + \frac{1}{4}\left(-w_{14} + 2w_{15}\right) m_u^2$$